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ON TRIPLE COMPLETENESS OF A PART OF EIGEN AND ADJOINT VECTORS OF A CLASS OF FOURTH ORDER POLYNOMINAL OPERATOR BUNDLES

Abstract

In the paper we prove a theorem on triple completeness of a part of eigen and adjoint vectors of a class of fourth order polynomial operator bundles. Notice that the principal part of the investigated bundles possesses a complex characteristic.

As in known, tests for multiple completeness of the system of eigen and adjoint vectors (e.a.v) of polynomial operator bundles are given in M.V. Keldyshs paper [1] (in detail see[2]). We can see different generalizations of these tests in the papers of J.A. Allahverdiev [3], M.G. Gasymov [4], S.S. Mirzoyev [5], Yu.A. Palant [6]. Notice that while solving many problems of mathematical physics and mechanics there arises necessity of multiple completeness of a part of e.a.v. of operator bundles. For that there are various methods stated in the papers [7]-[10]. One of such methods is investigation of an operator-differential equation corresponding to a polynomial operator bundle (see. [4],[8], developed in the papers [5],[9]).

In a separable Hilbert space H we consider a polynomial operator bundle

$$P(\lambda) = (\lambda E - A)(\lambda E + A)^3 + \sum_{s=1}^{3} \lambda^{4-s} A_s, \tag{1}$$

where E is a unit operator, $A, A_s,\ s = 1,2,3$, are linear operators in H, here A is a self-adjoint positive-definite operator, the operators $A_s A^{-s},\ s = 1,2,3$, are bounded in H.

We denote by H_α a scale of Hilbert spaces, generated by the operator A, i.e. $H_\alpha = D(A^\alpha), \alpha \geq 0,\ (x,y)_{H_\alpha} = (A^\alpha x, A^\alpha y),\ x,y \in D(A^\alpha)$. For $\alpha = 0$ we'll assume that $H_0 = H,\ (x,y)_{H_0} = (x,y)_H,\ x,y \in H$.

With the bundle (1) we connect the boundary value problem

$$P(d/dt)\, u(t) = 0,\ t \in R_+ = [0; +\infty), \tag{2}$$

$$\frac{d^k u(0)}{dt^k} = \varphi_k, \quad \varphi_k \in H_{7/2-k}, \quad k = 0,1,2. \tag{3}$$

We assume that $u(t) \in W_2^4(R_+; H)$, where

$$W_2^4(R_+; H) = \left\{ u(t) : \frac{d^4 u(t)}{dt^4} \in L_2(R_+; H),\ A^4 u(t) \in L_2(R_+; H) \right\}$$

with the norm

$$\|u\|_{W_2^4(R_+;H)} = \left(\left\| \frac{d^4 u}{dt^4} \right\|^2_{L_2(R_+;H)} + \|u\|^2_{L_2(R_+;H_4)} \right)^{1/2}$$

(see. [11]), definition of the space $L_2(R_+;H)$ is, for example in [12].

Definition 1. *If for any $\varphi_k \in H_{7/2-k}, k=0,1,2$ there exists a vector-function $u(t) \in W_2^4(R_+;H)$ satisfying the equation (2) almost everywhere in R_+, and boundary conditions (3) in the sense*

$$\lim_{t\to 0}\left\| \frac{d^k u(t)}{dt^k} - \varphi_k \right\|_{H_{7/2-k}} = 0, \quad k=0,1,2,$$

and it holds the inequality

$$\|u\|_{W_2^4(R_+;H)} \le const \sum_{k=0}^{2} \|\varphi_k\|_{H_{7/2-k}},$$

then $u(t)$ will be said to be a regular solution of the equation (2), the problem (2),(3) regular solvable.

Definition 2. *If there exists a non-zero solution of the equation $P(\lambda_0)\psi_0 = 0$, then λ_0 is said to be an eigen value, and ψ_0 an eigen vector of the operator bundle $P(\lambda)$.*

Definition 3. *Let λ_0 be an eigen value, and ψ_0 one of the responding eigen vectors. The system $\psi_1, \psi_2, ..., \psi_n$ is said to be a chain of vectors adjoint to the eigen vector ψ_0, if they satisfy the following equations:*

$$\sum_{m=0}^{4} \frac{1}{m!} \frac{d^m}{d\lambda^m} P(\lambda)\bigg|_{\lambda=\lambda_0} \psi_{p-m} = 0, \quad p=1,2,...,n,$$

$$\psi_{-1} = \psi_{-2} = \psi_{-3} = 0.$$

Traditionally, by $\sigma_\infty(H)$ we denote a set of completely continuous operators acting in H, and by $L(H)$ a set of linear bounded operators acting in H.

If we suppose $A^{-1} \in \sigma_\infty(H)$, the bundle $P(\lambda)$ has a discrete spectrum and if λ_n $(\mathrm{Re}\,\lambda_n < 0)$ is an eigen value, $\psi_{0,n}, \psi_{1,n}, ..., \psi_{m,n}$ are e.a.v. of the bundle $P(\lambda)$ responding to λ_n, then the vector-functions

$$u_{h,n}(t) = e^{\lambda_n t}\left(\psi_{h,n} + \frac{t}{1!}\psi_{h-1,n} + ... + \frac{t^h}{h!}\psi_{0,n}\right), \quad h=0,1,...,m,$$

satisfy the equation $P(d/dt)\,u(t) = 0$ and are called elementary solutions of this homogeneous equation. It is clear that elementary solutions satisfy the following boundary conditions:

$$\frac{d^k}{dt^k} u_{h,n}(t)\bigg|_{t=0} \equiv \psi_{h,n}^{(k)}, \quad k=0,1,2; h=0,1,...,m.$$

Now, let's define the vector

$$\widetilde{\psi}_{h,n} = \left\{\psi_{h,n}^{(k)}\right\}_{k=0}^{2} \in \widetilde{H} \equiv \mathop{\oplus}_{k=0}^{2} H_{7/2-k}$$

and denote by

$$K(\Pi_-) = \left\{\widetilde{\psi}_{h,n}\right\}_{n=0}^{\infty}, \quad h=0,1,...,m.$$

We'll call $K(\Pi_-)$ a system of e.a.v. responding to eigen values from the left half-plane Π_- of the bundle $P(\lambda)$.

We introduce the following denotation. If $Q \in \sigma_\infty(H)$, then $(Q^*Q)^{1/2}$ is a completely continuous self-adjont operator in H. The eigen values of the operator $(Q^*Q)^{1/2}$ will be called s numbers of the operator Q. Non-zero s-numbers of the operator Q will be enumerated in decrease order with regart to their multiplicity. Denote by

$$\sigma_p = \left\{ Q : Q \in \sigma_\infty(H);\ \sum_{r=1}^{\infty} s_r^p(Q) < \infty \right\},\ 0 < p < \infty.$$

In the present paper we prove a theorem on triple completeness of a part of e.a.v. of the bundle $P(\lambda)$ that respond to eigen values from the left half-plane.

Notice that in the course of the paper we assume that all the above-mentioned conditiones on operator coefficients of the bundle (1) are satisfied.

At first we formulate the following statement that easily proved using the Keldysh lemma (see. [2]) on expansion of a resolvent near eigen values.

Lemma. *In order the system $K(\Pi_-)$ be complete in the space H, it is necessary and sufficient that for any vectors $\eta_k \in H_{7/2-k}$, $k = 0,1,2,$ from holomorphy of the vector-function*

$$R(\lambda) = \sum_{k=0}^{2} \left(A^{7/2-k} P^{-1}\left(\overline{\lambda}\right) \right)^* \ \lambda^k A^{7/2-k} \eta_k$$

in the half-plane Π_- yield $\eta_k = 0$, $k = 0,1,2.$

Now, we give conditions of regular solvability of the boundary value problem (2),(3).

Theorem 1. *Let the inequality*

$$\sum_{s=1}^{3} a_s \left\| A_{4-s} A^{-(4-s)} \right\|_{H \to H} < 1,$$

where

$$a_1 = \frac{1}{\sqrt{3}},\ a_2 = \frac{1}{2\sqrt{3}},\ a_3 = \frac{3\sqrt{3}}{16},$$

be fulfilled.

Then the boundary value problem (2),(3) is regularly solvable.

We briefly outline the proof of the theorem. At first it is easy to show that the boundary value problem (2),(3) for $A_s = 0$, $s = 1,2,3,$ is regularly solvable. Then assuming that even if one of A_s, $s = 1,2,3,$ differs from zero, we prove regular solvability of the boundary value problem (2),(3). For that we must look for a regular solution of the boundary value problem (2),(3) in the form $u(t) = u_0(t) + \vartheta(t)$, where $u_0(t)$ is a regular solution of the problem (2),(3) for $A_s = 0$, $s = 1,2,3,$ and $\vartheta(t) \in W_2^4(R_+; H)$. Then the boundary value problem (2),(3) is reduced to the following problem with respect to $\vartheta(t)$:

$$P(d/dt)\,\vartheta(t) = f(t)\,, \tag{4}$$

$$\frac{d^k \vartheta(0)}{dt^k} = 0, \quad k = 0,1,2, \tag{5}$$

where $f(t) \in L_2(R_+;H)$. The case when the conditions of the theorem are satisfied, the solvability of the problem (4),(5) is indicated in the author's paper [13].

The following main theorem is valid

Theorem 2. *Let the conditions of theorem 1 be fulfilled. If one of the following conditions:*

1) $A^{-1} \in \sigma_p,\ 0 < p \leq 1;$

2) $A^{-1} \in \sigma_p,\ 0 < p < \infty,\ A_s A^{-s} \in \sigma_\infty(H),\ s = 1,2,3,$

is fulfilled, the system $K(\Pi_-)$ *is complete in the space* $\widetilde{H}$.

Proof. Prove by contracdiction. Let the system $K(\Pi_-)$ be not complete in $\widetilde{H}$. Then there exists a non-zero vector $\eta = (\eta_0, \eta_1, \eta_2) \in \widetilde{H}$, for which orthogonality condition $\left(\eta, \widetilde{\psi}_{h,n}\right)_{\widetilde{H}} = 0,\ n = 1,2,...$ is fulfilled. Therefore, it follows from Keldysh's lemma [2] that the vector-function $R(\lambda)$ is holomorphic in H. From the condition of the theorem (by theorem 1) it follows that the boundary value problem (2),(3) has a regular solution representable in the form

$$u(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} P^{-1}(\lambda)\sum_{j=0}^{3} B_j u^{(3-j)}(0)\, e^{\lambda t} d\lambda, \tag{6}$$

where

$$B_0 = E,\ B_1 = \lambda E + C,\ B_2 = \lambda^2 E + \lambda C + A_2,$$

$$B_3 = \lambda^3 E + \lambda^2 C + \lambda A_2 + A_3 - 2A^3,\ C = 2A + A_1.$$

Further, as in the paper [4], in the formula (6) for $t > 0$ we can replace the integration contour $(-i\infty; i\infty)$ by $\Gamma_{\pm\theta} = \left\{\lambda : \lambda = re^{\pm i\left(\frac{\pi}{2}+\theta\right)},\ r > 0\right\}$. As a result, for $t > 0$ we get:

$$\sum_{k=0}^{2}\left(\frac{d^k u(t)}{dt^k}, \eta_k\right)_{H_{7/2-k}} = \frac{1}{2\pi i}\times$$

$$\times\int_{\Gamma_{\pm\theta}}\sum_{k=0}^{2}\left(A^{7/2-k}P^{-1}(\lambda)\lambda^k\sum_{j=0}^{3} B_j u^{(3-j)}(0), A^{7/2-k}\eta_k\right)e^{\lambda t}d\lambda =$$

$$= \frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}}\sum_{j=0}^{3}\left(B_j u^{(3-j)}(0), R\left(\overline{\lambda}\right)\right)e^{\lambda t}d\lambda = \frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}} g(\lambda)\, e^{\lambda t} d\lambda,$$

where

$$g(\lambda) = \sum_{j=0}^{3}\left(B_j u^{(3-j)}(0), R\left(\overline{\lambda}\right)\right).$$

Taking into attention in the case 1) the estimates of the resolvent of the bundle (1) from the paper [14], in the case 2) the Keldysh theorem [2] with using Fragmen-Lindelöff theorem we get that $g(\lambda)$ is a polynomial. Since for $t > 0$

$$\int_{\Gamma_{\pm\theta}} g(\lambda)\, e^{\lambda t} d\lambda = 0,$$

then consequently for $t > 0$

$$\sum_{k=0}^{2} \left(\frac{d^k u(t)}{dt^k}, \eta_k \right)_{H_{7/2-k}} = 0.$$

Here passing to limit as $t \to 0$, we get

$$\sum_{k=0}^{2} (\varphi_k, \eta_k)_{H_{7/2-k}} = 0.$$

Since the choice of vectors φ_k, $k = 0, 1, 2$, is arbitrary, then $\eta_k = 0$, $k = 0, 1, 2$, therefore $\eta = 0$. We get contradiction. The theorem is proved.

Notice that conditions of double completeness of a part of e.a.v. of fourth order polynomial operator bundles differ from the bundle (1) by characteristical properties are given in the papers [5],[15].

Remark. Under conditions of theorem 2 we can prove completeness of the system of elementary solutions of the operator differential equation (2), but we'll speak about this fact in another paper.

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