Togrul R. MURADOV

ON BASICITY OF PERTURBED SYSTEM OF EXPONENTS IN LEBESGUE SPACES WITH VARIABLE SUMMABILITY INDEX

Abstract

In this paper basis properties of some system of exponents in generalized Lebesque spaces $L_p(x)$ are investigated.

Recently interest to studying of the various questions connected with Lebesgue spaces with variable summability index has increased [1-3].

The monographs of Sharapudinov, Bilalov, Guseynov [4-6] are devoted to questions of basicity of a classical system of exponents, and also studying basis properties of exponential system of the following type in generalized Lebesgue spaces $L_p(x)$:

$$\{e^{i(n+\alpha\cdot signn)x}\}_{n\in Z}.$$

Investigation of basis properties of the following system of cosines and sines

$$\left\{\cos\left[\sqrt{n^2+\alpha}\cdot x\right]\right\}_{n\geq 0}\cup\left\{\sin\left[\sqrt{n^2+\alpha}\cdot x\right]\right\}_{n\geq 0}, \tag{1}$$

where $\alpha\in C$ is a complex parameter, in connection with the application to a specific problem of mechanics in different functional spaces is of special interest. The necessary and sufficient condition to parameter α, when this system forms Riesz basis in $L_p(-\pi+\delta,\pi+\delta)$, where $\delta\in R$ is a real parameter, was found by Yu.A.Kazmin [6].

In the offered work we consider the following generalization of system (1):

$$1\cup\{e^{\pm i\sqrt[m]{P_m(n)}\cdot x}\}_{n\in N} \tag{2}$$

in $L_p(x)$ spaces, where $P_m(n)$ is an m-degree polynomial

$$P_m(n)\equiv a_m\cdot n^m+...+a_o,\ a_m\neq 0, a_m>0,$$

with complex $a_i\in C, i=\overline{0,m-1}$ coefficients. $\sqrt[m]{z}$ means the branch for which $\sqrt[m]{1}=1$.

We'll always assume that the following condition holds:

$$P_m(n)\neq 0,\forall n\in N\ P_m(k)\neq P_m(l), k\neq l. \tag{3}$$

It should be noted that basis properties of system (2) in L_p have been earlier studied in [7].

1. Necessary concepts and facts. We will consider the Lebesgue measure on R. Denote by $meas\Omega$ the measure of $\Omega\subset R$. Let's mention some facts.

Thus, let $\Omega\subset R$ be a measurable subset and $meas\Omega>0$.

$$E\equiv\{u:\exists measu\in\Omega\}.$$

Let $p \in E$. Always assume that $u \in E$ and $\varphi(x,s) = s^{p(x)}, \forall x \in \Omega, s \geq 0$,

$$\rho(u) = \rho_{p(x)}(u) = \int_\Omega \varphi(x,|u|)dx = \int_\Omega |u(x)|^{p(x)}dx,$$

$$L_{p(x)}(\Omega) = \{u \in E : \lim_{\lambda \to +0} \rho(\lambda u) = 0, \quad L^0_{p(x)}(\Omega) = \{u \in E : \rho(u) < \infty,$$

$$L^1_{p(x)}(\Omega) = \{u \in E : \forall \lambda > 0, \rho(\lambda u) < \infty\}, \quad L^+_\infty(\Omega) = \{u \in L_\infty(\Omega) : ess \inf_\Omega u \geq 1\}.$$

From properties of the function $\varphi(x,s)$ it follows that

$$L_{p(x)}(\Omega) = \{u \in E : \exists \lambda > 0, \rho(\lambda u) < \infty\}.$$

Theorem 1. *The following two conditions are equivalent:*
1) $p \in L^+_\infty(\Omega)$*;*
2) $L^1_{p(x)}(\Omega) = L_{p(x)}(\Omega)$.

Henceforth let's consider a case when $p \in L^+_\infty(\Omega)$, i.e. $1 \leq p^- \leq p^+ < +\infty$, where $p^- = ess \inf_{x \in \Omega} p(x), p^+ = ess \sup_{x \in \Omega} p(x)$.

For simplicity we will write

$$E_\rho = L_{p(x)}(\Omega) = L^0_{p(x)}(\Omega) = L^1_{p(x)}(\Omega),$$

and call $L_{p(x)}(\Omega)$ a generalized Lebesgue space.

Norm $\|u\|_{L_{p(x)}(\Omega)}$ in E_ρ (denote it by $\|u\|_\rho$) is defined as follows:

$$\|u\|_\rho = \inf\{\lambda > 0 : \rho(\frac{u}{\lambda}) \leq 1\},$$

and $(E_\rho, \|u\|_\rho)$ forms a Banach space.

Let $\varphi_p(x,s) = \frac{1}{p(x)} \cdot s^{p(x)}$. Then

$$\rho_p(u) = \int_\Omega \varphi_p(x,|u(x)|)dx,$$

$$\|u\|_{\rho_p} = \inf\{\lambda > 0 : \rho_p(\frac{u}{\lambda}) \leq 1.$$

$\|u\|_{\rho_p}$ is an equivalent norm on $L_{p(x)}(\Omega)$.

Then, the conjugate function for φ_p will be the function

$$\varphi^*_p(x,s) = \frac{1}{q(x)} \cdot s^{q(x)},$$

where $q(x)$ is a conjugate to $p(x)$ function in the sense $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

Obviously, $(\varphi^*_p)^* = \varphi_p$ and q^-, q^+ are conjugated to p^-, p^+, respectively.

Then

$$\rho^*_p(v) = \int_\Omega \frac{1}{q(x)} \cdot |v(x)|^{q(x)}dx = \int_\Omega \varphi^*_p(x,|v(x)|)dx;$$

$$E^*_{\rho_p} = \{v \in E : \lim_{\lambda \to +0} \rho^*_p(\lambda v) = 0\}.$$

We have:

$$E^*_{\rho_p} = L_{q(x)}(\Omega) = L^0_{q(x)}(\Omega) = \{v \in E : \int\limits_{\Omega} |v(x)|^{q(x)} dx < \infty\}.$$

The following theorem is true.

Theorem 2. *$(L_{p(x)}(\Omega))^* = L_{q(x)}(\Omega)$, i.e.*

1) $\forall v \in L_{q(x)}(\Omega)$ function f defined as

$$f(u) = \int\limits_{\Omega} u(x)v(x)dx, \forall u \in L_{p(x)}(\Omega), \tag{4}$$

is a linear continuous functional on $L_{p(x)}(\Omega)$;

2) For any linear continuous functional f on $L_{p(x)}(\Omega)$ there exists a unique element $v \in L_{q(x)}(\Omega)$ such that f is defined exactly by the formula (4).

Theorem 3. *Let $u \in E_\rho$, then:*

1) $\|u\|_\rho < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$;

2) if $\|u\|_\rho > 1$, then $\|u\|_\rho^{p^-} \le \rho(u) \le \|u\|_\rho^{p^+}$;

3) if $\|u\|_\rho < 1$, then $\|u\|_\rho^{p^+} \le \rho(u) \le \|u\|_\rho^{p^-}$.

Under the class H^{ln} we'll understand a class of measurable on $[-\pi,\pi]$ functions $f(x)$ for which the inequality

$$|f(x_1) - f(x_2)| \le \frac{A}{ln\frac{1}{|x_1 - x_2|}}, |x_1 - x_2| \le \frac{1}{2}$$

is fulfilled. A is a constant dependent only on f.

Theorem 4. *Let $1 \le p^- \le p^+ < +\infty$. Then $C^0_\infty[-\pi,\pi]$ is dense in $L_{p(x)}$. In particular, $C[-\pi,\pi]$ is dense in $L_{p(x)}$.*

In sequel, we'll need the following fact

Theorem 5. (Paley-Wiener). *Let $\{x_n\}_{n=1}^\infty$ forms a basis in Banach space X with norm $\|\cdot\|$ and $\{x_n^*\}_{n=1}^\infty \subset X^*$ corresponding biorthogonal system. If $\exists \theta \in [0,1)$ is such that for $\forall x \in X$ the inequality*

$$\| \sum_n (x_n - y_n) x_n^*(x) \| \le \theta \|x\|$$

holds for any finite sum $\sum\limits_n$, then the system $\{y_n\}_{n=1}^\infty \subset X$ forms a basis in X isomorphic to $\{x_n\}_{n=1}^\infty$.

Let's prove the following lemma.

Lemma 1. *Let the sequence $\{x_n\}_{n\in N}$ forms a basis in some Banach space B.*

Then, if $M = card\{n : x_n \ne y_n\} < +\infty$, where $\{y_n\}$ is some sequence, the following statements are equivalent:

1) $\{y_n\}_{n\in N}$ forms basis in B isomorphic to $\{x_n\}_{n\in N}$;

2) the system $\{y_n\}_{n\in N}$ is complete in B;

3) the system $\{y_n\}_{n\in N}$ is minimal in B;

4) $\{y_n\}_{n\in N}$ is ω-linear independent in B.

Proof. Actually, consider the following operator:

$$Tx = \sum_{n\in M} x_n(x)(y_n - x_n).$$

It is clear that operator T is finite-dimensional and continuous, $T : B \to B$.

Consequently, T is a completely continuous operator. Therefore the operator $F = I + T$ is a Fredholm operator, where I is a unit operator.

It's easy to see that

$$Fx_n = y_n, \forall n \in N.$$

If at least one condition from conditions 1-4 of lemma 1 is fulfilled, then it follows that the operator F is invertible and as a result system $\{y_n\}_{n\in N}$ forms basis in B isomorphic to $\{x_n\}_{n\in N}$.

Lemma is proved.

2. Basic result. Consider system (2).

Without losing generality, we will assume that $a_m = 1$.

Let $a_{m-k} = 0, k = \overline{1,l}; a_{m-l} \neq 0, 1 \leq l \leq m$.

Then

$$\frac{[P_m(n)]^{1/m}}{n} = (1 + a_{m-l} \cdot n^{-l} + ... + a_0 \cdot n^{-m})^{1/m} =$$

$$= (1 + \underline{\underline{0}}(n^{-l}))^{1/m} = 1 + \underline{\underline{O}}(n^{-l}).$$

$$\sqrt[m]{P_m(n)} = n[1 + \underline{\underline{O}}(n^{-l})], n \in N.$$

Consequently,

$$\sqrt[m]{P_m(n)} = n[1 + a_{m-l} \cdot n^{-l} + \underline{\underline{O}}(n^{-l-1})], n \to \infty,$$

i.e.

$$\sqrt[m]{P_m(n)} = n + a_{m-l} \cdot n^{1-l} + \underline{\underline{O}}(n^{-l}), n \to \infty.$$

We investigate a case, when $l > 1$, i.e. $a_{m-1} = 0$.

In this case $\sqrt[m]{P_m(n)} = n + \underline{\underline{O}}(\frac{1}{n}) \equiv \lambda_n$. Denote $\delta_n = \lambda_n - n$.

It's obvious that $|\delta_n| \leq c \cdot \frac{1}{n}$, where c is a constant.

Let's estimate the following expression:

$$|e^{i\lambda_n x} - e^{inx}| = |e^{inx}(e^{i(\lambda_n - n)x} - 1)| = |e^{i(\lambda_n - n)x} - 1| = |\sum_{k=1}^{\infty} \frac{(i\delta_n)^k}{k} \cdot x^k| \leq$$

$$\leq \sum_{k=1}^{\infty} \frac{|\delta_n|^k}{k!} \cdot \pi^k \leq \sum_{k=1}^{\infty} \frac{c^k \cdot \pi^k}{k!} \cdot \frac{1}{n} \leq \frac{1}{n} \sum_{k=1}^{\infty} \frac{c \cdot \pi^k}{k!} = (e^{c\pi} - 1) \cdot \frac{1}{n} = \frac{c_1}{n}.$$

Denote by $e_n(x) = e^{\pm i \sqrt[m]{P_m(n)}x}$ if $|n| \geq n_0$ and $e_n(x) = e^{inx}$ if $k = 0; \pm 1; ...; \pm(n_0 - 1)$

Then

$$\|e_n(x) - e^{inx}\|_\rho \leq \frac{c_2}{n}. \tag{5}$$

Let $p : 1 < p \leq \min\{p^-; 2\}$ be an arbitrary number. From the formula (5) it directly follows that

$$\sum_n \|e_n(x) - e^{inx}\|_\rho^p \leq +\infty.$$

Let $\forall f \in L_{p(x)}$. It's clear that $f \in L_p$. Consider the following expression

$$\sum_n (e_n(x) - e^{inx}) f_n, \qquad (6)$$

where $\{f_n\}_{n\in Z}$ are the Fourier coefficients of the function f by the system $\{e^{inx}\}_{n\in Z}$. Estimating (6) we have:

$$\|\sum_n (e_n(x) - e^{inx}) f_n\|_\rho \le \sum_n \|e_n(x) - e^{inx}\|_\rho \cdot |f_n| \le$$

$$\le (\sum_n \|e_n(x) - e^{inx}\|_\rho^p)^{1/p} \cdot (\sum_n |f_n|^q)^{1/q}.$$

From the Hausdorff-Young theorem we have:

$$(\sum_n |f_n|^q)^{1/q} \le m\|f\|_p,$$

where m is a constant dependent only on p. Thus,

$$\|\sum_n (e_n(x) - e^{inx}) f_n\|_\rho \le m(\sum_n \|e_n(x) - e^{inx}\|_\rho^p)^{1/p} \cdot \|f\|_p.$$

From embedding $L_{p(x)} \subset L_p$ it follows that there exists a constant $c > 0$:

$$\|f\|_p \le c\|f\|_\rho.$$

Thus, for the expression (6) we have:

$$\|\sum_n (e_n(x) - e^{inx}) f_n\|_\rho \le cm(\sum_n \|e_n(x) - e^{inx}\|_\rho^p)^{1/p} \cdot \|f\|_\rho. \qquad (7)$$

From the formula (7) it directly follows that $\sum_n (e_n(x) - e^{inx}) f_n$ is some function from $L_{p(x)}$, since $L_{p(x)}$ is a Banach space.

It's obvious that $\exists n_0 \in N$:

$$\sum_{n\ge n_0} \|e_n(x) - e^{inx}\|_\rho^p < \frac{1}{(cm)^p}.$$

Let's introduce the function $\tilde{e_n}(x) \equiv e_n(x)$ if $|n| \ge n_0$ and $\tilde{e_n}(x) \equiv e^{inx}$ if $|n| < n_0$.

Define the operator T in the following way:

$$Tf = \sum_n (\tilde{e_n}(x) - e^{inx}) f_n,$$

where $\{f_n\}_{n\in Z}$ are the coefficients defined earlier.

It's clear that the operator T is correctly defined. Moreover, from estimate (7) it directly follows that

$$\|Tf\|_\rho \le \delta\|f\|_\rho,$$

where $\delta = cm(\sum\limits_{n} \|\widetilde{e_n}(x) - e^{inx}\|_{\rho}^{p})^{1/p} < 1$, whence it follows that $\|T\| < 1$.

As a result, the operator $(I+T)$ is invertible in $L_{p(x)}$, where I is an identity operator.

It's easy to see that

$$(I+T)[e^{inx}] = \widetilde{e_n}(x).$$

Hence, the system $\{\widetilde{e_n}(x)\}_{n=-\infty}^{+\infty}$ forms a basis in $L_{p(x)}$ space.

To prove the basicity of the system $\{e_n(x)\}_{n=-\infty}^{+\infty}$, owing to lemma 1, it suffices to show completeness or minimality of this system in $L_{p(x)}$.

From the results of monograph [7] it follows that the system $\{e_n(x)\}$ is complete in $L_p(-\pi,\pi), \forall p : 1 < p < +\infty$, and, in particular, in $L_{p^+}(-\pi,\pi)$.

From the continuous embedding $L_{p^+} \subset L_{p(x)}$ and density of $C_0^{\infty}(-\pi,\pi)$ in $L_{p(x)}$ we have the density of L_{p^+} in $L_{p(x)}$.

From the last statement it follows that the system $\{e_n(x)\}$ is complete in $L_{p(x)}$.

Theorem. *Let $p^- > 1, p \in H^{ln}, a_m = 1, a_{m-1} = 0$ and condition (3) hold. Then system (2) forms a basis in generalized Lebesgue space $L_{p(x)}(-\pi,\pi), \forall a_k \in C$, $k = \overline{2, m-2}$.*

REFERENCES

[1]. Kokilashvili V., Samko S. *Singular integrals in weighted Lebesgue spaces with variable exponent* Georgian Math. J., 2003, vol.10, No 1, pp.145-156.

[2]. Fan X., Zhao D. *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$* Jour. of Math. An. And Appl., 2001, No 263, pp.424-446.

[3]. Sharapudinov I.I. *On basicity of Haar system in $L^{p(x)}([0,1])$ and principle of mean localization* Mat. Sbor., 1986, 130(172), No 2(6), pp. 275-283 (Russian).

[4]. Bilalov B.T., Guseynov Z.G. *Bases from exponents in Lebesgue spaces of functions with variable summability exponent* Trans. of NAS Az., vol.XXVIII, 2008, No 1, pp.43-48.

[5]. Bilalov B.T. *On basis of some systems of exponents, cosines and sines in L_p* Dokl. Ak. Nauk, 2001, vol.379, No2, pp.158-160 (Russian).

[6]. Kazmin Yu.A. *Approximate properties in L_2 of some functional sequences* Mat. Zametki., 1988, vol.43, No5, pp.593-603 (Russian).

[7]. Muradov T.R. *Bases from the exponents in L_p* Trans. of NAS Az., vol.XXV, 2005, No 7, pp.101-106.

Togrul R. Muradov
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F. Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 47 20 (off.)
e-mail: togrul@europe.com

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