Hidayat M.HUSEYNOV, Jalala A. OSMANLI

UNIQUENESS OF THE SOLUTION OF THE INVERSE SCATTERING PROBLEM FOR DISCONTINUOUS STURM-LIOUVILLE OPERATOR

Abstract

In the paper we obtain the basic equation of the inverse problem for discontinuous Sturm-Liouville operator and prove uniqueness of the determination of the potential by the scattering data.

Let's consider a boundary value problem generated by the differential equation

$$-y'' + q(x)\,y = \lambda^2 y, \qquad 0 < x < +\infty, \tag{1}$$

boundary condition

$$y(0) = 0, \tag{2}$$

and discontinuity conditions at some point $a \in (0, +\infty)$

$$y(a-0) = \alpha y(a+0),$$
$$y'(a-0) = \alpha^{-1} y'(a+0), \tag{3}$$

where $\alpha > 0$, $\alpha \neq 1$, $q(x)$ is a real-valued function satisfying the condition

$$\int_0^{+\infty} x\,|q(x)|\,dx < +\infty. \tag{4}$$

In the paper [4] it is shown that under condition (4), the boundary value problem (1)-(3) has a restrictied solution for $\lambda \in \mathbb{R} \setminus \{0\}$ and $\lambda = i\chi_k$ $(k = 1, 2, .., n)$, moreover, as $x \to \infty$ these solutions satisfy the asymptotic formulae:

$$u(x,\lambda) = e^{-i\lambda x} + S(\lambda)\,e^{i\lambda x} + o(1) \quad (\lambda \in \mathbb{R} \setminus \{0\}), \tag{5}$$

$$u_k(x, i\chi_k) = m_k e^{-\chi_k x}(1 + o(1)) \quad (k = 1, 2, .., n). \tag{6}$$

A totality of quantities $\{S(\lambda), \chi_k, m_k\}$ is said to be scattering data of problem (1)-(3).

Thus, the scattering data completely determines the behavior of the normed eigen functions of problem (1)-(3).

The inverse scattering problem for boundary value problem (1)-(3) consists of reconstruction of the function $q(x)$ by the scattering data.

In the present paper we obtain the basic equation of the inverse problem and investigating this equation we prove the uniqueness of the restoration of the function $q(x)$ by the scattering data.

The case $\alpha = 1$ was considered in the paper [1] (see also [2]).

1. Derivation of the basic equation

In the paper [3] it is proved that equation (1) provided (3),(4) has a Iost type solution $e(x,\lambda)$, regular with respect to λ in the upper half-plane $\operatorname{Im}\lambda > 0$, continuous for $\operatorname{Im}\lambda \geq 0$ and representable in the form

$$e(x,\lambda) = e_0(x,\lambda) + \int_x^{+\infty} K(x,t)\, e^{i\lambda t} dt, \tag{7}$$

where

$$e_0(x,\lambda) = \begin{cases} e^{i\lambda x}, & x > a, \\ \alpha^+ e^{i\lambda x} + \alpha^- e^{i\lambda(2a-x)}, & 0 < x \leq a, \end{cases} \qquad \alpha^{\pm} = \frac{1}{2}\left(\alpha \pm \frac{1}{\alpha}\right),$$

and the functions $K(x,t)$, $K'_x(x,t)$, $K'_t(x,t)$ are continuous for $t \neq 2a - x$, $K(x,\cdot) \in L_1(x,+\infty)$ and

$$K(x,x) = \frac{\alpha^+}{2} \int_x^{+\infty} q(t)\, dt, \qquad x \in (0,a),$$

$$K(x,x) = \frac{1}{2} \int_x^{+\infty} q(t)\, dt, \qquad x \in (a,+\infty), \tag{8}$$

$$K(x,2a-x+0) - K(x,2a-x-0) =$$
$$= \frac{\alpha^-}{2} \int_a^{+\infty} q(t)\, dt - \frac{\alpha^-}{2} \int_x^{a} q(t)\, dt, \quad x \in (0,a).$$

According to the paper [4], equation (1) has the solution $S(x,\lambda)$ satisfying the conditions (3) and $S(0,\lambda) = 0$, $S'(0,\lambda) = 1$, and this solution is connected with the Iost solutions $e(x,\lambda)$ by the formulae

$$\frac{-2i\lambda S(x,\lambda)}{e(0,\lambda)} = e(x,-\lambda) - S(\lambda)\, e(x,\lambda), \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

From here, taking into account representation (7) for the solution $e(x,\lambda)$, we have:

$$-2i\lambda S(x,\lambda) \left\{ \frac{1}{e(0,\lambda)} - \frac{1}{e_0(0,\lambda)} \right\} - \frac{2i\lambda}{e_0(0,\lambda)} \{S(x,\lambda) - S_0(x,\lambda)\} =$$

$$= \int_x^{+\infty} K(x,t)\, e^{-i\lambda t} dt + \{S_0(\lambda) - S(\lambda)\} \times$$

$$\times \left\{ e_0(x,\lambda) + \int_x^{+\infty} K(x,t)\, e^{i\lambda t} dt \right\} - S_0(\lambda) \int_x^{+\infty} K(x,t)\, e^{i\lambda t} dt, \tag{9}$$

where

$$S_0(\lambda) = \frac{e_0(0,-\lambda)}{e_0(0,\lambda)} = \frac{\alpha^+ + \alpha^- e^{-2ia\lambda}}{\alpha^+ + \alpha^- e^{2ia\lambda}},$$

$$S_0(\lambda)=\begin{cases}\frac{\sin\lambda x}{\lambda}, & 0<x\le a,\\ \alpha+\frac{\sin\lambda x}{\lambda}+\alpha\frac{\sin\lambda(x-2a)}{\lambda}, & x>a,\end{cases}$$

is the solution of the equation $-y''=\lambda^2 y$ satisfying conditions (3) and $S_0(0,\lambda)=0$, $S_0'(0,\lambda)=1$.

As it was shown in the paper [4] (see lemma 2), $S_0(\lambda)-S(\lambda)$ is a Fourier transform of the function

$$F_s(y)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left[S_0(\lambda)-S(\lambda)\right]e^{i\lambda y}d\lambda. \tag{10}$$

Further, since

$$S_0(\lambda)=\frac{\alpha^++\alpha^-e^{-2i\lambda a}}{\alpha^+}\cdot\frac{1}{1+\dfrac{\alpha^-}{\alpha^+}e^{2i\lambda a}}=$$

$$=\left(1+\frac{\alpha^-}{\alpha^+}e^{-2i\lambda a}\right)\left(1-\frac{\alpha^-}{\alpha^+}e^{2i\lambda a}+\left(\frac{\alpha^-}{\alpha^+}\right)^2e^{4i\lambda a}-\cdots\right)=$$

$$1+\frac{\alpha^-}{\alpha^+}e^{-2i\lambda a}-\left(\frac{\alpha^-}{\alpha^+}\right)^2-\frac{\alpha^-}{\alpha^+}e^{2i\lambda a}+\left(\frac{\alpha^-}{\alpha^+}\right)^3e^{2i\lambda a}+\left(\frac{\alpha^-}{\alpha^+}\right)^2e^{4i\lambda a}-\cdots,$$

we have

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}S_0(\lambda)e^{i\lambda t}dt=\delta(t)+\frac{\alpha^-}{\alpha^+}\delta(t-2a)-\left(\frac{\alpha^-}{\alpha^+}\right)^2\delta(t)-$$

$$-\left[\frac{\alpha^-}{\alpha^+}-\left(\frac{\alpha^-}{\alpha^+}\right)^3\right]\delta(t+2a)+\cdots. \tag{11}$$

Now, multiplying the both hand sides of equality (9) by $\dfrac{1}{2\pi}e^{i\lambda y}$ and integrating with respect to $\lambda\in(-\infty,+\infty)$ and taking into account (10) and (11) in the right hand side, we get

$$K(x,y)+F_{1s}(x,y)+\int\limits_x^{+\infty}K(x,t)F_s(t+y)dt-\left[1-\left(\frac{\alpha^-}{\alpha^+}\right)^2\right]K(x,-y)-$$

$$-\frac{\alpha^-}{\alpha^+}K(x,2a-y)+\left[\frac{\alpha^-}{\alpha^+}+\left(\frac{\alpha^-}{\alpha^+}\right)^3\right]K(x,-y-2a)-\cdots, \tag{12}$$

where

$$F_{1s}(x,y)=\begin{cases}F_s(x+y), & x>a,\\ \alpha^+F_s(x+y)+\alpha^-F_s(2a-x+y), & 0<x\le a.\end{cases} \tag{13}$$

If we put $y > x$, the expression (12) takes the form

$$K(x,y) + F_{1s}(x,y) + \int_x^{+\infty} K(x,t) F_s(t+y)\,dt - \frac{\alpha^-}{\alpha^+} K(x, 2a-y). \qquad (14)$$

If remains to calculate the integral

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ -2i\lambda S(x,\lambda) \left[\frac{1}{e(0,\lambda)} - \frac{1}{e_0(0,\lambda)} \right] - \right.$$

$$\left. - \frac{2i\lambda}{e_0(0,\lambda)} \left[S(x,\lambda) - S_0(x,\lambda) \right] \right\} e^{i\lambda y} d\lambda,$$

when $y > x$. Show that this integral may be calculated by means of the contour integration. Really, the first term in this integral is regular everywhere at the upper half-plane except the finite number of points $i\chi_k$, that are the simple zeros of the function $e(0,\lambda)$ (see [4]). The second term in this integral is regular at the upper half-plane and the function $\lambda [S(x,\lambda) - S_0(x,\lambda)] e^{i\lambda y}$ is restricted for $\operatorname{Im}\lambda \geq 0$, $y > x$. Consequently, using the Jordan lemma, for $y > x$ we get

$$I = \sum_{k=1}^{n} \frac{2i\chi_k S(\chi, i\chi_k) e^{\chi_k y}}{e(0, i\chi_k)} = -\sum_{k=1}^{n} m_k^2 e(x, i\chi_k) e^{-\chi_k y} =$$

$$= -\sum_{k=1}^{n} m_k^2 e_0(x, i\chi_k) e^{-\chi_k y} - \int_x^{+\infty} K(x,t) - \sum_{k=1}^{n} m_k^2 e^{-\chi_k (t+y)} dt.$$

Equating I and the expressions from (14) we get the basic equations of the inverse problem for $K(x,y)$

$$K(x,y) - \frac{\alpha^-}{\alpha^+} K(x, 2a-y) + F_1(x,y) + \int_x^{\infty} K(x,t) F(t+y)\,dt = 0, \quad y > x, \qquad (15)$$

where

$$F(y) = F_s(y) + \sum_{k=1}^{n} m_k^2 e^{-\chi_k y},$$

$$F_1(x,y) = \begin{cases} F(x+y), & x > a, \\ \alpha^+ F(x+y) + \alpha^- F(2a - x + y), & 0 \leq x \leq a. \end{cases} \qquad (16)$$

Thus, we proved the following

Theorem. *For each $x \geq 0$ the kernel $K(x,y)$ from representation (5) satisfies the functional-integral equation (15),(16).*

Equation (15) plays an important part in solving the inverse problem of the scattering theory. Really, if equation (15), constructed only by the scattering data (see. (10),(16)), has a unique solution $K(x,y)$, equation (1), i.e. the function $q(x)$ may be found from formula (8).

2. Solvability of the basic equation

From the property of the function $F_s(x)$ and from the form of $F(x)$ it follows that for each fixed $x \geq 0$ the operator

$$(F_x f)\, y = \int\limits_x^{+\infty} F(t+y)\, f(t)\, dt,$$

acting in the space $L_1(x, +\infty)$ (also in $L_2(x, +\infty)$) is completely continuous.

In the basic equation we'll consider the kernel $K(x,t)$ as unknown and consider it as Fredholm type equation in the space $L_2(x, +\infty)$ (or $L_1(x, +\infty)$) for each fixed x.

It holds

Theorem. *For each fixed $x \geq 0$ the basic equation (15) has a unique solution $K(x, \cdot)$ from the space $L_2(x, +\infty)$.*

Proof. At first we show that for each fixed $x \geq 0$ the operator

$$(M_x f)(y) = \begin{cases} f(y), & x > a, \\ f(y) - \dfrac{\alpha^-}{\alpha^+} f(2a - y), & 0 \leq x \leq a, \end{cases}$$

acting in the space $L_2(x, +\infty)$ is invertible.

It suffices to consider the case $0 \leq x \leq a$. Let's consider the equation

$$f(y) - \frac{\alpha^-}{\alpha^+} f(2a - y) = g(y). \tag{17}$$

Making substitution $y \to 2a - y$, hence we have

$$f(2a - y) - \frac{\alpha^-}{\alpha^+} f(y) = g(2a - y). \tag{18}$$

From the system of equations (17)-(18) we get

$$f(y) = \frac{(\alpha^+)^2}{(\alpha^+)^2 + (\alpha^-)^2} \left[g(y) + g(2a - y) \right],$$

i.e. the operator M_x has the inverse. From the last formula we have

$$\int\limits_x^{+\infty} |f(y)|^2\, dy \leq C \int\limits_x^{+\infty} |g(y)|^2\, dy,$$

where C are some constants. Thus, we proved that for each fixed $x \geq 0$, the operator M_x is invertible in the space.

Now,let's denote that the basic equation is equivalent to the equation

$$K(x, y) + M_x^{-1} F_1(x, y) + M_x^{-1} \left(F\ K(x, \cdot) \right)(y) = 0, \quad y > x$$

i.e. to the equation with completely continuous operator, since $M_x^{-1} F$ is a completely continuous operator.

Thus, in order to prove the theorem, it suffices to show that the homogeneous equation

$$f_x(y) + M_x^{-1}(Ff_x)(y) = 0,$$

i.e. the equation

$$f_x(y) - \frac{\alpha^-}{\alpha^+} f_x(2a-y) + \int_x^{+\infty} f_x(t) F(t+y)\, dt = 0,\ \ y > x, \tag{19}$$

has only a zero solution $f_x(\cdot) \in L_2(x, +\infty)$.

We multiply the equation (19) by $\overline{f_x(y)}$ and integrate with respect to y in the interval $(x, +\infty)$. As the result, according to (10) and (16) we get

$$\int_x^{+\infty} |f_x(y)|^2\, dy - \frac{\alpha^-}{\alpha^+} \int_x^{+\infty} f_x(2a-y)\, \overline{f_x(y)} dy +$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} [S_0(\lambda) - S(\lambda)]\, \widetilde{f}(-\lambda)\, \overline{\widetilde{f}(\lambda)} d\lambda +$$

$$+ \sum_{k=1}^{n} m_k^2 \left| \widetilde{f}(-i\chi_k)^2 \right| = 0, \quad \text{where } \widetilde{f}(\lambda) = \int_{-\infty}^{+\infty} f_x(y)\, e^{-i\lambda y} dy.$$

Here, taking into account the equality (see(11))

$$\frac{\alpha^+}{\alpha^-} \int_x^{+\infty} f_x(2a-y)\, \overline{f_x(y)} dy = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_0(\lambda)\, \widetilde{f}(-\lambda)\, \overline{\widetilde{f}(\lambda)} d\lambda,$$

and also Parseval equality

$$\int_x^{+\infty} |f_x(y)|^2\, dy = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widetilde{f}(\lambda)\, \overline{\widetilde{f}(\lambda)} d\lambda,$$

we finally have

$$\sum_{k=1}^{n} m_k^2 \left| \widetilde{f}(-i\chi_k) \right|^2 + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ \widetilde{f}(\lambda) - S(\lambda)\, \widetilde{f}(-\lambda) \right\} \overline{\widetilde{f}(\lambda)} d\lambda = 0. \tag{20}$$

Since $|S(\lambda)| = 1$, then by the Cauchy-Bunyakovskii inequality

$$\left| \int_{-\infty}^{+\infty} S(\lambda)\, \widetilde{f}(-\lambda)\, \overline{\widetilde{f}(\lambda)} d\lambda \right| \leq \int_{-\infty}^{+\infty} \left| \widetilde{f}(\lambda) \right|^2 d\lambda.$$

Consequently, the second term at the left hand side of (20) is non-negative. Therefore, from equality (20) we have

$$\widetilde{f}(-i\chi_k) = 0, \quad k = 1, 2, .., n,$$

$$\int_{-\infty}^{+\infty} \left\{ \widetilde{f}(\lambda) - S(\lambda)\widetilde{f}(-\lambda) \right\} \overline{\widetilde{f}(\lambda)} d\lambda = 0.$$

Assuming $z(\lambda) = \widetilde{f}(\lambda) - S(\lambda)\widetilde{f}(-\lambda)$, we see that this function is orthogonal to the function $\widetilde{f}(\lambda)$. But then

$$\left\| \widetilde{f}(\lambda) \right\|^2 = \left\| S(\lambda)\widetilde{f}(-\lambda) \right\|^2 = \left\| \widetilde{f}(\lambda) - z(\lambda) \right\|^2 = \left\| \widetilde{f}(\lambda) \right\|^2 + \|z(\lambda)\|^2,$$

that is possible only for $z(\lambda) = 0$. So, we have

$$\widetilde{f}(-i\chi_k) = 0, \quad k = 1, 2, .., n, \tag{21}$$

$$\widetilde{f}(\lambda) = S(\lambda)\widetilde{f}(-\lambda). \tag{22}$$

By definition (see [2])

$$S(\lambda) = \frac{e(0, -\lambda)}{e(0, \lambda)}.$$

Substituting at in (22), we have

$$\frac{\widetilde{f}(\lambda)}{e(0, -\lambda)} = \frac{\widetilde{f}(-\lambda)}{e(0, \lambda)}, \quad -\infty < \lambda < +\infty.$$

Hence, it follows that $\dfrac{\widetilde{f}(-\lambda)}{e(0, \lambda)}$ is a meromorphic function on the whole of complex plane and has poles of first order in the zeros of the function $e(0, \lambda)$.

Consequently,

$$\frac{\widetilde{f}(-\lambda)}{e(0, \lambda)} = \sum_{k=1}^{n} \frac{1}{\lambda - i\chi_k} \cdot \frac{\widetilde{f}(-i\chi_k)}{e(i\chi_k)} + \psi(\lambda),$$

where $\psi(\lambda)$ is an entire function. It follows from (21) that

$$\frac{\widetilde{f}(-\lambda)}{e(0, \lambda)} = \psi(\lambda).$$

Obviously, as $\lambda \to \infty$ the left hand side tends to zero. Consequently, $\psi(\lambda) \equiv 0$. Then we have $\widetilde{f}(-\lambda) = 0$, i.e. $f_x(y) = 0$.

The theorem is proved.

Corollary. *The potential $q(x)$ is uniquely determined by the scattering data.*

References

[1]. Huseynov H.M., Osmanova J.A. *On Iost solution of Sturm-Liouville equation with discontinuity conditions.* Transactions of NAS of Azerbaijan, 2007, vol. XXVII, No 1, pp.63-70.

[2]. Osmanova J.A. *On scatterring data for discontinuous Sturm-Liouville operator.* Transactions of NAS of Azerbaijan, 2007, vol. XXVII, No 4, pp.73-80.

[3]. Marchenko V.A. *Renewal of potential energy by the phases of scattering waves.* Dokl. AN SSSR, 1995, 104, No 5, pp. 695-698 (Russian).

[4]. Marchenko V.A. *Sturm-Liouville operators and their applications.* Kiev, "Naukova Dumka", 1977, 331 p. (Russian).

Hidayat M.Huseynov, Jalala A.Osmanli
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F. Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 47 20 (off.).

Received September 01, 2008 ; Revised November 25, 2008.