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ON BASES FROM LINEAR PHASE EXPONENTS IN LEBESGUE SPACES WITH VARIABLE EXPONENT

Abstract

In the paper we consider the systems of exponents $\{\exp i\,(n-\alpha sign n)\,t\}_{n\in Z}$, $1\cup\{\exp i\,(n-\alpha sign n)\,t\}_{n\neq 0}$, *cosines* $\{\cos(n-\alpha)\,t\}_{n\geq 0}$ $\left(1\cup\{\cos(n-\alpha)\,t\}_{n\geq 1}\right)$ *and sines* $\{\sin(n-\alpha)\,t\}_{n\geq 1}$. *The basis properties of these systems are completely studied in the space* L_{p_t} *with variable exponent* $p(t)$.

The goal of the paper is to establish the basicity of the following systems of exponents

$$\left\{e^{i[(n-\alpha\cdot sign n)t-\beta\cdot sign n]}\right\}_{n\in Z}, \tag{1}$$

$$1\cup\left\{e^{i[(n-\alpha\cdot sign n)t-\beta\cdot sign n]}\right\}_{n\neq 0}, \tag{2}$$

in Lebesgue spaces of functions with variable exponent $p(t)$ denoted as L_{p_t}, where $\alpha,\beta\in C$ are complex parameters.The systems (1),(2) are model systems in order to study spectral properties of some differential operators. They are obtained from an ordinary system of exponents by linear perturbation. Apprently such known mathematicians as Paley-Weiner [1], N. Levinsion [2] and others first began to study basis properties of these systems.In ordinary Lebesgue spaces $L_p\,(p(t)\equiv const)$ the basis properties of systems (1),(2) have been completely studied.Concerning these issues we can consider the papers [2-4]. Recently in connection with consideration of some concrete problems of mechanics and mathematical physics (see f.e. [5,6]) interest to studying of such or other problems in the spaces L_{p_t} and $W_{p_t}^k$ increases.

In the present paper we study the basicity of systems (1),(2) in $L_{p_t}\equiv L_{p_t}(-\pi,\pi)$ under definite conditions on the function $p:[-\pi,\pi]\to[1,+\infty)$.

1. Necessary notation and facts. Let $p:[-\pi,\pi]\to[1,+\infty)$ be some function measurable by Lebesgue. By $\mathcal{L}_0$ we denote a class of all functions measurable on $[-\pi,\pi]$ (with respect to Lebesgue measure). Accept the denotation

$$I_p(f)\overset{def}{\equiv}\int_{-\pi}^{\pi}|f(t)|^{p(t)}\,dt.$$

Let $\mathcal{L}\equiv\{f\in\mathcal{L}_0: I_p(f)<+\infty\}$. With respect to ordinary linear operations of addition of functions and multiplication by the number, $\mathcal{L}$ turns into a linear space $p^+=\sup\underset{[-\pi,\pi]}{vrai}\,p(t)$. By then the norm if .

$$\|f\|_{p_t}\overset{def}{\equiv}\inf\left\{\lambda>0: I_p\left(\frac{f}{\lambda}\right)\leq 1\right\},$$

$\mathcal{L}$ is a Banach space and denote it by L_{p_t}.

Denote

$$H^{\ln} \overset{def}{\equiv} \left\{ p : p(\pi) = p(-\pi) \text{ and } \exists C > 0;\ \forall t_1, t_2 \in [-\pi, \pi], |t_1 - t_2| \le \frac{1}{2} \Longrightarrow \right.$$

$$\left. \Longrightarrow |p(t_1) - p(t_2)| \le \frac{C}{-\ln|t_1 - t_2|} \right\}.$$

Everywhere $q(t)$ denotes a function adjoint to $p(t)$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Accept $p^- = \inf\limits_{[-\pi,\pi]} vrai\, p(t)$,It holds Holder's generalized inequality:

$$\int_{-\pi}^{\pi} |f(t) g(t)|\, dt \le C\left(p^-; p^+\right) \|f\|_{p_t} \cdot \|g\|_{q_t},$$

where $C(p^-; p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

The following property follows directly from the definition

Property A**.** If $|f(t)| \le |g(t)|$ a.e. on $(-\pi, \pi)$, then $\|f\|_{p_t} \le \|g\|_{p_t}$.

By $C[-\pi, \pi]$ as usual we denote a space of continuous on $[-\pi, \pi]$ functions.We easily prove the following

Statement 1. Let $p \in H^{\ln}; p(t) > 0$, $\forall t \in [-\pi, \pi]$ and $\{\alpha_i\}_1^m \subset R$ ($-$ is a real axis). The function $\omega(t) = \prod\limits_{i=1}^{m} |t - t_i|^{\alpha_i}$ belongs to the space L_{p_t}, iff $\alpha_i > -\frac{1}{p(t_i)}$, $\forall i = \overline{1, m}$; where $\{t_i\}_1^m \subset [-\pi, \pi]$, $t_i \ne t_j$ for $i \ne j$.

In sequel, we'll need the following facts.

Property B **[5].** If $p(t) : 1 < p^- \le p^+ < +\infty$, the class $C_0^\infty(-\pi, \pi)$ (finite and infinitely differentiable) is everywhere dense in L_{p_t}.

By S we denote a singular integral:

$$Sf = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \Gamma,$$

Let $\rho : [-\pi, \pi] \to [1, +\infty)$ be some weight function. Let's define a weight class L_{p_t, ρ_t} :

$L_{p_t,\rho_t} \overset{def}{\equiv} \{f : \rho \cdot f \in L_{p_t}\}$ with the norm:

$$\|f\|_{p_t,\rho_t} \overset{def}{\equiv} \|\rho f\|_{p_t}$$

The following statement is proved in [11].

Statement [11]. Let $p \in H^{\ln}, 1 < p^-$ and $p(t) = \prod\limits_{k=1}^{m} |t - \tau_k|^{\alpha_k}$, where $\{\tau_k\}_1^m \subset [-\pi, \pi]$, $\tau_i \ne \tau_j$ for $i \ne j$. Then a singular operator S boundedly acts from L_{p_t,ρ_t} to L_{p_t,ρ_t} iff

$$-\frac{1}{p(\tau_k)} < \alpha_k < \frac{1}{q(\tau_k)}, \quad k = \overline{1, m}$$

are fulfilled.

When we establish the basicity we'll need the following classes of analytic functions.

2. Hardy classes with variable exponent. These classes were considered in the papers [7,8]. Let $U \equiv \{z : |z| < 1\}$ be a unit ball on a complex plane and $\Gamma = \partial U$ be a unit circle. For a harmonic in U function $u(z)$ we have

$$\|u\|_{p_t} \equiv \sup_{0<r<1} \left\|u\left(re^{it}\right)\right\|_{p_t},$$

where $p : [-\pi, \pi] \to [1, +\infty)$ is some measurable function. Denote

$$h_{p_t} \equiv \left\{u : \Delta u = 0 \text{ in } U \text{ and } \|u\|_{p_t} < +\infty\right\}.$$

The continuous imbeddings $h_{p^+} \hookrightarrow h_{p_t} \hookrightarrow h_{p^-}$ are true. The following theorem is valid.

Theorem [8]. *Let* $1 < p^- \le p^+ < +\infty$. *If*

$$u \in h_{p_t}, \text{ then } \exists f \in L_{p_t} :$$

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_r(\theta - t) f(t)\, dt, \tag{3}$$

where $p_r(\alpha) = \dfrac{1 - r^2}{1 + r^2 - 2r\cos\alpha}$ is a Poisson kernel.

Vice versa, if $f \in L_{p_t},\ p \in H^{\ln}$ then (3) belongs to h_{p_t}.

In a similar way, the Hardy class $H^+_{p_t}$ is introduced:

$$H^+_{p_t} :\equiv \left\{f : f \text{ analytic in } U \text{ and } \|f\|_{H^+_{p_t}} < +\infty\right\},$$

where $\|f\|_{H^+_{p_t}} = \sup\limits_{0<r<1} \left\|f\left(re^{it}\right)\right\|_{p_t}$.

It is easy to notice that $f \in H^+_{p_t} \Longleftrightarrow \operatorname{Re} f;\ \operatorname{Im} f \in h_{p_t}$, where $\operatorname{Re} z; \operatorname{Im} z$ are real and imaginary parts of z, respectively.

Using the previous theorem it is easy to prove the following refined variant of theorem 5 of the paper [7].

Theorem 1. *Let* $p \in H^{\ln}$ *and* $p^- > 1$. *Then*

$$F \in H^+_{p_t} \Longleftrightarrow \exists f \in L_{p_t} :$$

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} f(t)\, dt}{e^{it} - z}.$$

Smirnov's known theorem is also valid.

Theorem [7]. *Let* $p_i(t) : 0 < p_i^- \le p_i^+ < +\infty,\ \ i = 1, 2;\ p_1(t) \le p_2(t)$, *a.e. on* $[-\pi, \pi]$ *be measurable functions;* $F \in H^+_{p_{1t}}$; $p_2 \in H^{\ln}$ *and* $p_2^- > 1$. *Then, if* $F^+ \in L_{p_{2t}} \Longrightarrow F \in H^+_{p_{2t}}$.

Let's define the Hardy class $_mH^-_{p_t}$*,analytic outside of a unit circle functions of order* $\le m$ *at infinity. Let* $f(z)$ *be a function analytic in* $C\backslash\overline{U}$ $\left(\overline{U}=U\cup\Gamma\right)$*, having a finite order* $\le m$ *at infinity, i.e.* $f(z)=f_1(z)+f_2(z)$*, where* $f_1(z)$ *is a polynomial of power* $\le m$*,* $f_2(z)$ *is a tame part of expansion of* $f(z)$ *in Lorentz series in the vicinity of a point at infinity. If the function* $\varphi(z)\equiv\overline{f_2\left(\frac{1}{\overline{z}}\right)}$ *(*$(\bar{\cdot})$ *is a complex conjugation) belongs to the class* $H^+_{p_t}$*, we'll say that the function* $f(z)$ *belongs to the class* $_mH^-_{p_t}$*.*

3. Riemann's problem in the classes $H^\pm_{p_t}$**.** Let's formulate general statement of the Riemann problem in these classes, even we'll need very special case. Let a complex valued on $[-\pi,\pi]$ function $G(t)$ satisfy the conditions:

1) $|G|\in L_{r_t}, |G|^{-1}\in L_{\omega_t}; r:0<r^-\le r^+<+\infty, \omega:0<\omega^-\le\omega^+<+\infty$, are some measurable functions.

2) the argument $\theta(t)\equiv\arg G(t)$ has an expansion of the form $\theta(t)=\theta_0(t)+\theta_1(t)+\theta_2(t)$, where $\theta_0(t)\in C[-\pi,\pi]$; $\theta_1(t)$ is a bounded variation function on $[-\pi,\pi]$; $\theta_2(t)$ is a merasurable part.

It is required to find a piece-wise analytic function $F^\pm(z)$ on C with section Γ satisfying the conditions:

a) $F^+\in H^+_{p_t}:\ 0<p^-\le p^+<+\infty$;

b) $F^-\in\ _mH^-_{v_t}:0<v^-\le v^+<+\infty$;

c) non-tangential boundary values $F^\pm\left(e^{it}\right)$ on a unit circle Γ a.e.satisfy the relation:

$$F^+\left(e^{it}\right)+G(t)\,F^-\left(e^{it}\right)=g(t)\,,\ \text{a.e. } t\in(-\pi,\pi)\,,$$

where $g\in L_{p_t}:0<p^-\le p^+<+\infty$. All the considered functions are measurable.

When summability indices are constant, theory of these problems was studied well (see. f.e. [9]). Consider the following Riemann problem:

$$\begin{cases} F^+(\tau)+G(\tau)\,F^-(\tau)=0,\ \tau\in\Gamma; \\ F^+\in H^+_{p_t}; F^-\in\ _mH^-_{p_t}. \end{cases} \tag{4}$$

Let's consider the following functions $X^\pm_i(z)$ analytic inside (the sign << + >>) and autside (the sign << − >>) of a unit circle.

$$X^\pm_1(z)\equiv\exp\left\{\pm\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\left|G\left(e^{it}\right)\right|\frac{e^{it}+z}{e-z}dt\right\},$$

$$X^\pm_2(z)\equiv\exp\left\{\pm\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta(t)\,\frac{e^{it}+z}{e-z}dt\right\}.$$

Let

$$Z_i(z)\equiv\begin{cases} X^+_i(z)\,, & |z|<1, \\ \left[X^-_i(z)\right]^{-1}, & |z|<1. \end{cases}$$

Denote $Z^\pm(z)\equiv Z^\pm_1(z)\cdot Z^\pm_2(z)$. In sequal, we'll require the following conditions to be fulfilled:

3) $\theta_2(t) \equiv 0$; $\theta_1(t)$ has a finite number of discontinnity points $\{s_k\}_1^r : -\pi < s_1 < ... < s_r < \pi$ $[-\pi, \pi]$.

4) $\left\{\frac{h_k}{2\pi} + \frac{1}{q(s_k)}\right\}_{k=0}^{r} \cap Z = \{\varnothing\}$ where $h_k = \theta(s_k+0) - \theta(s_k-0)$, $k = \overline{1,r}$; $h_0 = \theta(-\pi) - \theta(\pi)$ and Z is a set of integers.

Define $\{n_i\}_{i=1}^r \subset Z$ from the inequalities:

$$\begin{cases} -\dfrac{1}{q(s_k)} < \dfrac{h_k}{2\pi} + n_k - n_{k-1} < \dfrac{1}{p(s_k)}, \; k = \overline{1,r}; \\ n_0 = 0. \end{cases}$$

Let

$$\omega_r = \frac{h_0}{2\pi} + n_r.$$

Eearlier we proved

Theorem [10]. *Let $p \in H^{\ln}$, $1 < p^-$; the conditions 1)-4) be satisfied. Then, if it holds*

$$-\frac{1}{q(\pi)} < \omega_r < \frac{1}{p(\pi)},$$

the general solution of homogeneous problem (4) in the classes $\left(H^+_{p_t};_m H^-_{p_t}\right)$ is of the form

$$F(z) \equiv Z(z) \cdot P_m(z),$$

where $P_m(z)$ is an arbitrary polynomial of power $\leq m$.

Corollary 1. *Let all the requirements of the previous theorem be fulfilled. Then, provided $F^-(\infty) = 0$ Riemann's homogeneous problem (4) in the classes $\left(H^+_{p_t};_m H^-_{p_t}\right)$ has only a trivial solution, i.e. zero solution.*

Now, let's consider Riemann's inhomogeneous problem

$$\begin{cases} F^+(\tau) + G(\tau) F^-(\tau) = g(\tau), \; \tau \in \Gamma; \\ F^+ \in H^+_{p_t}; \; F^- \in \; _mH^-_{p_t}. \end{cases} \tag{5}$$

where $g(\tau) \in L_{p_t}$ is an arbitrary function. It is obvious the problem (5) has a unique solution (if it is solvable), iff appropriate homogeneous problem (4) has only a trivial solution. In a general case the solution $F(z)$ of the problem (5) is of the form

$$F(z) = F_0(z) + Z(z) \cdot P_m(z),$$

where $F_0(z)$ is one of the particular solutions of problem (5), $P_m(z)$ is a polynomial of power $\leq m$.

As $G(\tau)$ we take a concrete function $G\left(e^{it}\right) = e^{2i[\alpha t + \beta]}$, $t \in [-\pi, \pi]$.

Assume that $\alpha, \beta \in R$. The complex case is investigated in the similar way.

At first we assume that $g\left(e^{it}\right)$ is a Holder function on $[-\pi, \pi]$. We solve the problem (5) by the method worked out in [5]. We get a particular solution of $F_0(z)$ of the form:

$$F_0^+(z) = \frac{e^{i\beta}}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\alpha\theta} g\left(e^{i\theta}\right) d\theta}{\left(1+e^{i\theta}\right)^{2\alpha} \left(1 - z \cdot e^{-i\theta}\right)} (1+z)^{2\alpha},$$

$$F_0^-(z)=\frac{e^{-i\beta}}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\alpha\theta}g\left(e^{i\theta}\right)d\theta}{\left(1+e^{i\theta}\right)^{2\alpha}\left(1-z\cdot e^{-i\theta}\right)}z^{-2\alpha}\left(1+z\right)^{2\alpha}.$$

From Sokhotsky-Plemel formula it directly follows that $F_0(z)$ satisfies relation (5). Denote

$$h_n^+(t)=\frac{\left(1+e^{it}\right)^{-2\alpha}}{2\pi}e^{i(\alpha t+\beta)}\cdot\sum_{k=0}^{n}C_{2\alpha}^{n-k}\cdot e^{-ikt},\quad n=\overline{0,\infty};$$

$$h_m^-(t)=-\frac{\left(1+e^{it}\right)^{-2\alpha}}{2\pi}e^{i(\alpha t-\beta)}\cdot\sum_{k=1}^{m}C_{2\alpha}^{m-k}\cdot e^{ikt},\ m=\overline{1,\infty};$$

where $C_\beta^n=\dfrac{\beta(\beta-1)\dots(\beta-n+1)}{n!}$ are binomial coefficients. Expanding the functions $F_0^+(z)$;$F_0^-(z)$ respectively, in the vicinities of zero and a point at infinity in the series of z, we get:

$$F_0^+(z)=\sum_{n=0}^{\infty}a_n^+\cdot z^n,\quad F_0^-(z)=\sum_{n=1}^{\infty}a_n^-\cdot z^{-n},$$

where

$$a_n^+=\int_{-\pi}^{\pi}g\left(e^{i\theta}\right)\overline{h_n^+(\theta)}d\theta,\quad n\geq 0;$$

$$a_m^-=\int_{-\pi}^{\pi}g\left(e^{i\theta}\right)\overline{h_m^-(\theta)}d\theta,\quad m\geq 1.$$

Let $|2\alpha|<1$. It is easy to notice that $F_0^+\in H_1^+$; $F_0^-\in_{-1}H_1^-$. It follows from relations [9]

$$\int_{-\pi}^{\pi}\left|F_0^+\left(e^{it}\right)-F_0^+\left(re^{it}\right)\right|dt\to 0,\ r\to 1-0;$$

$$\int_{-\pi}^{\pi}\left|F_0^-\left(e^{it}\right)-F_0^-\left(re^{it}\right)\right|dt\to 0,\ r\to 1+0,$$

that

$$a_n^+=\frac{1}{2\pi}\int_{-\pi}^{\pi}F_0^+\left(e^{it}\right)e^{-int}dt,\quad\forall n\geq 0;$$

$$a_m^-=\frac{1}{2\pi}\int_{-\pi}^{\pi}F_0^-\left(e^{it}\right)e^{imt}dt,\quad\forall m\geq 1.$$

Using the representation of the Cauchy type integral with power character peruliarity in the vicinity of a discontinuity point of first kind density (see [12], p.74), it is easy to show that if the conditions $0<2\alpha<1$ and $g(1)=g(-1)=0$ hold, the functions $F_0^{\pm}(\tau)$ are continuous on a unit circle. Therefore, the Fourier series of these functions by the system of exponents $\left\{e^{int}\right\}_{n\in Z}$ converge to them on $[-\pi,\pi]$ uniformly, since they satisfy some conditions of Holder property on Γ. As the result we get

$$F_0^+\left(e^{it}\right)=\sum_{n=0}^{\infty}a_n^+e^{int};\quad F_0^-\left(e^{it}\right)=\sum_{n=1}^{\infty}a_n^-e^{-int},$$

uniformly on $[-\pi,\pi]$. Considering these relations in (5) we get (where $g(\tau)=f(\tau)\cdot e^{i(\alpha t+\beta)}$, $\tau=e^{it}$; $f\left(e^{it}\right)$ is a Holder function on $[-\pi,\pi]$):

$$f\left(e^{it}\right)=\sum_{n=0}^{\infty}a_n^+e^{i[(n-\alpha)t-\beta]}+\sum_{n=1}^{\infty}a_n^-e^{-i[(n-\alpha)t-\beta]},$$

uniformly on $[-\pi,\pi]$. It is proved in the paper [13] that the relations:

$$\left.\begin{array}{l}\displaystyle\int_{-\pi}^{\pi}e^{i[(n-\alpha)t-\beta]}\overline{h_m^+(t)}dt=\delta_{nm},\ \forall n,m\geq 0;\\ \displaystyle\int_{-\pi}^{\pi}e^{i[(n-\alpha)t-\beta]}\overline{h_m^-(t)}dt=0,\forall n\geq 0;\forall m\geq 1;\\ \displaystyle\int_{-\pi}^{\pi}e^{-i[(n-\alpha)t-\beta]}\overline{h_m^+(t)}dt=0,\ \forall n\geq 1;\forall m\geq 0;\\ \displaystyle\int_{-\pi}^{\pi}e^{-i[(n-\alpha)t-\beta]}\overline{h_m^-(t)}dt=\delta_{nm},\forall n,m\geq 1.\end{array}\right\}\qquad(6)$$

are fulfilled for $|\alpha|<\frac{1}{2}$.

It follows directly from Property A that, if $p(t)\in H^{\ln}$ and $p^->1$, then the system (1) belongs to L_{p_t}. In this case the space L_{q_t} is adjoint to the space L_{p_t} (see, f.e.[6]). Consequently, it follows from statement 1 and from representations for $h_n^{\pm}(t)$ that for $\alpha<\frac{1}{2q(\pi)}$ the system $\{h_n^+;h_m^-\}$ belongs to L_{q_t}. Then, from relations (6) we get that under fulfilment of the conditions formulated above, the system (1) and $\{h_n^+;h_m^-\}$ are conjugated and so (1) is minimal in L_{p_t}. Having paid attention to the Property B we get that for $\frac{1}{2}>\alpha\geq 0$ the system (1) is complete in L_{p_t}.Thus, if the inequality $0\leq\alpha<\frac{1}{2q^{\pi}}$ is fulfilled, then (1) is complete and minimal in L_{p_t}.

Denote:

$$I(z)=\int_{-\pi}^{\pi}\frac{e^{i\alpha\theta}g_0(\theta)\,d\theta}{\left(1+e^{i\theta}\right)^{2\alpha}\left(1-ze^{-i\theta}\right)},\quad g(\theta)=g\left(e^{i\theta}\right).$$

Then we can represent $F_0^{\pm}(z)$ in the form:

$$\left.\begin{array}{l}F_0^+(z)=\dfrac{e^{i\beta}}{2\pi}I(z)(1+z)^{2\alpha},\ |z|<1;\\ F_0^-(z)=\dfrac{e^{-i\beta}}{2\pi}I(z)\left(1+z^{-1}\right)^{2\alpha},\ |z|>1.\end{array}\right\}\qquad(7)$$

From the same reasonings we get that for finite functions $g_0(\theta)$ on $[-\pi,\pi]$, the Fourier series for boundary values $I^{\pm}\left(e^{i\theta}\right)$ converge to them uniformly on $[-\pi,\pi]$. Therewith, if $2\alpha > -\frac{1}{p(\pi)}$, the functions $\left(1+e^{i\theta}\right)^{2\alpha}$ and $\left(1+e^{-i\theta}\right)^{2\alpha}$ belong to the space L_{p_t} and by the results of the paper [14], the Fourier series of these functions converge to them in L_{p_t}. Again, it follows from the property B that for $-\frac{1}{2p(\pi)} < \alpha < \frac{1}{2}$ the system (1) is complete in L_{p_t}. Combining the obtained results we arrive at the following conclusion.

Statement 2. Let $p(t) \in H^{\ln}$; $p^- > 1$, and the inequality

$$-\frac{1}{2p(\pi)} < \alpha < \frac{1}{2q(\pi)} \tag{8}$$

be fulfilled.

Then the system (1) is complete and minimal in L_{p_t}.

Now let's study the basicity. Let (8) be fulfilled. Then the system (1) is minimal in L_{p_t} and let $\{h_n^+(t); h_m^-(t)\}_{n\geq 0; m\geq 1}$ be an appropriate conjugated system.Take $\forall f \in L_{p_t}$ and consider the partial sum S_m:

$$S_m[f] = \sum_{n=0}^{m} a_n^+ e^{i[(n-\alpha)t-\beta]} + \sum_{n=1}^{m} a_n^- e^{-i[(n-\alpha)t-\beta]},$$

where

$$a_n^+ = \int_{-\pi}^{\pi} f(t)\,\overline{h_n^+(t)}dt, \quad n \geq 0;$$

$$a_k^- = \int_{-\pi}^{\pi} f(t)\,\overline{h_n^-(t)}dt, \quad k \geq 1.$$

Let's consider the problem (5), where as the right hand side of $g(\tau)$ we take the function: $g\left(e^{i\theta}\right) = e^{i(\alpha t+\beta)} f(t)$, furthermore, require $F^-(\infty) = 0$. Then, as it follows from Corollary 1, the problem (5) has a unique solution $F_0^{\pm}(z)$ in the classes $\left(H_{p_t}^+;{}_{-1}H_{p_t}^-\right)$ and thus $F_0^{\pm}\left(e^{it}\right) \in L_{p_t}$.

Show that

$$\sup_{\substack{m \\ \|f\|_{p_t}=1}} \|S_m[f]\|_{p_t} < +\infty.$$

As we noticed

$$a_n^+ = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_0^+\left(e^{it}\right) e^{-int}dt, \quad \forall n \geq 0;$$

$$a_n^- = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_0^-\left(e^{it}\right) e^{ikt}dt, \quad \forall k \geq 1.$$

We have

$$\|S_m[f]\|_{p_t} \leq \left\| \sum_{n=0}^{m} a_n^+ \cdot e^{int} \cdot e^{-i(\alpha t+\beta)} \right\|_{p_t} + \left\| e^{i(\alpha t+\beta)} \sum_{n=1}^{m} a_n^- e^{-int} \right\|_{p_t}.$$

Since the classic system of exponents $\{e^{int}\}_{n\in Z}$ forms a basis in L_{p_t} (see.[14]),then taking into account the Property A we get:

$$\|S_m[f]\|_{p_t} \le M_1 \left\|F_0^+\left(e^{it}\right)\right\|_{p_t} + M_2 \left\|F_0^-\left(e^{it}\right)\right\|_{p_t},$$

where $M_i,\ i=1,2$; are some constants. Applying the Sokhotsky-Plemel formula to the expressions $F_0^+(z)$ and $F_0^-(z)$ we get:

$$F_0^+\left(e^{i\theta}\right) = ie^{i(\alpha\theta+\beta)} f(\theta) + S^+(f),$$

$$F_0^-\left(e^{i\theta}\right) = ie^{-i(\alpha\theta+\beta)} f(\theta) + S^-(f),$$

where $S^\pm(f)$ are appropriate integrals of singular type:

$$S^+(f) = \frac{e^{i\beta}}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\alpha\theta} f(\theta)\, d\theta}{\left(1+e^{i\theta}\right)^{2\alpha}\left(1-e^{i(s-\theta)}\right)} \cdot \left(1+e^{is}\right)^{2\alpha},$$

$$S^-(f) = \frac{e^{-i\beta}}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\alpha\theta} f(\theta)\, d\theta}{\left(1+e^{i\theta}\right)^{2\alpha}\left(1-e^{i(s-\theta)}\right)} \cdot \left(1+e^{-is}\right)^{2\alpha}.$$

Further, having paid attention to the statement [11] we get that the integral operators $S^+(f)$ and $S^-(f)$ boundedly act from L_{p_t} to L_{p_t}, i.e.

$$\left\|S^\pm(f)\right\|_{p_t} \le M\,\|f\|_{p_t}, \quad \forall f \in L_{p_t}.$$

As the result we have:

$$\|S_m[f]\|_{p_t} \le M_1\left(M_3\|f\|_{p_t} + \left\|S^+(f)\right\|_{p_t}\right) +$$

$$+M_2\left(M_4\|f\|_{p_t} + \left\|S^-(f)\right\|_{p_t}\right) \le M_5\|f\|_{p_t}, \ \ \forall f \in L_{p_t},$$

where $M_i, i=\overline{3,5}$ are some constants.

And it follows from the basicity criterium that the system (1) forms a basis in L_{p_t}, i.e. it is valid:

Theorem 2. *Let $p(t)\in H^{\ln}$; $p^->1$, and the inequality*

$$-\frac{1}{2p(\pi)} < \alpha < \frac{1}{2q(\pi)}$$

be fulfilled.

Then the system of exponents (1) forms a basis in L_{p_t}.

We separately consider the case $\alpha = -\dfrac{1}{2p(\pi)}$. In this case, it follows from relations (6) and expressions for $h_n^\pm(t)$ that system (1) is minimal in L_{p_t}, since it has a biorthogonal system.Represent system (1) in the form:

$$\left\{e^{i[(n+1-(\alpha+1)t-\beta]}; e^{-i[(m-\alpha)t-\beta]}\right\}_{n\ge0;m\ge1}. \tag{9}$$

Multiplication of each term of the system (9) by the function $e^{i\frac{t}{2}}$ doesn't affect its completeness in L_{p_t}. As the result we get the system $\left\{I_{n;m}^{\widetilde{\alpha}}(t)\right\}_{n\geq 1;m\geq 1}$, where $I_{n;m}^{\widetilde{\alpha}}(t) \equiv \left(e^{i[(n-\widetilde{\alpha})\cdot t-\beta]}, e^{-i[(m-\widetilde{\alpha})\cdot t-\beta]}\right)$, $\widetilde{\alpha} = \alpha + \frac{1}{2}$. It is easy to see that $\widetilde{\alpha} = \frac{1}{2q(\pi)} < \frac{1}{2}$. Then by the previous results we get that the system $\left\{I_{n;m}^{\widetilde{\alpha}}(t)\right\}_{n\geq 0;m\geq 1}$ is complete in L_{p_t}. It follows from expressions for $\{h_n^{\pm}(t)\}$ and statement 1 that in this case the system doesn't belong to the space L_{q_t}, since the system $\left\{I_{n;m}^{\widetilde{\alpha}}(t)\right\}_{n\geq 1;m\geq 1}$ is complete in L_{p_t}. Then from the uniqueness of biorthogonal system to the complete system we get that $\left\{I_{n;m}^{\widetilde{\alpha}}(t)\right\}_{n\geq 0;m\geq 1}$ is not minimal in L_{p_t}, and as a result of that the system $\left\{I_{n;m}^{\widetilde{\alpha}}(t)\right\}_{n;m\geq 1}$ so the system (1) is complete and minimal in L_{p_t}. The fact that in this case system doesn't form a basis in L_{p_t} is proved similar to the paper [3]. We arrive at the following conclusion: if $-\frac{1}{2p(\pi)} \leq \alpha < \frac{1}{2q(\pi)}$, the system (1) is complete and minimal in L_{p_t}. Now, let $\alpha < -\frac{1}{2p(\pi)}$, for example $-\frac{1}{2p(\pi)} - \frac{1}{2} \leq \alpha < -\frac{1}{2p(\pi)}$. In this case, it holds $-\frac{1}{2p(\pi)} \leq \widetilde{\alpha} < \frac{1}{2q(\pi)}$, so the system $\left\{I_{n;m}^{\widetilde{\alpha}}(t)\right\}_{n\geq 0;m\geq 1}$ is complete, and minimal in L_{p_t}. As the result system (1) is not complete, but minimal in L_{p_t}. We show similarly that for $\alpha \geq \frac{1}{2q(\pi)}$ the system is complete, but not minimal in L_{p_t}.

Combining all the obtained results, we have the following theorem.

Theorem 3. *Let $p(t) \in H^{\ln}$; $p^- > 1$. The system (1) is complete in L_{p_t} iff $\alpha \geq -\frac{1}{2p(\pi)}$; it is minimal in L_{p_t} only for $\alpha < -\frac{1}{2q(\pi)}$.*

Let the inequality $\alpha < \frac{1}{2q(\pi)}$ hold. By theorem 2, in this case the system (1) is minimal in L_{p_t}. It directly follows from analytical expressions for the adjoint system $\{h_n^{\pm}(t)\}$ that

$$h_n^{+}(t) = \frac{e^{i\beta}}{2\pi} \cdot \frac{e^{i\alpha t}}{(1+e^{it})^{2\alpha}}.$$

We have

$$\overline{c_0^{+}} = \int_{-\pi}^{\pi} \overline{h_0^{+}(t)} dt = \frac{e^{-i\beta}}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{(1+e^{-it})^{2\alpha} \cdot (e^{it})^{\alpha}} =$$

$$= \frac{e^{-i\beta}}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{\left(e^{i\frac{t}{2}} + e^{-i\frac{t}{2}}\right)^{2\alpha}} = \frac{e^{-i\beta}}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{\left(2\cos\frac{t}{2}\right)^{2\alpha}} \neq 0.$$

Consider the system $\{H_n^{+}; H_m^{-}\}_{n\geq 0;m\geq 1}$::

$$H_0^{+} = \frac{1}{c_0^{+}} h_0^{+}; \quad H_n^{\pm} = h_n^{\pm} - \frac{c_n^{\pm}}{c_0^{+}} h_0^{+}, \tag{10}$$

where $c_n^{\pm} = \int_{-\pi}^{\pi} h_n^{\pm}(t)\, dt$, $\forall n \geq 1$. It is easy to verify that the systems

$\left\{H_n^+; H_{n+1}^-\right\}_{n\geq 0}$ and (2) are biorthonormed. Thus, for $\alpha < \dfrac{1}{2q(\pi)}$ the system (2) is minimal in L_{p_t}. The remainig cases for the values of α are similarly proved.

Let $-\dfrac{1}{2p(\pi)} < \alpha < \dfrac{1}{2q(\pi)}$. Take $\forall f \in L_{p_t}$ and consider

$$S_m[f] = f_0^+ + \sum_{n=1}^{n} \left[f_n^+ e^{-i(\alpha t+\beta)} e^{int} + f_n^- e^{i(\alpha t+\beta)} e^{-int}\right],$$

where $f_n^{\pm}$ are biorthogonal coefficients of the function f by the system (2). Considering expression (10) for $H_n^{\pm}$ it is easy to show that $\|S_m(f) - f\|_{p_t} \to 0,\ m \to \infty$. This proves the basicity of the system (2) in the considered case. So, we proved.

Theorem 4.*Let $p(t) \in H^{\ln}$; $p^- > 1$. If the inequality $-\dfrac{1}{2p(\pi)} < \alpha < \dfrac{1}{2q(\pi)}$* is fulfilled, the system (2) forms a basis in L_{p_t}. *Moreover, it is complete in L_{p_t} only for $\alpha \geq -\dfrac{1}{2p(\pi)}$; it is minimal iff $\alpha < \dfrac{1}{2q(\pi)}$. For $\alpha = -\dfrac{1}{2p(\pi)}$ it is complete and minimal in L_{p_t} but doesn't form a basis in it.*

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