Suraya M. NAJAFOVA

THEOREMS ON CONTINUATION OF FUNCTIONS BELONGING TO $W-$SPACES TYPE GENERAL SPACES

Abstract

It is constructed a W-spaces type generalized space of functions $f = f(x)$ of points $x = (x_1; ...; x_s) \in E_n$, $x_k = (x_{k,1}; ...; x_{k,n_k})$ $(k = 1, 2, ..., s)$, determined in domain$G \subset E_n = E_{n_1} \times ... \times E_{n_s}$ $(1 \le s \le n = n_1 + ... + n_s)$ satisfying the "$\sigma-$ semihorn" condition.

The conditions under which theorems on continuation of functions $f = f(x)$ beyond the domain $G \subset E_n$ with preservation of appropriate smoothness properties are found.

1. Construction of W-spaces type functional spaces of functions

1.1. By Q denote a set of all possible vectors $i = (i_1, ..., i_s)$ with coordinates

$$i_k \in \{0, 1, 2, ..., n_k\} \quad (k = 1, 2, ..., s). \tag{1.1}$$

The amount of elements of the set Q equals

$$|Q| = \prod_{k=1}^{s} (1 + n_k) \tag{1.2}$$

Consequently,

$$(n + 1) \le |Q| \le 2^n \, (n = n_1 + ... + n_s), \tag{1.3}$$

moreover

$$\left.\begin{array}{ll} |Q| = n + 1 & \text{(in the case } s = 1), \\ |Q| = 2^n & \text{(in the case } s = n). \end{array}\right\} \tag{1.4}$$

By means of the set Q we determine a collection of the following vectors

$$r^i = \left(r_1^{i_1}; ...; r_s^{i_s}\right) \quad (i = (i_1, i_2, ..., i_s) \in Q), \tag{1.5}$$

with coordinate vectors

$$r_k^{i_k} = \left(r_k^{i_k}; ...; r_{k,n_k}^{i_k}\right) \quad (k = 1, 2, ..., s)$$

moreover

$$r_{k,s}^{i_k} \ge 0 \, (j = 1, 2, ..., n_k) \quad (k = 1, 2, ..., s). \tag{1.6}$$

Assume that the collection of vectors (1.5) satisfies the "$*$ - arrangement condition", i.e. assume that

$$e_k^i = \sup pr_k^{i_k} \supset \{i_k\} \quad (k = 1, 2, ..., s) \tag{1.7}$$

Consequently,

$$r_{k,s}^{i_k} \ge 0 \, (k = 1, 2, ..., s) \tag{1.8}$$

for each $i=(i_1,...,i_s)\in Q$.

1.2. Now, by means of the collection of vectors (1.5) we determine appropriate collection of "integer non-negative" vectors

$$\left[r^i\right]=\left(\left[r_1^{i_1}\right];...;\left[r_s^{i_s}\right]\right)\ \ (i=(i_1,...,i_s)\in Q), \tag{1.9}$$

with coordinate vectors

$$\left[r_k^{i_k}\right]=\left(\left[r_{k,1}^{i_k}\right];...;\left[r_{k,n_k}^{i_k}\right]\right)\ \ (k=1,2,...,s) \tag{1.10}$$

i.e. such that

$$\left[r_{k,j}^{i_k}\right]\geq 0\,(j=1,2,...,n_k) \tag{1.11}$$

for all $k=1,2,...,s$ and for each $i=(i_1,...,i_s)\in Q$, moreover $\left[r_{k,j}^{i_k}\right]$ is an entire part of appropriate coordinate $r_{k,j}^{i_k}$, for $j=1,2,...,n_k$ $(k=1,2,...,s)$ from the collection of vectors (1.5), consequently

$$0\leq r_{k,j}^{i_k}-\left[r_{k,j}^{i_k}\right]<1\ \ (j=1,2,...,n_k;\ k=1,2,...,s). \tag{1.12}$$

Notice that

$$e_{*,k}^{i}=\sup p\left(r_k^{i_k}-\left[r_k^{i_k}\right]\right)(i=(i_1,...,i_s)\in Q) \tag{1.13}$$

is a set of indices $j\in\{1,2,...,n_k\}$, for which $r_k^{i_k}-\left[r_k^{i_k}\right]>0$, consequently, $e_{*,k}^{i}$ is a set of indices $j\in\{1,2,...,n_k\}$ for which $r_{k,j}^{i_k}$ are non integral for appropriate $k=1,2,...,s$ and $i=(i_1,...,i_s)\in Q$.

1.3. Let

$$\omega^i=\left(\omega_1^{i_k};...;\omega_s^{i_s}\right)\ \ (i=(i_1,...,i_s)\in Q) \tag{1.14}$$

be a collection of "integer non-negative" vectors with coordinate vectors

$$\omega_k^{i_k}=\left(\omega_1^{i_k};...;\omega_{k,n_k}^{i_k}\right)\ \ (k=1,2,...,s),$$

where

$$\omega_{k,j}^{i_k}=1 \text{ or } \omega_{k,j}^{i_k}=0\ \ (j=1,2,...,n_k)$$

for all $k=1,2,...,s$ and $i=(i_1,...,i_s)\in Q$.

Assume

$$\sup p\omega_k^{i_k}=e_{*,k}^{i_k}=\sup p\left(r_k^{i_k}-\left[r_k^{i_k}\right]\right) \tag{1.15}$$

$$(k=1,2,...,s),i=(i_1,...,i_s)\in Q.$$

This equality (1.15) means that

$$\omega_{k,j}^{i_k}=\begin{cases}1 \text{ for} & j\in e_{*,k}^{i_k}\\ 0 \text{ for} & j\in\{1,2,...,n_k\}-e_{*,k}^{i_k}\end{cases} \tag{1.16}$$

for all $k = 1, 2, ..., s$ and $i = (i_1, ..., i_s) \in Q$.

Let

$$\Delta^1_{k,j}(t_{k,j})\, g(..., x_{k,j}, ...) = g(..., x_{k,j} + t_{k,j}, ...) - g(..., x_{k,j}, ...) \quad (1.17)$$

be a finite difference of first order functions $g = g(x)$ in the direction of variable $x_{k,j}$, with step $t_{k,j}$ for appropriate $j = 1, 2, ..., n_k$ $(k = 1, 2, ..., s)$.

Then

$$\Delta_k^{\omega_k^{i_k}}(t_k)\, D^{\left[r_k^{i_k}\right]} f(..., x_k, ...) = \left\{ \prod_{j \in e_{*,k}} \Delta^1_{k,j}(t_{k,j}) \right\} D^{\left[r_k^{i_k}\right]} f(..., x_k, ...), \quad (1.18)$$

where

$$D^{\left[r_k^{i_k}\right]} f(...; x_k; ...) = \frac{\partial^{\left[r_{k,1}^{i_k}\right]}}{\partial x_{k,1}^{\left[r_{k,1}^{i_k}\right]}} \cdots \frac{\partial^{\left[r_{k,n_k}^{i_k}\right]}}{\partial x_{k,n_k}^{\left[r_{k,n_k}^{i_k}\right]}} f(...; x_k; ...).$$

Notice that

$$\Delta^{\omega^i}(t)\, D^{[r^i]} f(x) = \left\{ \prod_{k \in e_s^*} \Delta_k^{\omega_k^{i_k}}(t_k) \right\} D_1^{\left[r_1^{i_1}\right]} ... D_s^{\left[r_s^{i_s}\right]} f(x_{1;} ...; x_s), \quad (1.19)$$

for each $i = (i_1, ..., i_s) \in Q$.

Let domain $G \subset E_n$, then

$$\Delta^{\omega^i}(t; G)\, D^{[r^i]} f(x) = \Delta^{\omega^i}(t)\, D^{[r^i]} f(x), \quad (1.20)$$

if a mixed difference of the function is constructed by the vertices of a polyhedron wholly belonging to domain G, otherwise, we assume

$$\Delta^{\omega^i}(t; G)\, D^{[r^i]} f(x) = 0. \quad (1.21)$$

1.4. Cite generally accepted denotation

$$\|f\|_{p,G} = \|f\|_{L_p,(G)} = \left(\int_G |f(x)|^p\, dx \right)^{\frac{1}{p}}$$

for $1 \le p < \infty$,

$$\|f\|_{\infty,G} = \|f\|_{L_\infty,(G)} = vrai \sup_{x \in G} |f(x)|$$

for $p = \infty$.

Give a semi-norm of functions $f = f(x)$ by the equality (for each $i = (i_1, ..., i_s) \in Q$)

$$\|f\|_{L_p^{<r^i>},(G;s)} = \left\{ \int_{E_{|\omega^i|}} \left\| \frac{\Delta^{\omega^i}(t; G)\, D^{[r^i]} f(\cdot)}{\prod_{k \in \xi_s^i} \prod_{j \in e^i_{*,k}} |t_{k,j}|^{r^{i_k}_{k,j} - \left[r^{i_k}_{k,j}\right]}} \right\|_{p,G}^p \frac{dt}{t} \right\}^{1/p} \quad (1.22)$$

for $1 \le p < \infty$ where

$$\frac{dt}{t} = \prod_{k\in\xi_s^i}\prod_{j\in e_{*,k}^i}\frac{dt_{k,j}}{t_{k,j}}, \quad E_{|\omega^i|} = \prod_{k\in\xi_s^i}\prod_{j\in e_{*,k}^i}\{t_{k,j}\in E_1\}$$

therewith

$$\xi_s^i = \left\{k\in\{1,2,...,s\}\,;\,e_{*,k}^i \neq \varnothing\right\}.$$

In the case $p=\infty$ we assume

$$\|f\|_{L_\infty^{<r^i>},(G;s)} = vrai\sup_{t\in E_{|\omega^i|}}\left\|\frac{\Delta^{\omega^i}(t;G)\,D^{\left[r^i\right]}f(\cdot)}{\prod\limits_{k\in\xi_s^i}\prod\limits_{j\in e_{*,k}^i}|t_{k,j}|^{r_{k,j}^{i_k}-\left[r_{k,j}^{i_k}\right]}}\right\|_{\infty,G} \tag{1.23}$$

for each $i=(i_1,...,i_s)\in Q$.

Definition. *The space*

$$W = \bigcap_{i\in Q} L_p^{<r^i>}(G;s) \tag{1.24}$$

is a closure of the set of sufficiently smooth and finite in E_n functions $f=f(x)$ with respect to

$$\sum_{i=(i_1,...,i_s)\in Q}\|f\|_{L_p^{<r^i>}(G;s)} < \infty, \tag{1.25}$$

where the sum is taken over all possible vectors $i=(i_1,...,i_s)\in Q$ with coordinates

$$i_k\in\{0,1,2,...,n_k\}\,(k=1,2,...,s).$$

Notice that the functional space (1.24) is a generalization of the known spaces $W_p^{r_1,..,r_n}(G)$ of S.L. Sobolev - S.N. Slabodetskiy in the case $s=1$. In the case $s=n$, these spaces (1.24) are generalizations of spaces $S_p^rW(G)$ of S.M. Nikolskiy cited in the papers of S.M. Nikolskiy, P.I.Lizorkin, A.J. Jabrailov and others.

1.5. Let the vector $\delta=(\delta_1;...;\delta_s)$ with coordinate vectors $\delta_k=(\delta_{k,1};...;\delta_{k,n_k})$ $(k=1,2,..,s)$ be such that

$$\delta_{k,j}=+1 \text{ or } \delta_{k,j}=-1 \;\; (j=1,2,...,n_k)$$

for all $k\in e_s=\{1,2,...,s\}$.

Let the vector $\sigma=(\sigma_1;...;\sigma_s)$ with coordinate vectors $\sigma_k=(\sigma_{k,1};...;\sigma_{k,n_k})$ $(k=1,2,..,s)$ be "positive", i.e. $\sigma_{k,j}>0$ $(j=1,2,...,n_k)$ for all $k\in e_s=\{1,2,...,s\}$.

By $R_\delta(\sigma;h)$ (see [4]) we denote a "σ - semihorn" with a vertex at the origin of coordinates

$$R_\delta(\sigma;h) = \bigcup_{\substack{0<v_k\le h_k\\(k\in e_s)}}\left\{\begin{array}{c} y\in E_n;\, c_{k,j}^* \le \dfrac{y_{k,j}\delta_{k,j}}{\vartheta_k\sigma_{k,j}} \le c_{k,j}^{**} \\ \\ (j=1,2,...,n_k;\;\; k=1,2,...,s)\end{array}\right\}$$

Then it is obvious that

$$x + R_{\delta}(\sigma; h) \tag{1.26}$$

is a "σ - semihorn" with a vertex at the point $x \in E_n$.

Definition. *A subdomain $\Omega \subset G$ is said to be a subdomain satisfying the "σ-semihorn" condition, if*

$$x + R_{\delta}(\sigma; h) \subset G$$

for all $x \in E_n$ (see [4]).

Definition. *A subdomain $\Omega \subset G$ is said to be a domain satisfying the "σ - semihorn"condition if there exists a finite collection of subdomains*

$$\Omega_1, \Omega_2, ..., \Omega_N \subset G, \tag{1.27}$$

satisfying the condition of appropriate "σ - semihorns" covering the domain G, i.e. such that

$$\bigcup_{\mu=1}^{N} \Omega_{\mu} = G. \tag{1.28}$$

Definition. *A domain $\Omega \subset G$ satisfying the "σ - semihorn" condition is said to be a domain satisfying the condition of "strong σ - semihorn", if alongside with condition (1.27) it holds the condition*

$$\bigcup_{\mu=1}^{N} \Omega_{\mu,\varepsilon} = G \tag{1.29}$$

for some $\varepsilon > 0$ where

$$\Omega_{\mu,\varepsilon} = \{y \in \Omega_{\mu}; \rho(y; G/\Omega_{\mu}) > \varepsilon\}$$

is a set of all possible points $y \in \Omega_{\mu}$ lagging from G/Ω_{μ} at a distance more than ε.

2. Basic results. The basic results of the paper are given in the form of theorems.

Theorem 1. *Let*

$$f \in \bigcap_{i=(i_1,...,i_s)\in Q} L_p^{<r^i>}(G; s), \tag{2.1}$$

where $1 < p < \infty$. It is assumed in (2.1) that

1)the collection of vectors

$$r^i = \left(r_1^{i_1}; ...; r_s^{i_s}\right) \quad (i = (i_1, ..., i_s) \in Q) \tag{2.2}$$

are " $*$ - arranged ", i.e. the coordinate vectors

$$r_k^{i_k} = \left(r_k^{i_k}; ...; r_{k,n_k}^{i_k}\right) \quad (k = 1, 2, ..., s),$$

for each $i = (i_1, ..., i_s) \in Q$ are subjected to the conditions

$$\sup pr_k^{i_k} \supset \{i_k\} \quad (k = 1, 2, ..., s). \tag{2.3}$$

2) domain $G \subset E_n$ satisfies the "σ - semihorn" condition.

Let the given "integer non-negative" vector $\nu = (\nu_1; ...; \nu_s)$ with coordinate vectors

$$\nu_k = (\nu_{k,1}; ...; \nu_{k,n_k}) \quad (k = 1, 2, ..., s),$$

satisfy the " $*$ - agreement " condition with vectors of the collection (2.2) for each $i = (i_1, ..., i_s) \in Q$ in the form

$$\left.\begin{array}{l} \nu_{k,j} \geq r^0_{k,j}\,(j = 1, 2, .., n_k) \quad (\text{case } i_k = 0), \\ \left\{\begin{array}{l} \nu_{k,j} \geq r^{i_k}_{k,j}\,(j \neq i_k) \\ \nu_{k,i_k} < r^{i_k}_{k,i_k}\,(j = i_k) \end{array}\right. , \quad (\text{case } i_k \neq 0) \end{array}\right\} \tag{2.4}$$

for all $k = 1, 2, ..., s$. Moreover, let

$$\text{œ}_{k,i_k} = r^{i_k}_{k,i_k}\sigma_{k,i_k} - (\nu_k, \sigma_k) - \left(\frac{1}{p} - \frac{1}{q}\right)|\sigma_k| > 0$$

$$(k = 1, 2, ..., s). \tag{2.5}$$

for each $i = (i_1, ..., i_s) \in Q$ where

$$|\sigma_k| = \sigma_{k,1} + ... + \sigma_{k,n_k}, \ 1 < p \leq q < \infty,$$

$$(\nu_k, \sigma_k) = \sum_{j=1}^{n_k} \nu_{k,j}\sigma_{k,j} \quad (k = 1, 2, .., s).$$

Then there exists a generalized derivative

$$D^\nu f \in L_q(G; s), \tag{2.6}$$

and it is constructed a function $f_\nu = f_\nu(x)$ determined on E_n such that

$$f_\nu|_G = D^\nu f, \tag{2.7}$$

and the integral inequalitics

$$\|f_\nu\|_{L_q(E_n;s)} \leq c \sum_{i=(i_1,...,i_s)\in Q} \left(\prod_{k=1}^{s} h_k^{\text{œ}_{k,i_k}}\right) \|f_\nu\|_{L_p^{<r^i>}(G;s)}, \tag{2.8}$$

are valid, where c is a constant independent of the function $f = f(x)$ and the vector $h = (h_1, ..., h_s)$.

Theorem 2. *Under conditions of theorem 1, let a "non-negative vector"*

$$\rho = (\rho_1; ...; \rho_s), \tag{2.9}$$

with coordinate vectors

$$\rho_k = (\rho_{k,1}; ...; \rho_{k,n_k}) \quad (k = 1, 2, .., s),$$

be such that

$$(\rho_k, \sigma_k) \le œ_{k,i_k}\ (k = 1, 2, .., s) \tag{2.10}$$

for each $i = (i_1, ..., i_s) \in Q$ *(see. (2.5)).*

Then, assuming that domain $G \subset E_n$ satisfies the "strong condition of σ - semihorn", we can construct the function $f_\nu = f_\nu(x)$ determined on the E_n, such that

$$f_\nu|_G = D^\nu f, \tag{2.11}$$

moreover, the integral inequalities

$$\|f_\nu\|_{L_q^{<\rho>}(E_n;s)} \le c \sum_{i=(i_1,...,i_s)\in Q} \prod_{k=1}^{s} h_k^{œ_{k,i_k}-(\rho_k,\sigma_k)} \|f_\nu\|_{L_p^{<\rho>}(G;s)}, \tag{2.12}$$

are valid, where C is a constant independent of the function $f = f(x)$ and the vector $h = (h_1, ..., h_s)$.

The proof of basic results cited in the theorems are proved by the integral representations method worked out by the academician S.L. Sobolev, new integral representations of the functions $f = f(x)$ given in the monograph [4] of A.J. Jabrailov are used. The function $f = f(x)$ may be assumed to be sufficiently smooth, consequently integral representations of this function is written in the form

$$D^\nu f(x) = \sum_{i\in Q} A_{i,\delta} f(x), \tag{2.13}$$

where integral operators

$$A_{i,\delta} f(x) = C_\ell \left(\prod_{k\in e_s/e^i} h_k^{-\alpha_{k,0}} \right) \times$$

$$\times \int_{\vec{0}}^{\vec{h}} \prod_{k\in e_*^i} \frac{dv_k}{v_k^{1+\alpha_{k,i_k}}} \int_{E_{|\omega^i|}} dz \int_{E_n} \left\{ \Delta^{\omega^i}(z) D^{[r^i]} f_{(x+y)} \right\} \Phi_{i,\delta}(\cdot\cdot)\, dy. \tag{2.14}$$

Here, in (2.14)

$$\alpha_{k,0} = |\sigma_k| + (\nu_k, \sigma_k) \quad (\text{case } i_k = 0)$$

$$\alpha_{k,i_k} = |\sigma_k| + (\nu_k, \sigma_k) - \left[r_{k,i_k}^{i_k} \right] \sigma_{k,i_k} + \sigma_{k,i_k} \quad (\text{case } i_k = 0)$$

for all $k = 1, 2, .., s$.

A collection of auxiliary functions $f_{\nu,\mu} = f_{\nu,\mu}(x)$ coinciding on $\Omega_{\mu,\varepsilon} + R_{\gamma\mu}(\sigma; h)$ with the function $D^\nu f(x)$ is determined by the equality

$$f_{\nu,\mu}(x) = \sum_{i\in Q} A^*_{i,\delta^\mu} f(x), \tag{2.15}$$

moreover, the integral operators standing in the right hand side of equality (2.15) are of the form:

$$A^*_{i,\delta^\mu} f(x) = C_i \left(\prod_{k\in e_s/e_*^i} h_k^{-\alpha_{k,0}} \right) \int_{\vec{0}}^{\vec{h}} \prod_{k\in e_*^i} \frac{dv_k}{v_k^{1+\alpha_{k,i_k}}} \times$$

$$\times \int\limits_{E_{|\omega^i|}} dz \int\limits_{E_n} \left\{ \Delta^{\omega^i} \left(z; \Omega_{\mu,\varepsilon} + R_{\delta^\mu}\right) D^{[r^i]} f_{(x+y)} \right\} \Phi_{i,\delta^\mu} \left(\cdot\cdot\right) dy \tag{2.16}$$

for all $\mu = 1, 2, ..., N$.

We construct the desired function $f_\nu = f_\nu(x)$ by the equality

$$f_\nu(x) = \sum_{\mu=1}^{N} \eta_\mu(x) f_{\nu,\mu}(x). \tag{2.17}$$

In (2.17), the collection of functions

$$\eta_\mu = \eta_\mu(x) \quad (\mu = 1, 2, ..., N)$$

determines a unit expansion, in domain $G \subset E_n$ in covering

$$\{\Omega_{\mu,\varepsilon}\} \quad (\mu = 1, 2, ..., N).$$

By means of reasonings cited in [5], [6], we see that

$$\|f_{\nu,\mu}\|_{L_q^{<\rho>}(E_n;s)} \leq c(n) \sum_{i \in Q} \|f\|_{L_p^{<r^i>}(\Omega_{\mu,\varepsilon}+R_{\delta^\mu};s)} \tag{2.18}$$

Then it is obvious that

$$\left\|\widetilde{f_\nu}\right\|_{L_q^{<\rho>}(E_n;s)} \leq c \sum_{\mu=1} \sum_{i \in Q} \|f\|_{L_p^{<r^i>}(\Omega_{\mu,\varepsilon}+R_{\delta^\mu};s)} \leq c \sum_{i \in Q} \|f\|_{L_p^{<r^i>}(G;s)},$$

that proves the results of theorem 2, and theorem 1 is proved by the similar but more simple reasonings.

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Suraya N.Najafova
Azerbaijan. State. Pedogogical University.
34, U. Hajibeyov str. Az 1000, Baku, Azerbaijan.
Tel.: (99412) 493 33 69 (off.)

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