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# THE EXISTENCE AND UNIQUENESS OF WEAK SOLUTION OF THE FIRST BOUNDARY VALUE PROBLEM FOR SECOND ORDER DEGENERATE ELLIPTICO-PARABOLIC EQUATIONS

**Abstract**

*In the article the first boundary value problem for the second order degenerate elliptico-parabolik equations in divergent form with small coefficients is considered.The unique weak solvability of the formulated problem is proved for some conditions on small coefficients.*

**1.Introduction.** Let $R_n$ be an $n$-dimensional Euclidean space of the points $x = (x_1, ...x_n)$, $\Omega \subset R_n$, be a bounded domain with the boundary $\partial\Omega$, $Q_T = \{(x,t) : x \in \Omega,\ 0 < t < T < \infty\}$, $S_T = \{(x,t) : x \in \partial\Omega,\ 0 \le t \le T\}$, $\Gamma(Q_T)$ - be a parabolic boundary of $Q_T$,i.e. $\Gamma(Q_T) = S_T \cup \{(x,t) : x \in \Omega,\ t = 0\}$.

Consider the the first bonndary value problem in $Q_T$

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \frac{\partial}{\partial t}\left(\varphi(T-t)\frac{\partial u}{\partial t}\right) +$$

$$+\sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)\,u - \frac{\partial u}{\partial t} = f(x,t), \tag{1}$$

$$u|_{\Gamma(Q_T)} = 0 \tag{2}$$

in assumption that $\|a_{ij}(x,t)\|$ is a real symmetric matrix,where for $(x,t) \in Q_T, \xi \in R_n$

$$\gamma|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\xi_j \le \gamma^{-1}|\xi|^2,\ \gamma \in (0,1] - const, \tag{3}$$

$$\varphi(0) = 0, \varphi(z) > 0, \varphi'(z) \ge 0, \varphi''(z) \ge 0, \varphi'(z) \ge \varphi(z)\,\varphi''(z);\ z \in (0,T), \tag{4}$$

$$b_i \in L_{n+2}(Q_T), i = 1, ..., n;\ c(x,t) \in L_s(Q_T),$$

$$s = \begin{cases} \max\left\{2; \dfrac{n+2}{2}\right\}, & n \ne 2 \\ 2+\nu, & n = 2 \end{cases}, \tag{5}$$

here $\nu$ some positive constant.

The aim of the present article is the proof of the unique weak solvability of the problem (1), (2) in corresponding Sobolev weight space for any $f(x,t) \in L_2(Q_T)$. Note that, for parabolic equations of divergent structure the analogous result was obtainedin [1]. As to the parabolic equations of non-divergent structure we indicate the papers [2-4].For the second order degenerate elliptico-parabolic equations of non-divergent structure the solvability of the first boundary value problem is studied in [5-6]. The unique strong (almost everywhere) and weak solvability of problem (1),

(2) for conditions $b_i(x,t) \equiv 0$, $i = 1,...n$, $c(x,t) \equiv 0$ is proved in [7-8]. In [9] a unique strong (almost everywhere) solvability of problem (1), (2) when the coefficients $a_{ij}(x,t)$ are continuosly differentiable in $x$ is proved.

**$1^0$. Some notations and definitions.**

Let $V_2(Q_T), W_2^{1,1}(Q_T), W_{2,\varphi}^{1,1}(Q_T)$ and $W_{2,\varphi}^{1,1}$ be Banach spaces of measurable functions given on $Q_T$ for which the norms

$$\|u\|_{V_2(Q_T)} = \left(\sup \int_\Omega u^2(x,t)\,dx + \int_{Q_T} \sum_{i=1}^n u_i^2 dxdt\right)^{\frac{1}{2}}$$

$$\|u\|_{W_2^{1,1}(Q_T)} = \left(\int_{Q_T} \left(u + \sum_{i=1}^n u_i^2 + u_t^2\right) dxdt\right)^{\frac{1}{2}}$$

$$\|u\|_{W_{2,\varphi}^{1,1}(Q_T)} = \left(\int_\Omega u^2(x,T)\,dx + \int_{Q_T} \left(\sum_{i=1}^n u_i^2 + \varphi(T-t)\,u_t^2\right) dxdt\right)^{\frac{1}{2}}$$

and

$$\|u\|_{W_{2,\varphi}^{2,2}(Q_T)} =$$

$$= \left(\int_{Q_T} \left(u^2 + \sum_{i=1}^n u_i^2 + \sum_{i,j=1}^n u_{ij}^2 + u_t^2 + 2\varphi(T-t)\sum_{i=1}^n u_{it}^2 + \varphi^2(T-t)\,u_{tt}^2\right) dxdt\right)^{\frac{1}{2}}$$

are finite, respectively, $\dot{W}_2^{1,1}(Q_T)$ and $\dot{W}_{2,\varphi}^{1,1}(Q_T)$ be subspaces of $W_2^{1,1}(Q_T)$, $W_{2,\varphi}^{1,1}(Q_T)$, dense set in which is totality of all functions from $C^\infty\left(\overline{Q}_T\right)$ vanishing on $\Gamma(Q_T)$. Here for $i,\ j = 1,...,n$ $u_i = \dfrac{\partial u}{\partial x_i}$, $u_{ij} = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$.

In the sequel the notation $C(...)$ means that the positive constant $C$ depends only on the contents of brackets.

Under the weak solution of equation (1) we shall understand the function $u(x,t) \in \dot{W}_{2,\varphi}^{1,1}(Q_T)$, such that for any $\vartheta(x,t) \in \dot{W}_2^{1,1}(Q_T)$ the integral identity

$$\int_{Q_T} \sum_{i,j}^n a_{ij} u_j \vartheta_i dxdt - \int_{Q_T} u\vartheta_t dxdt + \int_\Omega u(x,T)\,\vartheta(x,T)\,dx - \int_{Q_T} \sum_{i=1}^n b_i u_i \vartheta dxdt -$$

$$- \int_{Q_T} cu\vartheta dxdt + \int_{Q_T} \varphi(T-t)\,u_t \vartheta_t dxdt = - \int_{Q_T} f\vartheta dxdt \qquad (6)$$

is valid.

Under the weak solution of the boundary value problem (1) , (2) we shall understand the function $u(x,t) \in \dot{W}^{1,1}_{2,\varphi}(Q_T)$ which is the weak solution of equation (1).

By $\Phi_h$ we denote the Friedrichs averaging of function $\Phi$ with averaging radius $h$.

$2^0$. **Solvability of the first boundary value problem.**

**Theorem.** *Let the coefficients of the operator $L$ satisfying conditions (3)-(6) be defined in $Q_T$. Then for $f(x,t) \in L_2(Q_T)$ the first boundary value problem (1), (2) is uniquely weak solvable in $\dot{W}^{1,1}_{2,\varphi}(Q_T)$ for $T \le T_1(L, n, diam\Omega)$ and for its solution the estimate*

$$\|u\|_{W^{1,1}_{2,\varphi}(Q_T)} \le C_1 \|f\|_{L_2(Q_T)},$$

*is valid, where the constant $C_1 > 0$ depends only on $\gamma, \varphi, b_i, c, n$ and $diam\Omega$.*

**Proof.** At first we assume that $\partial\Omega \in C^2$.Consider for $h > 0$ the family of the following first boundary value problems

$$L^h u^h = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[(a_{ij})_h u_j^h\right] + \frac{\partial}{\partial t}\left(\varphi(T-t)\, u_t^h\right) +$$

$$+\sum_{i=1}^{n} b_i u_i^h + c u^h - u_t^h = f(x,t), \tag{7}$$

$$u^h\Big|_{\Gamma(Q_T)} = 0, \tag{8}$$

where $\varphi$ satisfies (4).

It's obvious,that $(a_{ij})_h \in C^\infty\left(\overline{Q}_T\right)$ and for all $h > 0$ relative to the $(a_{ij})_h$ the conditions (3) with constant $\gamma$ is satisfied.Then according to [9] there exists a unique strong solution $u^h(x,t) \in \dot{W}^{2,2}_{2,\varphi}(Q_T)$ of the problem (7)-(8). It's obvious,that $u^h(x,t) \in \dot{W}^{1,1}_{2}(Q_T)$.

We multiply both sides of the equation (7) by the function $\vartheta(x,t) \in \dot{W}^{1,1}_{2}(Q_T)$ and integrate over $Q_T$ :

$$\int_{Q_T} L^h u^h \vartheta dxdt = \int_{Q_T} f\vartheta dxdt. \tag{9}$$

Since $u^h(x,t) \in \dot{W}^{1,1}_{2}(Q_T)$ we can put $\vartheta = u^h$ in (9). Then we have:

$$\int_{Q_T} \sum_{i,j=1}^{n} (a_{ij})_h u_j^h u_i^h dxdt - \int_{Q_T} u^h u_t^h dxdt +$$

$$+\int_{\Omega} \left(u^h(x,T)\right)^2 dx - \int_{Q_T} \sum_{i=1}^{n} b_i u_i^h u^h dxdt -$$

$$-\int_{Q_T} c\left(u^h\right)^2 dxdt + \int_{Q_T} \varphi(T-t)\left(u_t^h\right)^2 dxdt = -\int_{Q_T} f u^h dxdt. \tag{10}$$

Let's estimate the first term of the left hand side of equality (10) from below using the condition (3) and represent the second term of the left hand side of equality (10) in the form

$$\int\limits_{Q_T} u^h u_t^h dxdt = \frac{1}{2}\int\limits_{\Omega}\left(u^h(x,t)\right)^2 dx\Bigg|_{t=0}^{t=T} = \frac{1}{2}\int\limits_{\Omega}\left(u^h(x,T)\right)^2 dx.$$

It gives us the inequality

$$\gamma\int\limits_{Q_T}\sum_{i=1}^{n}\left(u_i^h\right)^2 dxdt + \frac{1}{2}\int\limits_{\Omega}\left(u^h(x,T)\right)^2 dx + \int\limits_{Q_T}\varphi(T-t)\left(u_t^h\right)^2 dxdt \le$$

$$\le \int\limits_{Q_T}\left|fu^h\right| dxdt + \int\limits_{Q_T}\sum_{i=1}^{n}\left|b_i u_i^h u^h\right| dxdt + \int\limits_{Q_T}|c|\left(u^h\right)^2 dxdt \le$$

$$\le \frac{\varepsilon_1}{2}\int\limits_{Q_T}\left(u^h\right)^2 dxdt + \frac{1}{2\varepsilon_1}\int\limits_{Q_T} f^2 dxdt + \int\limits_{Q_T}\sum_{i=1}^{n}\left|b_i u_i^h u^h\right| dxdt + \int\limits_{Q_T}|c|\left(u^h\right)^2 dxdt,$$

where $\varepsilon_1 > 0$ will be chosen later.

From the Friedrichs inequality we have:

$$\int\limits_{Q_T}\left(u^h\right)^2 dxdt \le C_2\left(diam\Omega\right)\int\limits_{Q_T}\sum_{i=1}^{n}\left(u_i^h\right)^2 dxdt.$$

Now the number $\varepsilon_1 > 0$ can be chosen so small that the estimate

$$\left\|u^h\right\|_{W_{2,\varphi}^{1,1}(Q_T)} \le C_3\|f\|_{L_2(Q_T)} + \int\limits_{Q_T}\sum_{i=1}^{n}\left|b_i u_i^h u^h\right| dxdt + \int\limits_{Q_T}|c|\left(u^h\right)^2 dxdt \qquad (11)$$

is satisfied.

Let's estimate integrals

$$\int\limits_{Q_T}\sum_{i=1}^{n}\left|b_i u_i^h u^h\right| dxdt \text{ and } \int\limits_{Q_T}|c|\left(u^h\right)^2 dxdt$$

from above. We have:

$$\int\limits_{Q_T}\sum_{i=1}^{n}\left|b_i u_i^h u^h\right| dxdt \le \frac{1}{2\varepsilon_2}\left\|\sum_{i=1}^{n} b_i u^h\right\|_{L_2(Q_T)}^2 + \frac{\varepsilon_2}{2}\left\|\sum_{i=1}^{n} u_i^h\right\|_{L_2(Q_T)}^2 \qquad (12)$$

$$\left\|\sum_{i=1}^{n} b_i u^h\right\|_{L_2(Q_T)}^2 \le \left(\int\limits_{Q_T}\sum_{i=1}^{n}\left(b_i\right)^{n+2} dxdt\right)^{\frac{2}{n+2}}\left(\int\limits_{Q_T} u^{h\,\frac{2(n+2)}{n}} dxdt\right)^{\frac{n}{(n+2)}} \le$$

$$\leq \left\|\sum_{i=1}^{n} b_i\right\|^2_{L_{n+2}(Q_T)} \left\|u^h\right\|^2_{L_{\frac{2(n+2)}{n}}(Q_T)} \leq C_3 \left\|\sum_{i=1}^{n} b_i\right\|^2_{L_{n+2}(Q_T)} \left\|u^h\right\|^2_{V_2(Q_T)}. \quad (13)$$

$$\int_{Q_T} |c| \left(u^h\right)^2 dxdt \leq \|c\|^2_{L_{\frac{n+2}{2}}(Q_T)} \left\|u^h\right\|^2_{L_{\frac{2(n+2)}{n}}(Q_T)} \leq$$

$$\leq C_3 \|c\|^2_{L_{\frac{n+2}{2}}(Q_T)} \left\|u^h\right\|^2_{V_2(Q_T)}. \quad (14)$$

where $C_3 = C_3(n)$. Let $t_h \in (0,T)$ be such that

$$\int_{\Omega} \left(u^h(x,t_n)\right)^2 dx \geq \frac{1}{2} \sup_{0\leq t\leq T} \int_{\Omega} u^{h^2}(x,t)\, dx.$$

Two cases are possible:

1) The points $t_h$ are seperated from $T$.

2) The points $t_h$ are not seperated from $T$.

Case 1).

In this case there exists a subsequence $t_{h_k}$ such that $t_{h_k} \to T$. We have:

$$\sup_{0\leq t\leq T} \int_{\Omega} u^{h^2}(x,t)\, dx \leq 2\int_{\Omega} \left(u^h(x,t_n)\right)^2 dx =$$

$$= 2\int_{\Omega} \left(u^{h^2}(x,t_n) - u^{h^2}(x,T)\right) dx +$$

$$+2\int_{\Omega} u^{h^2}(x,T)\, dx \leq 2\varepsilon(h) + 2\int_{\Omega} u^{h^2}(x,T)\, dx \leq 2\alpha + 2\int_{\Omega} u^{h^2}(x,T)\, dx.$$

Thus

$$\left\|u^h\right\|^2_{V_2(Q_T)} \leq 2\alpha + 2\int_{\Omega} u^{h^2}(x,T)\, dx + \int_{Q_T} \sum_{i=1}^{n} \left(u_i^h\right)^2 dxdt,$$

where $\alpha = \sup_h \varepsilon(h)$.

We choose numbers $\varepsilon_2$ and $T_1$ such that for $T \leq T_1$

$$\int_{Q_T} \sum_{i=1}^{n} \left|b_i u_i^h u^h\right| dxdt \leq \frac{1}{4}\int_{\Omega} u^{h^2}(x,T)\, dx + \frac{1}{8}\int_{Q_T} \sum_{i=1}^{n} \left(u_i^h\right)^2 dxdt + \frac{\alpha}{2}, \quad (15)$$

$$\int_{Q_T} |c| \left(u^h\right)^2 dxdt \leq \frac{1}{4}\int_{\Omega} u^{h^2}(x,T)\, dx + \frac{1}{8}\int_{Q_T} \sum_{i=1}^{n} \left(u_i^h\right)^2 dxdt + \frac{\alpha}{2}. \quad (16)$$

Taking into account (15) , (16) in (11) we have

$$\left\|u^h\right\|_{W^{1,1}_{2,\varphi}(Q_T)} \leq C_4 \|f\|_{L_2(Q_T)} + \frac{\alpha}{2} \quad (17)$$

Case 2).

In this case there exists a subsequence $t_{h_k}$ such that $t_{h_k} \to \tau \neq T, \quad \tau \in (0,T)$.

We have

$$\sup_{0\le t\le T}\int_{\Omega} u^{h^2}(x,t)\,dx \le 2\int_{\Omega}\left(u^h(x,t_n)\right)^2 dx = \int_0^{t_h}\int_{\Omega}\left(u^h\right)_t^2 dxdt \le$$

$$\le \int_0^{t_h}\int_{\Omega}\left(u_t^h\right)^2 dxdt + \int_0^{t_h}\int_{\Omega}\left(u^h\right)^2 dxdt \le \frac{1}{\varphi(T-t_h)}\times$$

$$\times\int_{Q_T}\varphi(T-t)\left(u_t^h\right)^2 dxdt + C_5\int_{Q_T}\sum_{i=1}^n\left(u_i^h\right)^2 dxdt.$$

Thus

$$\left\|u^h\right\|^2_{V_2(Q_T)} \le \frac{1}{\varphi(T-t_h)}\int_{Q_T}\varphi(T-t)\left(u_t^h\right)^2 dxdt+$$

$$+C_5\int_{Q_T}\sum_{i=1}^n\left(u_i^h\right)^2 dxdt + \int_{Q_T}\sum_{i=1}^n\left(u_i^h\right)^2 dxdt.$$

Let $T_2$ be such that for $T \le T_2$

$$\int_{Q_T}\sum_{i=1}^n\left|b_i u_i^h u^h\right| dxdt \le \frac{1}{4}\int_{Q_T}\varphi(T-t)\left(u_t^h\right)^2 dxdt + \frac{1}{8}\int_{Q_T}\sum_{i=1}^n\left(u_i^h\right)^2 dxdt, \tag{18}$$

$$\int_{Q_T}|c|\left(u^h\right)^2 dxdt \le \frac{1}{4}\int_{Q_T}\varphi(T-t)\left(u_t^h\right)^2 dxdt + \frac{1}{8}\int_{Q_T}\sum_{i=1}^n\left(u_i^h\right)^2 dxdt. \tag{19}$$

Taking into account (18) , (19) in (11) we have:

$$\left\|u^h\right\|_{W^{1,1}_{2,\varphi}(Q_T)} \le C_6\left\|f\right\|_{L_2(Q_T)}, \tag{20}$$

where the constant $C_6 > 0$ depends only on $\gamma,\ b_i,\ c,\ \varphi,\ n$ and $diam\Omega$.

From (17), (20) it follows that the sequence $\left\{u^h(x,t)\right\}$ is strongly bounded in $\dot{W}^{1,1}_{2,\varphi}(Q_T)$. Consequently,this sequence is weakly compact in $\dot{W}^{1,1}_{2,\varphi}(Q_T)$. In other words, there exists a subsequence $\left\{u^h(x,t)\right\}$, $h_k \to 0$ as $k \to \infty$ and the function $u(x,t) \in \dot{W}^{1,1}_{2,\varphi}(Q_T)$ such that for any $\psi(x,t) \in C_0^{\infty}\left(\overline{Q}_T\right)$

$$\lim_{k\to\infty}\left(Lu^{h_k},\psi\right) = (Lu,\psi). \tag{21}$$

Now we show that the function $u(x,t)$ satisfies (7) for any $\vartheta(x,t) \in \dot{W}^{1,1}_2(Q_T)$. Since $u^{h_k} \in \dot{W}^{2,2}_{2,\varphi}(Q_T)$ is a weak solution of equation (7),then for any $\vartheta(x,t) \in \dot{W}^{1,1}_2(Q_T)$ the following equality

$$\int_{Q_T}\sum_{i,j=1}^n\left(a_{ij}\right)_{h_k}u_j^{h_k}\vartheta_i dxdt - \int_{Q_T}u^{h_k}\vartheta_t dxdt + \int_{\Omega}u^{h_k}(x,T)\,\vartheta(x,T)\,dx-$$

$$-\int\limits_{Q_T}\sum_{i=1}^{n} b_i u_i^{h_k}\vartheta dxdt-\int\limits_{Q_T} cu^{h_k}\vartheta dxdt+$$

$$+\int\limits_{Q_T}\varphi\left(T-t\right)u_t^{h_k}\vartheta_t dxdt=-\int\limits_{Q_T} f\vartheta dxdt. \qquad (22)$$

is valid.

Here,taking the limits as $k\to\infty$,by virtue of (21),it remains to prove that

$$\int\limits_{Q_T}\sum_{i,j=1}^{n}\left(a_{ij}\right)_{h_k}u_j^{h_k}\vartheta_i dxdt\to\int\limits_{Q_T}\sum_{i,j=1}^{n} a_{ij}u_j\vartheta_i dxdt \qquad \left(k\to\infty\right).$$

We have

$$\int\limits_{Q_T}\sum_{i,j=1}^{n}\left(a_{ij}\right)_{h_k}u_j^{h_k}\vartheta_i dxdt=\int\limits_{Q_T}\sum_{i,j=1}^{n}\left[\left(a_{ij}\right)_{h_k}-a_{ij}\right]u_j^{h_k}\vartheta_i dxdt+$$

$$+\int\limits_{Q_T}\sum_{i,j=1}^{n} a_{ij}u_j^{h_k}\vartheta_i dxdt. \qquad (23)$$

The first term of the right hand side of equality (23)tends to zero as $k\to\infty$. Indeed,

$$\int\limits_{Q_T}\sum_{i,j=1}^{n}\left[\left(a_{ij}\right)_{h_k}-a_{ij}\right]u_j^{h_k}\vartheta_i dxdt\le\sum_{i,j=1}^{n} ess\ \sup_{Q_T}\left|\left(a_{ij}\right)_{h_k}-a_{ij}\right|\int\limits_{Q_T}\left|u_j^{h_k}\vartheta_i\right|dxdt\le$$

$$\le\sum_{i,j=1}^{n} ess\ \sup_{Q_T}\left|\left(a_{ij}\right)_{h_k}-a_{ij}\right|\left(\int\limits_{Q_T}\left(u_j^{h_k}\right)^2 dxdt\right)^{\frac{1}{2}}\left(\int\limits_{Q_T}\vartheta_i^2 dxdt\right)^{\frac{1}{2}}\to 0\left(k\to\infty\right)$$

by virtue of the estimate (17) .

The second term of the right hand side of equality (23) can be represented in the form

$$\int\limits_{Q_T}\sum_{i,j=1}^{n} a_{ij}u_j^{h_k}\vartheta_i dxdt=\int\limits_{Q_T}\sum_{i,j=1}^{n} a_{ij}\left(u_j^{h_k}-u_j\right)\vartheta_i dxdt+\int\limits_{Q_T}\sum_{i,j=1}^{n} a_{ij}u_j\vartheta_i dxdt.$$

We have

$$\int\limits_{Q_T}\sum_{i,j=1}^{n} a_{ij}\left(u_j^{h_k}-u_j\right)\vartheta_i dxdt\to 0,\quad k\to\infty$$

by virtue of weak convergense of sequence $\left\{u^h\left(x,t\right)\right\}$ to the function $u\left(x,t\right)$ in the space $W_{2,\varphi}^{1,1}\left(Q_T\right)$. Thus,

$$\int\limits_{Q_T}\sum_{i,j=1}^{n}\left(a_{ij}\right)_{h_k}u_j^{h_k}\vartheta_i dxdt\to\int\limits_{Q_T}\sum_{i,j=1}^{n} a_{ij}u_j\vartheta_i dxdt \quad \left(k\to\infty\right).$$

Hence,the existence of weak solution of the boundary value problem (1)-(2) for $\partial\Omega \in C^2$ is proved.The case $\partial\Omega \notin C^2$ is considered analogously by approximation $\Omega$ from inside by sequences of domains with smooth boundaries and then taking the limit.

Now we shall prove the uniqueness of solution of the boundary value problem (1), (2). It suffices to prove that the homogeneous boundary value problem $Lu = 0$, $u|_{\Gamma(Q_T)} = 0$ has only trivial solution.

We put $f = 0$ in the equality (7) and then as $\vartheta(x,t)$ we take the function

$$\vartheta_{(\overline{h})}(x,t) = \frac{1}{h}\int\limits_{t-h}^{h} \vartheta(x,\tau)\,d\tau, \tag{24}$$

where $\vartheta(x,t)$ is arbitrary element of $\dot{W}_2^{1,1}\left(Q_{-h}^T = \Omega \times (-h,T)\right)$ which is equal to zero for $t \geq T-h$ and $t \leq 0$ and fix $h > 0$.

Consequently,we have:

$$\int\limits_{Q_{T-h}} \sum_{i,j=1}^{n} a_{ij}u_j \left(\vartheta_{(\overline{h})}\right)_i dxdt - \int\limits_{Q_{T-h}} u\left(\vartheta_{(\overline{h})}\right)_t dxdt +$$

$$+\int\limits_{Q_{T-h}} \varphi(T-t)\,u_t\left(\vartheta_{(\overline{h})}\right)_t dxdt -$$

$$-\int\limits_{Q_{T-h}} \sum_{i=1}^{n} b_i u_i \vartheta_{(\overline{h})} dxdt - \int\limits_{Q_{T-h}} cu\vartheta_{(\overline{h})} dxdt = 0. \tag{25}$$

In the all terms of equality (25) we transfer averaging of $(\ )_{\overline{h}}$ from $\vartheta$ to multipliers standing before it, besides, in the second term, we integrate by parts with respect to $t$.

Then we have

$$\int\limits_{Q_{T-h}} \sum_{i,j=1}^{n} \left(a_{ij}u_j\right)_{(h)} \vartheta_i dxdt + \int\limits_{Q_{T-h}} \left(u_{(h)}\right)_t \vartheta dxdt + \int\limits_{Q_{T-h}} \left(\varphi(T-t)\,u_t\right)_{(h)} \vartheta_t dxdt -$$

$$-\int\limits_{Q_{T-h}} \sum_{i=1}^{n} \left(b_i u_i\right)_{(h)} \vartheta dxdt - \int\limits_{Q_{T-h}} (cu)_{(h)} \vartheta dxdt = 0 \tag{26}$$

where

$$u_{(h)}(x,t) = \frac{1}{h}\int\limits_{t}^{t+h} u(x,\tau)\,d\tau.$$

We have

$$\left(u_{(h)}\right)_t = \left(\frac{1}{h}\int\limits_{t}^{t+h} u(x,\tau)\,d\tau\right)_t' = \frac{1}{h}\left(u(x,t+h) - u(x,t)\right).$$

Consequently, $u_{(h)} \in \dot{W}_2^{1,1}(Q_T)$.

Therefore in equality (25) the function $u_{(h)}$ can be taken instead of $\vartheta$. Then

$$\int_{Q_{T-h}} \sum_{i,j=1}^{n} (a_{ij}u_j)_{(h)} \left(u_{(h)}\right)_i dxdt+$$

$$+\int_{Q_{T-h}} \left(u_{(h)}\right)_t u_{(h)}dxdt + \int_{Q_{T-h}} (\varphi(T-t)\, u_t)_{(h)} (u_t)_{(h)}\, dxdt-$$

$$-\int_{Q_{T-h}} \sum_{i=1}^{n} (b_i u_i)_{(h)} u_{(h)}dxdt - \int_{Q_{T-h}} (cu)_{(h)} u_{(h)}dxdt = 0.$$

Since

$$\int_{Q_{T-h}} \left(u_{(h)}\right)_t u_{(h)}dxdt = \frac{1}{2}\int_{\Omega} \left(u_{(h)}(x,T)\right)^2 dx,$$

then

$$\int_{Q_{T-h}} \sum_{i,j=1}^{n} (a_{ij}u_j)_{(h)} \left(u_{(h)}\right)_i dxdt+$$

$$+\int_{Q_{T-h}} (\varphi(T-t)\, u_t)_{(h)} (u_t)_{(h)}\, dxdt + \frac{1}{2}\int_{\Omega} \left(u_{(h)}(x,T)\right)^2 dx-$$

$$-\int_{Q_{T-h}} \sum_{i=1}^{n} b_i u_i u_{(h)}dxdt - \int_{Q_{T-h}} \left(\sum_{i=1}^{n} (b_i u_i)_{(h)} - b_i u_i\right) u_{(h)}dxdt-$$

$$-\int_{Q_{T-h}} \left((cu)_{(h)} - cu\right) u_{(h)}dxdt - \int_{Q_{T-h}} cu\; u_{(h)}dxdt < 0.$$

Let's fix arbitrary $0 < h_0 < T$. Then in the previous inequality the domain $Q_{T-h}$ can be replaced by the domain $Q_{T-h_o}$, where $h \le h_0$.

$$\int_{Q_{T-h_0}} \sum_{i,j=1}^{n} (a_{ij}u_j)_{(h)} \left(u_{(h)}\right)_i dxdt+$$

$$+\int_{Q_{T-h_0}} (\varphi(T-t)\, u_t)_{(h)} (u_t)_{(h)}\, dxdt + \frac{1}{2}\int_{\Omega} \left(u_{(h)}(x,T)\right)^2 dx-$$

$$-\int_{Q_{T-h_0}} \sum_{i=1}^{n} b_i u_i u_{(h)}dxdt - \int_{Q_{T-h_0}} \left(\sum_{i=1}^{n} (b_i u_i)_{(h)} - b_i u_i\right) u_{(h)}dxdt-$$

$$-\int_{Q_{T-h_0}} \left((cu)_{(h)} - cu\right) u_{(h)}dxdt - \int_{Q_{T-h_0}} cu\; u_{(h)}dxdt < 0.$$

Taking into account (15), (16), (18),(19) and taking the limit as $h \to 0$, we have

$$\int_{Q_{T-h_0}} \sum_{i,j=1}^{n} a_{ij} u_j u_i dxdt + \int_{Q_{T-h_0}} \varphi(T-t) u_t^2 dxdt < 0,$$

which gives us $u(x,t) = 0$ a.e. on $Q_T$ by using (3), Friedrichs inequality and the fact that $h_0$ is arbitrary.

The theorem is proved.

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