Oktay M. MAMEDOV, Ali MOLKHASI

ON CONGRUENCE SCHEMES AND COMPATIBLE RELATIONS OF ALGEBRAS

Abstract

By using congruence schemes we present a new characterization of algebras with modular congruence lattices. Then we prove new properties of compatible reflexive relations for k-majority algebras (in other terminology, for algebras having k-ary near unanimity operation among its term operations, $k \geq 3$). For algebras in congruence n-permutable varieties we show that every binary $(n-1)$-pretransitive compatible relation is symmetric. Then we obtain some consequences for some special types of relations.

Congruence scheme is a very important idea in the study of congruence lattices of algebras (see [1] – [5]); in particular in [6] we find some properties of varieties with a finite number of congruence schemes and relationships with first order definable principal congruences. In the present paper by using congruence schemes we formulate new characterization of congruence modular algebras and find some new properties of k-majority algebras, in other terminology, algebras having a k-ary near unanimity term operation. Then we show that every binary compatible $(n-1)$-pretrasiitive relation defined in an algebra belonging to a congruence n-permutable variety is symmetric. As a corollary we obtain that every compatible reflexive relation in an algebra from permutable variety is a tolerance relation. The next corollary is related with a quasiorder relation. Then we introduce so-called right θ-bound for relation θ and a pair (a, b) and we establish a property of algebras having all right θ-bounds.

1^0. Congruence modularity. Let $Con\ \mathbb{A}$ denote the congruence lattice of an algebra $\mathbb{A} = (A, F)$. It is well-known that $Con\ \mathbb{A}$ is a complete algebraic lattice with $0 = \Delta$ (the diagonal relation) and $1 = \nabla$ (the total relation), see [7]. If for all $\theta, \varphi, \psi \in Con\ \mathbb{A}$, with $\theta \leq \varphi$, $\varphi \cap (\theta \vee \psi) = \theta \vee (\varphi \cap \psi)$ then $Con\ \mathbb{A}$ is called modular. Recall that a lattice is nonmodular iff it contains "pentagon" N_5 as a sublattice.

SCHEME-MOD

Theorem 1. *The congruence lattice Con A of an algebra* $A = (A, F)$ *is modular if and only if for each* $\theta, \varphi, \psi \in Con\ A$ *with* $\varphi \cap \psi \leq \theta \leq \varphi$ *and every* $a, b \in A$ *and for all sequences of elements* $c_1, ..., c_k \in A$, $k \geq 1$*the following SCHEME-MOD holds:*

Proof. ($\Rightarrow$) Suppose that $Con\ \mathbb{A}$ is modular and for $\theta, \varphi, \psi \in Con\ \mathbb{A}$, we have

$$\varphi \cap \psi \leq \theta \leq \varphi \ \& \ (a, b) \in \varphi \ \& \ a\theta c_1 \psi c_2 \theta c_3 ... c_{k-1} \theta c_k \psi b.$$

Then, by modularity $(a, b) \in \varphi \cap (\theta \vee \psi) = \theta \vee (\varphi \cap \psi) = \theta$, so $(a, b) \in \theta$, as required.

($\Leftarrow$) Conversely, assume that an algebra $\mathbb{A}$ satisfies SCHEME-MOD, but $Con\ \mathbb{A}$ is not modular. Then, as it noted above, $Con\ \mathbb{A}$ contains a sublattice isomorphic to "pentagon" N_5 :

Obviously here $\varphi \cap \psi \leq \theta \leq \varphi$. Take any $(a, b) \in \varphi$. Then $(a, b) \in \theta \vee \psi$ and consequently there exist $c_1, ..., c_k \in A$ such that $(a, c_1) \in \theta$, $(c_1, c_2) \in \psi$, $(c_2, c_3) \in \theta, ...$ $..., (c_k, b) \in \psi$. Applying SCHEME-MOD we get $(a, b) \in \theta$, implying $\varphi \leq \theta$, $--$ a contradiction. Thus $Con\ \mathbb{A}$ must be a modular lattice.

2^0. On k-majority algebras. A term function $m(x_1, ..., x_k)$ of an algebra $\mathbb{A} = (A, F)$ is called a k-majority term (or sometimes, k-ary near unanimity term; see [8], p.34) if $m(y, x, x, .., x) = m(x, y, x, x, ..., x) = ... = m(x, .., x, y) = x$ holds for all $x, y \in A$. Clearly, 3-majority term is exactly the majority term. Algebras with a k-majority term are called k-majority algebras. We mention that various characterizations of majority algebras is given in [5].

Let us consider the following conditions.

(a) $\mathbb{A}$ is a 4-majority algebra with a term function $m(x_1, ..., x_4)$,

(b) $\mathbb{A}$ is a 5-majority algebra with a term function $m(x_1, ..., x_5)$,

(c) For every $a, b, c, e \in A$ and any compatible reflexive relations $\alpha, \beta, \delta, \gamma$ on $\mathbb{A}$ the following SCHEME-4 is satisfied:

(d) For every $a, b, c, e, d \in A$and any compatible reflexive relations $\alpha, \beta, \delta, \gamma, \rho$ on $\mathbb{A}$ the following SCHEME-5 is satisfied:

(e) Any compatible reflexive binary relations $\alpha, \beta, \delta, \gamma \subseteq A \times A$ satisfy

$$\alpha \cap (\beta \circ \delta \circ \gamma) \subseteq (\alpha \cap \delta) \circ (\alpha \circ \delta \cap \delta \circ \alpha) \circ (\alpha \cap \gamma);$$

(f) Any compatible reflexive binary relations $\alpha, \beta, \delta, \gamma, \rho \subseteq A \times A$ satisfy

$$\alpha \cap (\beta \circ \delta \circ \gamma \circ \rho) \subseteq (\alpha \cap \delta) \circ (\delta \circ \gamma \circ \alpha \cap \delta \circ \alpha \circ \gamma \cap \alpha \circ \delta \circ \gamma) \circ (\alpha \cap \rho).$$

Proposition 2. Let $\mathbb{A}$ be an algebra. Then the following assertions hold:
(i)(a)→(c)→(e), and(ii)(b)→(d)→(f).

Proof. (i).(a)→(c). Suppose $(a, b) \in \alpha$, $(a, c) \in \beta$, $(c, e) \in \delta$ and $(e, b) \in \gamma$, where $a, b, c, e \in A$ and $\alpha, \beta, \delta, \gamma \subseteq A \times A$ are compatible reflexive relations of $\mathbb{A}$. Take $d_1 := m(a, a, c, b)$ and $d_2 := m(a, b, e, b)$. Then we obtain: $(a, d_1) = (m(a, a, a, b), m(a, a, c, b)) \in \beta$ and $(a, d_1) = (m(a, a, c, a),\ m(a, a, c, b)) \in \alpha$.

So, $(a, d_1) \in \alpha \cap \beta$. Similarly, we get: $(d_2, b) = (m(a, b, e, b),\ m(b, b, e, b)) \in \alpha$ and $(d_2, b) = (m(a, b, e, b),\ m(a, b, b, b)) \in \gamma$.

So, $(d_2, b) \in \alpha \cap \gamma$. It is also clear that $(d_1, c) = (m(a, a, c, b),\ m(c, c, c, b)) \in \beta$, $(e, d_2) = (m(a, e, e, e),\ m(a, b, e, b)) \in \gamma$,

$$d_1 = m(a, a, c, b)\, \delta \ \ m(a, a, e, b)\, \alpha \ \ m(a, b, e, b) = d_2$$

and $d_1 = m(a, a, c, b)\, \alpha \ \ m(a, b, c, b)\, \delta \ \ m(a, b, e, b) = d_2$. So, $(d_1, d_2) \in \delta \circ \alpha \cap \alpha \circ \delta$.

(c)→(e). Take $(a, b) \in \alpha \cap (\beta \circ \delta \circ \gamma)$. Then there are $c, e \in A$ with $(a, c) \in \beta$, $(c, e) \in \delta$ and $(e, b) \in \gamma$. As $(a, b) \in \alpha$, by applying SCHEME-4 we obtain $(a, b) \in (\alpha \cap \beta) \circ (\delta \circ \alpha \cap \alpha \circ \delta) \circ (\alpha \cap \gamma)$. Hence the inclusion in (e) is proved.

(ii).*(b)* → *(d)*. Let $(a, b) \in \alpha$, $(a, c) \in \beta$, $(c, e) \in \delta$, $(e, d) \in \gamma$, and $(d, b) \in \rho$; here$a, b, c, e, d \in A$ and $\alpha, \beta, \delta, \gamma, \rho \subseteq A \times A$ are compatible reflexive relations. Take

$$d_1 := m(a, a, a, c, b)$$

and

$$d_2 := m(a, b, b, e, b).$$

Then we have:

$$(a, d_1) = (m(a, a, a, a, b),\ m(a, a, a, c, b)) \in \beta$$

and

$$(a, d_1) = (m(a, a, a, c, a),\ m(a, a, a, c, b,)) \in \alpha.$$

Thus $(a, d_1) \in \alpha \cap \beta$. Then we obtain:

$$(d_2, b) = (m(a, b, b, d, b),\ m(a, b, b, b, b)) \in \rho$$

and

$$(d_2, b) = (m(a, b, b, d, b),\ m(b, b, b, d, b,)) \in \alpha.$$

So, $(d_2 b) \in \alpha \cap \rho$. Similarly, it is clear that

$$(d_1, c) = (m(a, a, a, c, b),\ m(c, c, c, c, b)) \in \beta,$$

$$(d, d_2) = (m(a, d, d, d, d),\ m(a, b, b, d, b,)) \in \rho.$$

$$d_1 = m(a, a, a, c, b)\, \delta\, m(a, a, a, e, b)\, \gamma\, m(a, a, a, d, b)\, \alpha\, m(a, b, b, d, b) = d_2,$$

$$d_1 = m(a, a, a, c, b)\, \delta\, m(a, a, a, e, b)\, \alpha\, m(a, b, b, e, b)\, \gamma\, m(a, b, b, d, b) = d_2$$

and

$$d_1 = m(a, a, a, c, b)\, \alpha\, m(a, b, b, c, b)\, \delta\, m(a, b, b, e, b)\, \gamma\, m(a, b, b, d, b) = d_2.$$

So, $(d_1, d_2) \in \delta \circ \alpha \circ \gamma \cap \delta \circ \gamma \circ \alpha \cap \alpha \circ \delta \circ \gamma$.

The proof of implication (d)→(f) is similar to that of (c)→(e) and is omitted.

3^0. On compatible relations. Recall that Hagemann and Mitschke proved that a variety $\mathbb{V}$ is n-permutable $(n \geq 2)$ if and only if there are ternary terms $p_1, ..., p_{n-1}$ such that $\mathbb{V}$ satisfies the identities

$$(*) \qquad \begin{cases} x = p_1(x, z, z), \quad p_n(x, x, z) = z \\ p_i(x, x, z) = p_{i+1}(x, z, z)\, \forall i. \end{cases}$$

Well-known permutability is just 2-permutability (it implies congruence modularity) and n-permutability implies $(n + 1)$-permutability, of course.

Definition 3. *Let θ be any binary relation which is compatible with the all operations of an algebra $A = (A, F)$. If the relation $\theta^n = \theta \circ \theta \circ ... \circ \theta$ (n times) is contained in θ, $\theta^n \subseteq \theta$, then we will say that θ is n-pretransitive relation of A. Of course, every relation is 1-pretransitive.*

Theorem 4. *If A is an algebra in a some n-permutable variety then every $(n-1)$-pretransitive relation of A is a symmetric relation.*

Proof. Let $(a, b) \in \theta$ and θ is an $(n-1)$-pretransitive (compatible) relation of $\mathbb{A}$. Assume that there exist (θ-preserving) terms $p_1, ..., p_{n-1}$ on $\mathbb{A}$ satisfying the identities (*) above for n-permutability. We must show that $(b, a) \in \theta$. Indeed, by (*) we have:

$$b = p_{n-1}(a, a, b)\, \theta\, p_{n-1}(a, b, b) = p_{n-2}(a, a, b)\, \theta\, p_{n-2}(a, b, b) = ...$$

$$... = p_2(a, a, b)\, \theta\, p_2(a, b, b) = p_1(a, a, b)\, \theta\, p_1(a, b, b) = a.$$

As $\theta^n \subseteq \theta$, we have $(b, a) \in \theta$, as required.

Corollary 5. *(see [11]). If A is an algebra in a permutable variety then every compatible relation of A is a symmetric relation. In particular, every compatible reflexive relation of A is a tolerance relation.*

Recall that I. Chajda [10] proved for a variety $\mathbb{V}$: $\mathbb{V}$ is congruence permutable iff $\mathbb{V}$ is tolerance trivial (i.e. for each algebra $\mathbb{A} \in \mathbb{V}$, $Tol(\mathbb{A}) = Con(\mathbb{A})$). Consequently the next corollary is true (a well-known result of H.Werner).

Corollary 6. *(see [11]). If A is an algebra in a permutable variety then every compatible relation of A is a congruence relation.*

Recall also, that a quasiorder of an algebra $\mathbb{A} = (A, F)$ is a reflexive, transitive binary relation which is compatible with the operations F of $\mathbb{A}$. Let $Quord(\mathbb{A})$ stands for the set of quasiorders of $\mathbb{A}$. It is easy to see that $(Quord\mathbb{A}, \subseteq)$ is an algebraic lattice where the meet operation is the usual set-theoretic intersection of binary relations.

Corollary 7. *If A is an algebra in any n-perm*utable variety then every quasiorder of $\mathbb{A}$ is a congruence of $\mathbb{A}$, so $Quord\,(\mathbb{A}) = Con\,(\mathbb{A})$.

Definition 8. *Let θ be a binary relation on a set A. If for a pair $(a, b) \in A \times A$ there is an element $c \in A$ such that $(a, c) \in \theta \& \, (b, c) \in \theta$ then we will say that (a, b) has a right θ-bound.*

Lemma 9. *Let a binary relation θ is compatible with the operations of an algebra A and let $p : A^3 \to A$ be a term of A satisfying the identity $p\,(x, x, y) = y$. If $(a, b) \in A \times A$ has a right θ-bound then for all $z \in A$ $\;p\,(a, b, z)\,\theta z$.*

The proof is obvious: indeed, let c be a right θ-bound for (a, b) in $\mathbb{A}$. Then for all $z \in A$ we have:$p\,(a, b, z)\,\theta\;\; p\,(c, c, z) = z$, so $p\,(a, b, z)\,\theta z$.

Lemma 10. *Let a binary relation θ is compatible with the operations of an algebra A and let $p : A^3 \to A$ be a term of A satisfying the identity $p\,(x, x, y) = y$. If every $(a, b) \in A \times A$ every has a right θ-bound then $p\,(x, y, z)\,\theta z$ for all $x, y, z \in A$.*

The proof follows directly from Lemma 9.

Now we immediately obtain the next result, but firstly we recall the Maltsev condition given by H.-P. Gumm: a variety $\mathbb{V}$ is congruence modular if and only if for some $n \geq 1$ there exist 3-ary terms $d_0, ..., d_n$ and p such that $\mathbb{V}$ satisfies

(G1) $d_0\,(x, y, z) = x$,

(G2) $d_i\,(x, y, x) = x$ for all i,

(G3) $d_i\,(x, x, y) = d_{i+1}\,(x, x, y)$ for i even,

(G4) $d_i\,(x, y, y) = d_{i+1}\,(x, y, y)$ for i odd,

(G5) $d_n\,(x, y, y) = p\,(x, y, y)$,

(G6) $p\,(x, x, y) = y$.

These identities reduce to Jonsson's identities for congruence distributivity when we have $p\,(x, y, y) = y$.

Proposition 11. Let a binary relation θ is compatible with the operations of an algebra $\mathbb{A}$ and assume that every pair $(a, b) \in A \times A$ has a right θ-bound. If the variety $var(\mathbb{A})$ generated by $\mathbb{A}$ is congruence modular then $\mathbb{A}$ satisfies the above-mentioned equations (G1)$\to$(G4) and the relation $(G5^{/})\forall x, y \quad d_n\,(x, y, y)\,\theta y$.

Note. In a special case when θ is an compatible ordering relation, congruence modularity implies congruence distributivity (see[9]).

References

[1]. Gumm H.-P., *Geometrical methods in congruence modular algebras,* Mem. Amer. Math. Soc. 45 (1983), N 286.

[2]. Chajda I., *A note on the triangular scheme,* East-West J.Math. 3(2001), N.1, pp. 79-80.

[3]. Chajda I., Czedli G., Horvath E.K., *Trapezoid lemma and congruence distributivity,* Math.Slovaca 53(2003), pp. 247-253.

[4]. Czedli G., Horvath E.K., *Congruence modularity permit tolerances,* Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. 41(2002), pp. 43-53.

[5]. Chajda I., Radeleczki S., *Congruence schemes and their applications,*46(2005), N.1, pp. 1-14.

[6]. Mamedov O.M., *On varieties with a finite number of congruence schemes,* Proc. Azer. Math. Soc., 2(1996), pp.141-149.

[7]. Burris S., Sankappanavar H.P. *A course in Universal Algebra,* Springer-Verlag, 1981.

[8]. SzendreiA., *Clones in Universal Algebra* , Montreal, 1986.

[9]. Davey B.A., *Monotone clones and congruence modularity,* Order, 6(1990), N.4, pp. 389-400.

[10]. Chajda I., *Recent results and trends in tolerances on algebras and varieties,* Coll. Math. Soc. J.Bolyai, 28, Finite Algebra and Multiple-valued Logic, Szeged, 1979, North - Holl., 1981, pp. 69-95.

[11]. Werner H., *A Mal'cev condition for admissible relations,* Algebra Universalis, 3/2 (1973), 263 p.

Oktay M. Mamedov, Ali Molkhasi
Institute of Mathematics and Mechanics NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 47 20 (off)

Received April 02, 2009; Revised June 18, 2009.