MATHEMATICS

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GLOBAL SOLVABILITY AND BEHAVIOR OF SOLUTION THE CAUCHY PROBLEM FOR THE QUASILINEAR HYPERBOLIC EQUATION WITH ANISOTROPIC ELLIPTIC PART AND INTEGRAL NONLINEARITY

Abstract

In this paper we investigate the Cauchy problem for a quasilinear hyperbolic equation with anisotropic elliptic part and integral nonlinearity. The theorem on global solvability is proved. The decrease order of solutions and their derivatives are obtainet as $t \to \infty$.

1. Statement of the problem.

We consider the Cauchy problem for the quasilinear hyperbolic equation

$$u_{tt} + u_t + \sum_{i=1}^{n} (-1)^{l_i} a_i(t, [u]_{l,i}) D_{x_i}^{2l_i} u = 0, t > 0, x \in R_n \tag{1}$$

with initial datas

$$u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \quad x \in R_n \tag{2}$$

where $l_1, l_2, ..., l_n \in \{1, 2, ...\}$, $[u]_{l,i} = \sum_{k=1}^{n} \beta_{ik} \int_{R_n} \left| D^{l_k} u \right|^2 dx, \quad \beta_{ik} \in R.$

In the paper [1] the solvability of a mixed problem for quasilinear Kirchoff equations in Gevrey classes is investigated. But in the paper [2] a class of quasilinear Kirchoff equations, for which the corresponding mixed problem with initial data from Sobolev space have a global solution, is chosen. Further, the quasilinear hyperbolic equations with integral nonlinearity are investigated in the papers of the other autors (see [3], [4])

In this paper we investigate the global solvability of the problem (1), (2) with small initial datas in the anisotropic Sobolev spaces.

Suppose that the following conditions are satisfied:

1) $a_i(t, \xi) = 1 + a_{1i}(t, \xi)$ are defined for all $(t, \xi) \in [0, \infty) \times (-b, b)$ and continuously differentiable with respect to t, ξ. The function $a'_{1i_\xi}(t, \xi)$ are continuously differentiable with respect to t, where $b > 0$.

2) For any $t \in [0, \infty)$, $\xi \in (-b, b)$ the following inequalities are satisfied:

$$|a_{1i}(t,\xi)| \le c\,|\xi|^p, \quad |a_{1i_t}(t,\xi)| \le c\,|\xi|^p, \quad \left| a'_{1i_\xi}(t,\xi) \right| \le c\,|\xi|^{p-1}$$

where $p>1$ and $c\geq 0$ some constants.

By $W_2^{rl}(R_n)$ $r=1,2,...$ we denote the anisotropic Sobolev spaces:

$$\|u\|_{W_2^{rl}(R_n)}=\sqrt{\langle u,u\rangle_{W_2^{rl}(R_n)}},$$

where

$$\langle u,v\rangle_{W_2^{rl}(R_n)}=\int\limits_{R_n}u(x)v(x)dx+\sum_{i=1}^n\int\limits_{R_n}D_{x_i}^{rl_i}u(x)D^{rl_i}v(x)dx$$

$l=(l_1,...,l_n),\ \ rl=(rl_1,...,rl_n)$, at $r=0$ $W^{0.l}(R_n)=L_2(R_n)$.

By $H_r=W_2^{(r+1)l}(R_n)\times W_2^{rl}(R_n)$ denote the Hilbert space with the scalar product:

$$\left\langle w^1,w^2\right\rangle_{H_r}=\left\langle u^1,u^2\right\rangle_{W_2^{(r+1)l}(R_n)}+\left\langle v^1,v^2\right\rangle_{W_2^{rl}(R_n)}$$

and the corresponding norm $\|\cdot\|_{H_r}=\langle\cdot,\cdot\rangle_{H_r}^{\frac12}$, where $w^1=\begin{pmatrix}u^1\\ v^1\end{pmatrix}$, $w^2\begin{pmatrix}u^2\\ v^2\end{pmatrix}$, $r=0,1$.

By U_δ^r we denote the ball of radius $\delta>0$ in the space H_r, i.e.

$$U_\delta^r=\left\{(v_1,v_2)\in H_r\ \ \|v_1\|_{W_2^{(r+1)l}(R_n)}+\|v_2\|_{W_2^{rl}(R_n)}<\delta\right\},r=0,1.$$

The main results of this paper is the following theorem on global solvability and asymptotic behavior of solutions at $t\to+\infty$.

Theorem. *Suppose that the condition 1), 2) are satisfied. Then there exists a real number* $\delta_0>0$, *such that, for any* $(\varphi,\psi)\in U_{\delta_0}^1$ *the problem (1), (2) has a unique solution* $u\in C([0,\infty):W_2^{2l}(R_n))\cap C^1([0,\infty);W_2^l(R_n))\cap\ C^2([0,\infty);L_2(R_n))$ *which satisfies the estimates*

$$\|D^\alpha u(t,\cdot)\|_{L_2(R_n)}\leq c_{\delta_0}(1+t)^{-\left|\frac{\alpha}{l}\right|}$$

where $\alpha=(\alpha_1,...,\alpha_n),\ \alpha_1,...,\alpha_n\in\mathbb{N}\cup\{0\}$, $\left|\dfrac{\alpha}{l}\right|=\dfrac{\alpha_1}{l_1}+...+\dfrac{\alpha_n}{l_n}, c_{\delta_0}>0$ *is some constant independent of* $t>0$.

Proof of the theorem. By substitution $v_1=u,\ v_2=u_t$ we can reduce problem (1), (2) to the Cauchy problem

$$w'=A(t,w)w\tag{3}$$

$$w(0)=w_0\tag{4}$$

in Hilbert space H_0, where

$$w=\begin{pmatrix}v_1\\ v_2\end{pmatrix},\ w_0=\begin{pmatrix}\varphi\\ \psi\end{pmatrix},\ D(A(t,w))=H_1,$$

$$A(t,w)=\begin{pmatrix}0 & I\\ -\sum\limits_{i=1}^n(-1)^{l_i}a_i(t,[v_1]_{l,i})D_{x_i}^{2l_i} & -I\end{pmatrix}.$$

In the space H_0 we introduce the system of bilinear form

$$\langle h^1, h^2 \rangle_{H(t,w)} = \sum_{i=1}^{n} a_i(t, [v_1]_{l,i}) \int_{R_n} D_{x_i}^{l_i} h_1^1 \cdot D_{x_i}^{l_i} h_2^1 dx + \int_{R_n} h_1^1 \cdot h_2^1 dx + \int_{R_n} h_2^1 \cdot h_2^2 dx,$$

where $t \in [0, \infty),\ w = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in U_\delta^0,\ h^i = \begin{pmatrix} h_1^i \\ h_2^i \end{pmatrix} \in H_0,\ i = 1, 2$

By 2) it follows that the following inequalities is valid

$$\left[1 - c \left(\sum_{i,k=1}^{n} |\beta_{ik}| \cdot \|w\|_{H_0} \right)^p \right] \|h\|_{H_0} \leq \|h\|_{H(t,w)} \leq$$

$$\leq \left[1 + c \left(\sum_{i,k=1}^{n} |\beta_{ik}| \cdot \|w\|_{H_0} \right)^p \right] \|h\|_{H_0},$$

$$w \in \cup_\delta^0,\ t \in [0, \infty),\ h \in H_0,\ \|h\|_{H(t,w)} = \sqrt{\langle h, h \rangle_{H(t,w)}}.$$

Further by embedding theorem (see [9 pp. 137-158]) it follows that for the sufficiently small $\delta' \in (0, \delta)$ the following inequalities is having

$$c_1(\delta') \|h\|_{H_0} \leq \|h\|_{H(t,w)} \leq c_2(\delta') \|h\|_{H_0},$$

where $w \in U_{\delta'}^0,\ 0 < c_1(\delta') < c_2(\delta')$ independent from w, h and $t > 0$.

Taking into account the conditions 1), 2) we have that the bilinear form $\langle \cdot, \cdot \rangle_{H(t,w)}$ define the system of equivalent scalar products in the space H_0 (see [6]). Denote by $H(t, w)$ the space H_0 with the scalar product $\langle \cdot, \cdot \rangle_{H(t,w)}$.

The following lemma obtained with use conditions 1), 2).

Lemma 1. *For any $h \in H_0$ the mapping $(t, w) \to \|h\|_{H(t,w)}$:$[0, \infty) \times U_\delta^0 \to [0, \infty)$ satisfies the local Lipshitz condition, i.e. for any $t_1, t_2 \in [0, \infty),\ w^1, w^2 \in U_\delta$ the following inequality is fulfilled*

$$\left| \|h\|_{H(t_1,w^1)} - \|h\|_{H(t_2,w^2)} \right| \leq c_3(\delta) \cdot \left[|t_1 - t_2| + \left\| w^1 - w^2 \right\|_{H_0} \right] \|h\|_{H_1},$$

where $c_3(\delta) > 0$ is independent of t_1, t_2, w^1, w^2 and h.

From definition of $A(t, w)$ and the scalar product in $H(t, w)$, and also from condition 1), 2) follows that the next lemma 1 is hold.

Lemma 2. *There exists $\delta'' \in (0, \delta')$ and $\omega > 0$ so that for any $w \in U_{\delta'}$ the operator $A(t, w) + \omega I$ generates the strong conditions contraction semigroup in the space $H(t, w)$ where I-is a unique operator in $H(t, w)$.*

Lemma 3. *The mapping $(t, w) \to A(t, w)$: $[0, \infty) \times U_\delta^0 \to L(H_0; H_1)$ satisfies the local Lipchitz condition, i.e. for any (t_1, w^1), $(t_2, w^2) \in [0, \infty) \times U_\delta$ and $h \in H_1$ the following inequality is fulfilled*

$$\left\| \left[A(t_1, w^1 - A(t_2, w^2 \right] h \right\|_{H_0} \leq c_4(\delta) \left[|t_1 - t_2| + \left\| w^1 - w^2 \right\|_{H_0} \right] \|h\|_{H_1},$$

where $L(H_0, H_1)$ *is the set all bounded operator from* H_0*to* H_1, $c_4(\delta) \in C(R_+; R_+)$.

Thus from Lemmas 1-3 and results of paper [6], there exists $\delta_1 \in (0, \delta'']$ such that for any $w_0 \in U^1_{\delta_1}$ the problem (3), (4) has a unique solution $w(t) \in C([0, T_0]; H_1) \cap C([0, T_0]; H_0)$ where $T_0 = T_0(w_0)$ some positive number depend on w_0 and δ_1.

Now we prove following statement:

There exists such $\delta_0 \in (0, \delta_1)$ that for any $w_0 \in U^1_{\delta_0}$ the problem (3), (4) has a global solution

$$w(t) \in C([0, \infty); H_1) \cap C^1([0, \infty); H_0)$$

The solution of the problem (1), (2) can be represented as the following form:

$$u(t, x) = u_1(t, x) * \varphi(x) + u_2(t, x) * \psi(x) + \int_0^t u_2(t - \tau, x) \times$$

$$\times \left[\sum_{i=1}^{n} a_i(\tau, [u]_{l,i}) D^{2li}_{x_i} u(\tau, x) \right] d\tau, \tag{5}$$

where $u_i = F^{-1}[\widehat{u}_i]$, $i = 1, 2$. Here F^{-1} is a inverse Fourier transformation, $*$ is convolution to x, $\widehat{u}_1(t, x)$ and $\widehat{u}_2(t, x)$ are solutions of the following Cauchy problem

$$L_\xi \widehat{u}_1(t, \xi) = 0, \quad \widehat{u}_1(0, \xi) = 1, \quad \widehat{u}_{1_t}(0, \xi) = 0, \tag{6}$$

$$L_\xi \widehat{u}_2(t, \xi) = 0, \quad \widehat{u}_2(0, \xi) = 0, \quad \widehat{u}_{2_t}(0, \xi) = 1, \tag{7}$$

$$L_\xi \widehat{u}(t, \xi) = \widehat{u}_{tt}(t, \xi) + \widehat{u}_t(t, \xi) + \sum_{i=1}^{n} \xi_i^{2l_i} \widehat{u}(t, \xi).$$

Using the Fourier transformation, Plancherel theorem and Hausdorf-Young inequality we have

$$\| D_t (u_1(t, x) * \varphi(x) \| + \sum_{i=1}^{n} \left\| D^l_{x_i} (u_1(t, x) * \varphi(x)) \right\|_{L_2(R_n)} \leq$$

$$c(1 + t)^{-1} \left[\sum_{i=1}^{n} \left\| D^{l_i}_{x_i} \varphi(x)) \right\|_{L_2(R_n)} + \| \varphi(x) \|_{L_2(R_n)} \right], \tag{8}$$

$$\| D_t u_2(t, x) * \psi(x) \| + \sum_{i=1}^{n} \left\| D^{l_i}_{x_i} (u_2(t, x) * \psi(x)) \right\|_{L_2(R_n)} \leq$$

$$\leq c(1 + t)^{-1} \| \psi(x) \|_{L_2(R_n)} \tag{9}$$

Let $[0, T')$ be a maximal interval of existence of solutions $w(t) \in C([0, T'); H_1) \cap C^1([0, T'); H_0)$, for the problem (3), (4).

Taking into account (8), (9) from (5) we obtain that

$$\| u_t(t, x) \|^2_{L_2(R_n)} + \sum_{i=1}^{n} \left\| D^{l_i}_{x_i} u(t, x) \right\|_{L_2(R_n)} \leq c(1 + t)^{-1} \times$$

$$\times\left[\sum_{i=1}^{n}\left\|D_{x_i}^{l_i}\varphi(x)\right\|_{L_2(R_n)}+\|\varphi(x)\|_{L_2(R_n)}+\|\psi(x)\|_{L_2(R_n)}\right]+ \tag{10}$$

$$+c\int_0^t(1+t-\tau)^{-1}\sum_{i=1}^{n}a_i(\tau,[u]_{l,i}\left\|D_{x_i}^{2l_i}u(\tau,\cdot)\right\|_{L_2(R_n)}d\tau, t\in[0,T'].$$

Further taking into account condition 2) from (10) we have

$$\sum_{i=1}^{n}\left\|D_{x_i}^{l_i}u(t,\cdot)\right\|_{L_2(R_n)}\le c(1+t)^{-1}\times$$

$$\times\left[\|\varphi\|_{W_2^l(R_n)}+\|\psi\|_{W_2^l(R_n)}\right]+c_1\sum_{i=1}^{n}\int_0^t(1+t-\tau)^{-1}\times \tag{11}$$

$$\times\left(\sum_{k=1}^{n}\beta_{ik}\int_{R_n}\left|D_{x_k}^{l_k}u(\tau,x)\right|^2dx\right)^p\left\|D_{x_i}^{2l_i}u(\tau,\cdot)\right\|_{L_2(R_n)}d\tau, t\in[0,T'].$$

Denoting by

$$E_l(t)=(1+t)\left[\sum_{i=1}^{n}\left\|D_{x_i}^{l_i}u(t,\cdot)\right\|_{L_2(R_n)}+\|u_t(t,\cdot)\|_{L_2(R_n)}\right]$$

$$E_{2_l}(t)=\sum_{i=1}^{n}\left[\left\|D_{x_i}^{2l_i}u(t,\cdot)\right\|_{L_2(R_n)}+\left\|D_{x_i}^{l_i}u_t(t,\cdot)\right\|_{L_2(R_n)}\right], t\in[0,T'),$$

we can rewrite the inequality in following form

$$E_l(t)\le c_1\|w_0\|_{H_0}+c_2(1+t)\int_0^t(1+t-\tau)^{-1}(1+\tau)^{-2p}\times \tag{12}$$

$$\times E_l^{2p}(\tau)E_{2l}(\tau)d\tau, t\in[0,T').$$

Since $2p>1$ then

$$(1+t)\int_0^t(1+t-\tau)^{-1}(1+\tau)^{2p}d\tau\le c_3, t>0.$$

Taking into account the last inequality from (12) we have

$$\xi_l(t)\le c_1\|w_0\|_H+c_2c_3\xi_l^{2p}(t)\xi_{2l}(t), t\in[0,T') \tag{13}$$

where

$$\xi_l(t)=\sup_{0\le\tau\le t}E_l(\tau),\quad \xi_{2l}(t)=\sup_{0\le\tau\le t}E_{2l}(\tau), t\in[0,T')$$

Multiplying the both hand sides of (1) by $\sum_{k=1}^{n}(-1)^{l_k} D_{x_k}^{2l_k} u_t$ and integrating the both hand sides respect domain $[0,T]\times R^n$, where $t\in(0,T')$. After simple transformation we obtain identity

$$\sum_{i=1}^{n}\int_{R_n}\left|D^{l_i}u_t\right|^2dx+2\sum_{i=1}^{n}\int_0^t\int_{R_n}\left|D_{x_i}^{l_i}u_\tau\right|^2dxd\tau+$$
$$+\sum_{i,k=1}^{n}a_i(t,[u]_{l,i})\int_{R_n}\left|D_{x_i}^{l_i}D_{x_k}^{l_k}u\right|^2dx=\sum_{i=1}^{n}\int_{R_n}\left|D_{x_i}^{l_i}\psi(x)\right|^2dx+$$
$$+\sum_{i,k=1}^{n}a_i(0,[\varphi]_{l,i})\int_{R_n}\left|D_{x_i}^{l_i}D_{x_k}^{l_k}\varphi(x)\right|^2dx+ \tag{14}$$
$$+\int_0^t\int_{R_n}\sum_{i,k=1}^{n}\left(\frac{d}{d\tau}a_i(\tau,[u]_{l,i})\right)\int_{R_n}\left|D_{x_i}^{l_i}D_{x_k}^{l_k}u\right|^2dxd\tau.$$

Taking into account the condition 2) we can easily note that

$$\sum_{i=1}^{n}\left|\frac{d}{d\tau}a_i(\tau,[u]_{l,i})\right|\le c\left|\sum_{k=1}^{n}\beta_{ik}\int_{R_n}\left|D_{x_k}^{l_k}u(\tau,x)\right|^2dx\right|^p+$$
$$+c\sum_{i=1}^{n}\left|2\beta_{ik}\int_{R_n}D_{x_k}^{l_k}u(\tau,x)\cdot D_{x_k}^{l_k}u_\tau(\tau,x)dx\right|\times$$
$$\times\left(c\sum_{k=1}^{n}\left|\beta_{ik}\int_{R_n}D_{x_k}^{l_k}u(\tau,x)^2dx\right|\right)^{p-1}\times \tag{15}$$
$$\times\le c(1+\tau)^{-2p}\times\left[\xi_l^{2p}(\tau)+\xi_l^{2p-1}(\tau)\xi_{2l}(\tau)\right].$$

Analogously

$$\sum_{k=1}^{n}a_i(\tau,[u]_{l,i})\ge 1-c\sum_{i,k=1}^{n}|\beta_{ik}|\left(\int_{R_n}\left|D_{x_k}^{l_k}u(\tau,x)dx\right|^2\right)^p\ge 1-c_1\xi_l^{2p}(\tau). \tag{16}$$

holds.

From (14)-(16) it follows that

$$\xi_{2l}^2(t)\le c_0\xi_{2l}^2(0)+c_1\xi_l^{2p}(t)\xi_{2l}^2(t)+c_2\int_0^t(1+\tau)^{-2p}\times$$

$$\times\left[\xi_l^{2p}(\tau)+\xi_l^{2p-1}(\tau)\xi_{2l}(\tau)\right]\xi_{2l}^2(\tau)d\tau\le c_0\xi_{2l}(0)+$$

$$+c_3\xi_l^{2p}(t)\xi_{2l}^2(t)+c_3\left[\xi_l^{2p}(t)\xi_{2l}^2(t)+\xi_l^{2p-1}(t)\xi_{2l}^3(t)\right]\int\limits_0^t(1+\tau)^{-2p}d\tau,$$

that is

$$\xi_{2l}^2(t)\le c_0\xi_{2l}^2(0)+c_4\left(\xi_l^{2p}(t)\xi_{2l}^2(t)+\xi_l^{2p-1}(t)\cdot\xi_{2l}^2(t)\right) \tag{17}$$

We denote by $Y(t)=\xi_l(t)+\xi_{2l}^2(t)$. From (13) and (17) it follow that

$$Y(t)\le c_5Y(0)+c_6Y^{2p+\frac{1}{2}}(t)+c_7Y^{2p+1}(t).$$

On the other hand $Y^{2p+\frac{1}{2}}(t)\le\varepsilon+c_\varepsilon Y^{2p+1}(t)$, $c_\varepsilon=\dfrac{4p+1}{4p+2}\cdot[\varepsilon(4p+2)]^{-\frac{1}{4p+1}}$, hence

$$Y(t)\le(c_5Y(0)+c_6\varepsilon+(c_5Y(0)+c_6\varepsilon+c_7)\,Y^{2p+1}(t).$$

From its follows that for the sufficiently small $c_5Y(0)+c_6\varepsilon$

$$Y(t)\le M,\quad t\in[0,T')$$

holds, where $M>0$ independent of $t\in[0,T')$.

Consequently

$$E_{2l}(t)\le M,\quad t\in[0,T'), \tag{18}$$

$$E_l(t)\le M,\quad t\in[0,T'). \tag{19}$$

It follows from (18) that $T'=+\infty$.

Indeed, let $T'<+\infty$ and $w(t)\in C([0,T');H_1)\cap C^1([0,T');H_0$ be a solution of problem (3), (4). Then it is obvious that $w(t)=\begin{pmatrix}u(t,x)\\u_t(t,x)\end{pmatrix}$, where $u(t,x)$ is a solution of problem (1), (2).

From (18) it follows that

$$u(t,x)\in L_\infty(0,T';W_2^{2l}(R_n),u_t(t,x)\in L_\infty(0,T';W_2^l(R_n)),$$

$$u_{tt}\in L_\infty(0,T';L_2(R_n)). \tag{20}$$

Using conditions 1), 2) and traces theorem (se [5, ch.I]) from (20) we have that the function $\widetilde{a}_k(t)=a_k(t,[u]_{p,k})$ $k=1,2,...,n$ are defined on $[0,T']$ and satisfy the Lipchitz condition. Then by virtue of theory solvability of Cauchy problems for the linear hyperbolic equations we have

$$u(t,x)\in C\left([0,T'];W_2^l(R_n)\right)\cap C^1([0,T'];L_2(R_n)), \tag{21}$$

(see [5, ch. 3, 8, ch. 9].

It follows from (20) and (21) that $w(T')\in U_{\delta_1}$.

On the basis of above mentions, the Cauchy problem

$$z'(t) = A(t, z_{(t)})z(t), \quad z(T') = w(T') \tag{22}$$

has the solution

$$z(t) \in C([T', T''); H_1) \cap C^1([T', T''); H_0), \tag{23}$$

where $T'' \in (T', +\infty)$ depends on $w(T')$.

Thus in view of (21) and (22) the function

$$\widetilde{w}(t) = \begin{cases} w(t), & 0 \le t < T \\ z(t), & T' \le t < T'' \end{cases}$$

belongs to the class

$$C([0, T''); H_1) \cap C^1([0, T'); H_0).$$

The function $\widetilde{w}(t)$ is a solution the definition of T'.

Thus there exists such $\delta_0 > 0$ that at any $\delta_0 > 0$ the problem (1), (2) has a unique solution

$$u(t, x) \in C([0, +\infty); W_2^{2l}(R_n)) \cap C^1(R_n; W_2^l(R_n)) \cap C^2([0, \infty); L_2(R_n)),$$

and in view of (19) for the function $u(t, x)$ the following estimates are valid

$$\sum_{i=1}^{n} \left\| D^{2l_i} u(t, \cdot) \right\|_{L_2(R_n)} \le M, \quad t \in [0, \infty), \tag{24}$$

$$\|u_t(t, \cdot)\|_{L_2(R_n)} + \sum_{i=1}^{n} \left\| D^{l_i} u(t, \cdot) \right\|_{L_2(R_n)} \le M(1+t)^{-1}, \quad t \in [0, \infty), \tag{25}$$

where M depends only on $r_0 = \|\varphi\|_{W_2^l(R_n)} + \|\psi\|_{L_2(R_n)}$.

Taking into account conditions 2) from (1), (10), (24) and (25) we obtain that

$$\begin{aligned} \|u(t, \cdot)\|_{L_2(R_n)} &\le c_\delta \left[\|\varphi\|_{L_2(R_n)} + \|\psi\|_{L_2(R_n)} \right] + \|u_t(t, \cdot)\|_{L_2(R_n)} + \\ &+ \sum_{i,k=1}^{n} \int_0^t 2c \left|\beta_{ik}\right| \left(\int_{R_n} \left| D^{l_k} u(\tau, x) \right|^2 dx \right)^p \left\| D^{2l_i} u(\tau, \cdot) \right\|_{L_2(R_n)} d\tau \le \\ &\le c_9 \left[\sum_{i=1}^{n} \left\| D^{l_i} \varphi \right\|_{L_2(R_n)} + \|\varphi\|_{L_2(R_n)} + \|\psi\|_{L_2(R_n)} \right] + \\ &+ c_{10} M \int_0^t (1+\tau)^{-2p} d\tau \le c_{r_0}, \end{aligned} \tag{26}$$

where c_{r_0} depends only $c_0 = \|\varphi\|_{W_2^p(R_n)} + \|\psi\|_{L_2(R_n)}$.

Further using the multiplicative inequality for anisotropic Sobolev spaces (se [9, s. 245-246]) we obtain

$$\|D^{\alpha}u(t,\cdot)\|_{L_2(R_n)} \leq M_{1,r_0}(1+t)^{-\left|\frac{\alpha}{l}\right|},$$

where $\alpha = (\alpha_1, ..., \alpha_n), \alpha_i \in N \cup \{0\}, i = 1, 2, ..., n,$ $\left|\frac{\alpha}{l}\right| = \sum\limits_{k=1}^{n} \frac{\alpha_k}{\beta_k} \leq 1,$ M_{1,r_0} depends only on r_0.

Remark. Analogously we can prove that

$$\varphi_0 \in W_2^{(s+1)l}, \ \psi_1 \in W_2^{sl}(R_n)$$

then

$$u(t,x) \in C([0,\infty); W^{(s+1)l}(R_n) \cap C^1([0,\infty); W_2^{sl}(R_n)) \cap C^2([0,\infty); L_2(R_n))$$

and

$$\|D^{\alpha}u(t,\cdot)\|_{L_2(R_n)} \leq c_{r_s}(1+t)^{-\left|\frac{\alpha}{l}\right|},$$

where $\left|\frac{\alpha}{l}\right| \leq s,$ c_{r_s} depends on $r_s = \|u_0\|_{W_2^{(s+1)l}(R_n)} + \|u_1\|_{W_2^{sl}(R_n)}$.

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