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ON A BOUNDARY VALUE PROBLEM FOR A SINGULARLY PERTURBED QUASILINEAR EQUATION OF NON-CLASSIC TYPE

Abstract

In a rectangular domain we consider a boundary value problem for a non-classic type quasilinear equation of arbitrary odd order, containing a small parameter at the higher derivatives. Complete asymptotic of the solution of the considered problem by small parameter with boundary layer functions near the three sides of a rectangle is constructed and residual term is estimated.

In some applied problems it is necessary to construct asymptotics of the solution of boundary value problems for singularly perturbed differential equations. Such non-classic equations have been studied enough. In the papers [1] [2] the asymptotics of the solution of boundary value problems for non-classic type linear equations is constructed in a rectangular domain with four viscous boundaries.

It should be noted that the construction of the solution of boundary value problems for non-linear equations reduces to some analytic calculations.

The papers devoted to nonlinear singularly perturbed differential equations are few. Here we note the papers [3]-[8].In these papers nonlinear classic equations are investigated.

In the present paper, in a rectangular domain

$$D = \{(t,x)\,|0 \le t \le 1,\ 0 \le x \le 1\}$$

we consider the following boundary value problem

$$L_\varepsilon U = (1)^m \varepsilon^{2m} \frac{\partial^{2m+1} U}{\partial t^{2m+1}} - \varepsilon \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)^p - \\ -\varepsilon \frac{\partial^2 U}{\partial x^2} + \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} + aU - f(t,x) = 0, \tag{1}$$

$$U\,|_{t=0} = \frac{\partial U}{\partial t}\,|_{t=0} = ... = \frac{\partial^m U}{\partial t^m}\,|_{t=0} = 0, \\ \frac{\partial^{m+1} U}{\partial t^{m+1}}\,|_{t=1} = \frac{\partial^{m+2} U}{\partial t^{m+2}}\,|_{t=1} = ... = \frac{\partial^{2m} U}{\partial t^{2m}}\,|_{t=1} = 0, \tag{2}$$

$$U\,|_{x=0} = U\,|_{x=1} = 0, \tag{3}$$

where $\varepsilon > 0$ is a small parameter $p = 2k+1$, k and m are arbitrary natural numbers, $a > 0$ is a constant, $f(t,x)$ is a given function.

Our goal is to construct asymptotic expansion of the solution of boundary value problem (1)-(3) by a small parameter $\varepsilon > 0$.

In the first iteration process we'll look for the approximate solution of equation (1) in the form

$$W = W_0 + \varepsilon W_1 + ... + \varepsilon^n W_n, \tag{4}$$

and the functions $W_i(t,x); i = 0,1,...,n$ will be chosen so that

$$L_\varepsilon W = 0(\varepsilon^{n+1}). \tag{5}$$

Substituting expression (4) for W into (5), for defining $W_i; i = 0,1,...,n$ we'll get the following recurrently connected equations:

$$L_0 W_0 \equiv \frac{\partial W_0}{\partial t} + \frac{\partial W_0}{\partial x} + aW_0 = f(t,x), \tag{6}$$

$$L_0 W_j = f_j(t,x); j = 1,2,...,n, \tag{7}$$

where the functions $f_j(t,x)$ depend on the derivatives $W_0, W_1, ..., W_{j-1}$.

For the equations (6), (7) with respect x the first condition from (3), i.e.

$$W_j\,|_{x=0} = 0;\ j = 0,1,...,n \tag{8}$$

should be used.

Below we'll write boundary conditions with respect to t for the equations (6), (7). Now we note that with respect to t we'll use the first condition from (2) for $t = 0$. For such choice of boundary conditions for equations (6), (7) on the boundary $S_1 = \{(t,x)\,|t = 0,\ 0 \le x \le 1\}$ in conditions from $m+1$ boundary conditions (2) for $t = 0$, on the boundary $S_2 = \{(t,x)\,|t = 1,\ 0 \le x \le 1\}$ all the m conditions from (2) for $t = 1$ and on the boundary $S_3 = \{(t,x)\,|0 \le t \le 1, x = 1\}$ the second condition from (3) will be lost. For compensating the lost boundary conditions, boundary layer functions near the boundaries S_1, S_2, S_3 should be constructed. Therefore, it is necessary to write new decompositions of the operator L_ε near these boundaries.

For writing new decompositions of the operator L_ε near the boundary S_1 we make change of variables: $t = \varepsilon\xi,\ x = x$. The decomposition of the operator L_ε in the coordinates (ξ, x) has the form:

$$L_{\varepsilon,1}U \equiv \varepsilon^{-1}\left\{(-1)^m \frac{\partial^{2m+1}U}{\partial\xi^{2m+1}} + \frac{\partial U}{\partial\xi} + \varepsilon\left(\frac{\partial U}{\partial x} + aU\right) - \right.$$

$$\left. -\varepsilon^2 \frac{\partial^2 U}{\partial x^2} - \varepsilon^{p+1}\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)^p\right\}. \tag{9}$$

A boundary layer function η near the boundary S_1 is found in the form

$$\eta = \varepsilon\left(\eta_0 + \varepsilon\eta_1 + ... + \varepsilon^{n+m-1}\eta_{n+m-1}\right), \tag{10}$$

as a solution of the equation

$$L_{\varepsilon,1}(W + \eta) - L_{\varepsilon,1}W = 0(\varepsilon^{n+m}). \tag{11}$$

It follows from (9) that the left hand side of (11) is of the form

$$L_{\varepsilon,1}(W+\eta)-L_{\varepsilon,1}W=\varepsilon^{-1}\left\{(-1)^m\frac{\partial^{2m+1}\eta}{\partial\xi^{2m+1}}+\frac{\partial\eta}{\partial\xi}+\varepsilon\left(\frac{\partial\eta}{\partial x}+a\eta\right)-\right.$$
$$\left.-\varepsilon^2\frac{\partial^2\eta}{\partial x^2}-\varepsilon^{p+1}\frac{\partial}{\partial x}\left[\left(\frac{\partial(W+\eta)}{\partial x}\right)^p-\left(\frac{\partial W}{\partial x}\right)^p\right]\right\}. \tag{12}$$

Expanding each function $W_i(\varepsilon\xi,x); i=0,1,...,n$ by Taylor's formula at the point $(0,x)$ we get a new expansion of W in powers of ε in the form

$$W=\sum_{j=0}^{n+m}\varepsilon^j\omega_j^{(0)}(\xi,x)+0(\varepsilon^{n+m+1}), \tag{13}$$

where $\omega_0^{(0)}=W_0(0,x)$ is independent of ξ, and the other functions $\omega_k^{(0)}$ are determined by the formula

$$\omega_k^{(0)}=\sum_{i+j=k}\frac{1}{i!}\frac{\partial^i W_j(0,x)}{\partial t^i}\xi^i; k=1,2,...,n+m. \tag{14}$$

Following (10)-(13) we get the following equations for determining $\eta_j; j==0,1,...,n+m-1$:

$$A\eta_0\equiv(-1)^m\frac{\partial^{2m+1}\eta_0}{\partial\xi^{2m+1}}+\frac{\partial\eta_0}{\partial\xi}=0, \tag{15}$$

$$A\eta_1=-\frac{\partial\eta_0}{\partial x}-a\eta_0, \tag{16}$$

$$A\eta_s=-\frac{\partial\eta_{s-1}}{\partial x}-a\eta_{s-1}+\frac{\partial^2\eta_{s-2}}{\partial x^2}; s=2,3,...,p+1, \tag{17}$$

$$A\eta_k=-\frac{\partial\eta_{k-1}}{\partial x}-a\eta_{k-1}+\frac{\partial^2\eta_{k-2}}{\partial x^2}+h_k;$$
$$k=p+2,p+3,...,n+m-1, \tag{18}$$

where h_k are the known functions that polynomially depend on the first and second derivatives of the functions $\omega_0^{(0)},\omega_1^{(0)},...,\omega_{k-p-2}^{(0)};\eta_0,\eta_1,...,\eta_{k-p-2}$. We can write obvious forms of h_k but their expressions are of bulky form. Here we indicate the expressions for h_{p+2},h_{p+3}:

$$h_{p+2}=\frac{\partial}{\partial x}\left[p\left(\frac{\partial\omega_0^{(0}}{\partial x}\right)^{p-1}\frac{\partial\eta_0}{\partial x}\right],$$

$$h_{p+3}=\frac{\partial}{\partial x}\left[p\left(\frac{\partial\omega_0^{(0}}{\partial x}\right)^{p-1}\frac{\partial\eta_1}{\partial x}+p(p-1)\left(\frac{\partial\omega_0^{(0}}{\partial x}\right)^{p-2}\times\right.$$

$$\times\frac{\partial\omega_1^{(0)}}{\partial x}\frac{\partial\eta_0}{\partial x}+\frac{p(p-1)}{2!}\left(\frac{\partial\omega_0^{(0}}{\partial x}\right)^{p-2}\left(\frac{\partial\eta_0}{\partial x}\right)^2\Bigg].$$

The boundary conditions for equations (14)-(17) are obtained from the requirement of satisfaction of the sum $W+\eta$ the conditions (2) for $t=0$ except the first condition, i.e.

$$\frac{\partial}{\partial t}(W+\eta)\left|_{t=0}=\frac{\partial^2}{\partial t^2}(W+\eta)\right|_{t=0}=...=\frac{\partial^m U}{\partial t^m}(W+\eta)\left|_{t=0}=0.\right. \quad (19)$$

Substituting the expressions for W,η from (4), (10) into (19) and comparing the terms at the same degrees with respect to ε, we get $(n+m-1)\ m$ boundary conditions, that may be given by a general formula

$$\frac{\partial^k\eta_j}{\partial\xi^k}\left|_{\xi=0}=-\frac{\partial^k W_{j+1-k}}{\partial t^k}\right|_{t=0};k=1,2,...,m;j=0,1,...,n+m-1. \quad (20)$$

In the equalities (20) the function W_r with negative indices and for $r>n$ should be considered identity zeros.

Now, we find boundary conditions for the equations (6), (7) with respect to t. For that we substitute the expansions (4) and (10) into the equality

$$(W+\eta)\left|_{t=0}=0\right.$$

and equate to zero the coefficients for ε, whose degrees are small than $n+1$. Then we have

$$W_0\left|_{t=0}=0,W_j\right|_{t=0}=-\eta_{j-1}\left|_{\xi=0};j=1,2,...,n.\right. \quad (21)$$

It should be noted that if the functions $W_i;i=0,1,...,n$ will satisfy the conditions (21), the sum $W+\eta$ will satisfy the first boundary condition from (2) to within ε^{n+1}, i.e.

$$(W+\eta)\left|_{t=0}=\varepsilon^{n+1}\varphi_\varepsilon(x),\right. \quad (22)$$

and the function $\varphi_\varepsilon(x)$ is determined by the formula

$$\varphi_\varepsilon(x)=(\eta_n+\varepsilon\eta_{n+1}+...+\varepsilon^{m-1}\eta_{n+m-1})\left|_{\xi=0}.\right. \quad (23)$$

Now, let's construct the functions $W_i,i=0,1,...,n$ and $\eta_j;j=0,1,...,n+m-1$. From (8) and (21) we have that the function W_0 is a solution of the equation (6), satisfying the boundary conditions

$$W_0\left|_{t=0}=0,W_0\right|_{x=0}=0. \quad (24)$$

The following lemma is valid.

Lemma 1. *Let the function $f(t,x)\in C^s(D)$ and satisfy the condition*

$$\frac{\partial^i f(t,x)}{\partial t^{i_1}\partial x^{i_2}}\left|_{t=x}=0;i=i_1+i_2;i=0,1,...,s,(0\leq t\leq 1).\right. \quad (25)$$

Then the problem (6), (24) has a unique solution, moreover $W_0(t,x)\in$

$\in C^s(D)$ *and satisfies the conditions*

$$\frac{\partial^i W_0(t,x)}{\partial t^{i_1}\partial x^{i_2}}\,|_{t=x}=0; i=i_1+i_2; i=0,1,...,s,(0\le t\le 1) \tag{26}$$

where s is an arbitrary natural number.

Proof. The solution of the problem (6), (24) is represented by the formula

$$W_0(t,x)=\begin{cases}\int\limits_0^x f(t-x+\tau,\tau)\exp\left[a(\tau-x)\right]d\tau & \text{for } 0\le x<t\le 1,\\ \int\limits_0^t f(\tau,\tau+x-t)\exp\left[a(\tau-t)\right]d\tau & \text{for } 0\le t<x\le 1.\end{cases} \tag{27}$$

Obviously, if $f(t,x)$ is a smooth function in D, the function $W_0(t,x)$ determined by the formula (27) will also be a smooth function in D for $t\neq x$. Assume that the function $f(t,x)$ satisfies the condition (25). Then, we can easily prove that the solution of the problem (6), (24) will be a smooth function in D and satisfy the condition (26). Lemma 1 is proved.

The natural number s contained in the conditions of lemma 1 should be chosen so that the smoothness of the function $W_0(t,x)$ and condition (26) allow to construct the remaining functions $W_1,W_2,...,W_n$. For that it suffices to assume $s=2m+2n+2$.

From (21) for $j=1$ we get that before to construct the function W_1 the function η_0 should be determined. Notice that in sequel, the functions $W_1,\eta_1,W_2,\eta_2,...,$ $W_n,\eta_n,\eta_{n+1},...,\eta_{n+m-1}$ will be determined in turn.

Let's write boundary conditions for η_0, for that we put in (20) $j=0$:

$$\frac{\partial\eta_0}{\partial\xi}\,|_{\xi=0}=-\frac{\partial W_0}{\partial t}\,|_{t=0}\,,\ \frac{\partial^2\eta_0}{\partial\xi^2}\,|_{\xi=0}=0,...,\frac{\partial^m\eta_0}{\partial\xi^m}\,|_{\xi=0}=0. \tag{28}$$

Thus, η_0 is a boundary layer type solution of the equation (14), satisfying the boundary conditions (28). A characteristic equation that corresponds to the ordinary differential equation (14) has m roots with negative real parts, that are denoted by $\lambda_1,\lambda_2,...,\lambda_m$. Obviously, the boundary layer solution of the problem (14), (28)is of the form

$$\eta_0=-\frac{\partial W_0(0,x)}{\partial t}(C_{01}e^{\lambda_1\xi}+C_{02}e^{\lambda_2\xi}+...+C_{0m}e^{\lambda_m\xi}), \tag{29}$$

where C_{0i} are the known numbers.

Since the functions W_0,η_0 are known, we can determine the function W_1 from the problem (7), (8) and (21) for $j=1$. We can look for the solution of this problem in the form: $W_1=W_1^{(1)}+W_1^{(2)}$ where $W_1^{(1)}$ and $W_1^{(2)}$ are the solutions of the following problems:

$$\frac{\partial W_1^{(1)}}{\partial t}+\frac{\partial W_1^{(1)}}{\partial x}+aW_1^{(1)}=\frac{\partial^2 W_0}{\partial x^2}, W_1^{(1)}\,|_{t=0}=0, W_1^{(1)}\,|_{x=0}=0, \tag{30}$$

$$\frac{\partial W_1^{(2)}}{\partial t}+\frac{\partial W_1^{(2)}}{\partial x}+aW_1^{(2)}=0, W_1^{(2)}|_{t=0}=\varphi_1(x), W_1^{(2)}|_{x=0}=0, \tag{31}$$

moreover, instead of the right hand side of the equation (7) for $j=1$ its obvious form $f_1=\dfrac{\partial^2 W_0}{\partial x^2}$ is substituted, and from (21) for $j=1$ and (29) it follows that $\varphi_1(x)$ is determined by the equality:

$$\varphi_1(x)=\left(\sum_{i=1}^{m} C_{0i}\right)\frac{\partial W_0(0,x)}{\partial t}. \tag{32}$$

The right hand side of the equation for $W_1^{(1)}$ satisfies the condition of lemma 1 for $s=2m+2n$. Therefore, by this lemma the problem (30) has a unique solution, moreover $W_1^{(1)}\in C^{2m+2n}(D)$ and satisfies the condition

$$\frac{\partial^k W_1^{(1)}(t,x)}{\partial t^{k_1}\partial x^{k_2}}|_{t=0}=0; k=k_1+k_2; k=0,1,...,2m+2n. \tag{33}$$

The solution of the problem (31) is of the form

$$W_1^{(2)}(t,x)=\begin{cases} 0 & \text{for } 0\le x<t\le 1,\\ \\ \varphi_1(x-t)\exp(-at) & \text{for } 0\le t<x\le 1. \end{cases} \tag{34}$$

By lemma 1 and from (32) it follows that $\varphi_1(x)\in C^{2m+2n+1}[0;1]$. Therefore, the function $W_1^{(2)}(t,x)$ for $t\neq x$ will be smooth in D. It follows from the formula (26) and (32) that

$$\varphi_1^{(k)}(0)=0; k=0,1,...,2m+2n+1. \tag{35}$$

Considering (35), the smoothness of the function $W_1^{(2)}(t,x)$ for $t=x$ is obtained directly from (34).

The function W_1 being the sum of $W_1^{(1)}, W_1^{(2)}$ belongs to the space $C^{2m+2n}(D)$ and following (33)-(35) satisfies the condition

$$\frac{\partial^k W_1(t,x)}{\partial t^{k_1}\partial x^{k_2}}|_{t=x}=0; k=k_1+k_2; k=0,1,...,2m+2n.$$

The remaining functions $W_2, W_3, ..., W_n$ entering into the right hand side of (4) are constructed by the similar reasonings carried out for W_1, by lemma 1.

While constructing the functions $\eta_1, \eta_2, ..., \eta_{p+1}$ we use the following statement.

Lemma 2. *The functions η_s being the of boundary layer type solutions of equations (16), (17) are determined by the formula*

$$\eta_s=\sum_{i=1}^{m}\left[C_{s0}^{(i)}(x)+C_{s1}^{(i)}(x)\xi+...+C_{ss}^{(i)}(x)\xi^s\right]e^{\lambda_i\xi}; s=1,2,...,p+1, \tag{36}$$

and the coefficients $C_{sj}^{(i)}(x)$ are expressed uniformly by the function

$$\frac{\partial^k W_r(0,x)}{\partial t^{1+k_1}\partial x^{k_2}}; k=k_1+k_2+1; r=0,1,...,S;$$
$$k_1=0,1,...,m-1; k_1+k_2+r=S. \tag{37}$$

Proof. At first we determine the function η_1. It follows from (16) and (29) that η_1 is a solution of the following equation:

$$(-1)^m \frac{\partial^{2m+1}\eta_1}{\partial\xi^{2m+1}} + \frac{\partial\eta_1}{\partial\xi} = \left[\frac{\partial^2 W_0(0,x)}{\partial t\partial x} + a\frac{\partial W_0(0,x)}{\partial t}\right]\left(\sum_{i=1}^{m} C_{0i}e^{\lambda_i\xi}\right). \tag{38}$$

From (20) for $j=1$ we find the boundary conditions for η_1

$$\frac{\partial\eta_1}{\partial\xi}\mid_{\xi=0} = -\frac{\partial W_1}{\partial t}\mid_{t=0}, \frac{\partial^2\eta_1}{\partial\xi^2}\mid_{\xi=0} = -$$
$$-\frac{\partial^2 W_0}{\partial t^2}\mid_{t=0}, \frac{\partial^3\eta_1}{\partial\xi^3}\mid_{\xi=0} = 0, ..., \frac{\partial^m\eta_1}{\partial\xi^m}\mid_{\xi=0} = 0. \tag{39}$$

So, η_1 is a boundary layer type solution of the equation (38) satisfying the boundary conditions (39). We can easily show that the function

$$\eta_1^{(1)} = \left[\frac{\partial^2 W_0(0,x)}{\partial t\partial x} + a\frac{\partial W_0(0,x)}{\partial t}\right]\xi\left(\sum_{i=1}^{m} d_{0i}e^{\lambda_i\xi}\right). \tag{40}$$

is a boundary layer type special solution of the equation (38). Here d_{0i} are the known numbers that are determined by the formula:

$$d_{0i} = \frac{C_{0i}}{(-1)^m(2m+1)\lambda_i^{m+1}} li = 1,2,...,m.$$

Represent η_1 in the form $\eta_1 = \eta_1^{(1)} + \eta_1^{(2)}$. Then $\eta_1^{(2)}$ will be a boundary layer type solution of the problem

$$(-1)^m \frac{\partial^{2m+1}\eta_1^{(2)}}{\partial\xi^{2m+1}} + \frac{\partial\eta_1^{(2)}}{\partial\xi} = 0, \tag{41}$$

$$\frac{\partial\eta_1^{(2)}}{\partial\xi}\mid_{\xi=0} = \varphi_1(x), \frac{\partial^2\eta_1^{(2)}}{\partial\xi^2}\mid_{\xi=0} =$$
$$= \varphi_2(x), ..., \frac{\partial^m\eta_1^{(2)}}{\partial\xi^m}\mid_{\xi=0} = \varphi_m(x). \tag{42}$$

where

$$\varphi_1(x) = -\frac{\partial W_1(0,x)}{\partial t} + \varphi(x)\cdot d_1, \varphi_2(x) = -\frac{\partial^2 W_0(0,x)}{\partial t^2} +$$
$$+d_2\varphi(x), \varphi_j(x) = d_j\cdot\varphi(x); j = 3,4,...,m;$$
$$d_s = -s\sum_{i=1}^{m} d_{0i}\lambda_i^{m-1}; s = 1,2,...,m; \varphi(x) = \frac{\partial^2 W_0(0,x)}{\partial t\partial x} + a\frac{\partial W_0(0,x)}{\partial t}.$$

Obviously, a boundary layer type solution of the problem (41), (42) is of the form

$$\eta_1^{(2)} = C_1(x)e^{\lambda_1\xi} + C_2(x)e^{\lambda_2\xi} + ... + C_m(x)e^{\lambda_m\xi}, \tag{43}$$

and the functions $C_i(x)$ are expressed by the functions W_0, W_1 in the following way:

$$C_i(x) = C_i^{(1)} \frac{\partial W_1(0,x)}{\partial t} + C_i^{(2)} \frac{\partial^2 W_0(0,x)}{\partial t^2} + \\ + C_i^{(3)} \left[\frac{\partial^2 W_0(0,x)}{\partial t \partial x} + a \frac{\partial W_0(0,x)}{\partial t} \right], \tag{44}$$

where $C_i^{(1)}, C_i^{(2)}, C_i^{(3)}, i = 1, 2, ..., m$ are the known numbers.

We get from (40) and (43) that the function η_1 is a sum of $\eta_i^{(1)}, \eta_i^{(2)}$ and is determined by the formula

$$\eta_1 = \sum_{i=1}^{m} \left[C_i(x) + d_{0i} \varphi(x) \xi \right] e^{\lambda_i \xi}. \tag{45}$$

Introducing the denotation

$$C_{10}^{(i)}(x) = C_i(x), C_{11}^{(i)}(x) = \\ d_{0i} \left[\frac{\partial^2 W_0(0,x)}{\partial t \partial x} + a \frac{\partial W_0(0,x)}{\partial t} \right]; i = 1, 2, ..., m \tag{46}$$

we can write formula (45) in the following way:

$$\eta_1 = \sum_{i=1}^{m} \left[C_{10}^{i}(x) + C_{11}^{(i)}(x) \xi \right] e^{\lambda_i \xi}. \tag{47}$$

It follows from (44), (46) and (47) that the statement of lemma 2 is true for $s = 1$. Now, let's assume that the statement of lemma 2 is true for $s \le r - 1$ and prove that it is true for $s = r \le p + 1$. From (17) for $s = r$ and from (20) for $j = r$ we have

$$(-1)^m \frac{\partial^{2m+1} \eta_r}{\partial \xi^{2m+1}} + \frac{\partial \eta_r}{\partial \xi} = - \frac{\partial \eta_{r-1}}{\partial x} - \eta_{r-1} + \frac{\partial^2 \eta_{r-2}}{\partial x^2},$$

$$\frac{\partial^k \eta_r}{\partial \xi^k} |_{\xi=0} = \frac{\partial^k W_{r+1-k}}{\partial t^k} |_{t=0}; k = 1, 2, ..., m.$$

The right hand side of the equation for η_r contains the functions η_{r-1}, η_{r-2} that by assumption are determined by the formula (36). Repeating the similar reasonings carried out for determining the function η_1 we can affirm that η_r is also determined by the formula (36).

Lemma 2 is proved.

By lemma 2 we can assume that the functions $\eta_0, \eta_1, ..., \eta_{p+1}$ are already constructed. The right hand side of the equation (18) for $\eta_j; j = p+2, p+3, ..., n+m-1$ contains some of previous functions $\eta_0, \eta_1, ..., \eta_{j-1}$ in a nonlinear way. In this connection we should clarify if these equations have boundary layer solutions. For example, it is seen from the obvious form of the function h_{p+3} that in the right hand side of the equation (18) for $k = p + 3$ there is a member $\frac{\partial}{\partial x}\left[\left(\frac{\partial \omega_0^{(0)}}{\partial x}\right)^{p-2}\left(\frac{\partial \eta_0}{\partial x}\right)^2\right]$. At the expense of this number the formula for η_{p+3}, in addition to the members in (36) will contain one more complementary member of the form

$$\frac{\partial}{\partial x}\left(\frac{\partial \omega_0^{(0)}}{\partial x}\right)^{p-2}\Big[C_1 e^{2\lambda_1 \xi} + C_2 e^{2\lambda_2 \xi} + ... + C_m e^{2\lambda_m \xi} +$$

$$+ C_{m+1} e^{(\lambda_1+\lambda_2)\xi} + C_{m+2} e^{(\lambda_1+\lambda_3)\xi} + ... + C_{\frac{m(m+1)}{2}} e^{(\lambda_{m-1}+\lambda_m)\xi}\Big],$$

where C_i are constants. In a general form we can express it so that the formula determining the functions $\eta_j; j = p + 3, p + 4, ..., n + m - 1$, in addition to the members in (36) will contain the members of the form

$$P_j^{(0)}(\omega_0^{(0)}, \omega_1^{(0)}, ..., \omega_{j-1}^{(0)}) e^{(k_r \lambda_r + k_s \lambda_s)\xi}, \tag{48}$$

where $r, s = 1, 2, ..., m; k_r, k_s$ are natural numbers, $P_j^{(0)}(\omega_0^{(0)}, \omega_1^{(0)}, ..., \omega_{j-1}^{(0)})$ are the known functions dependent on $\omega_0^{(0)}, \omega_1^{(0)}, ..., \omega_{j-1}^{(0)}$ and their first and second derivatives, and this dependence is polynomial and uniform. Hence and from (14) it follows that the function $P_j^{(0)}$ is a polynomial with respect to ξ. Since the real parts of all the members $\lambda_1, \lambda_2, ..., \lambda_m$ are negative, the members of the form (48) exponentially decrease as $\xi \to +\infty$.

Thus, the equations (18) also have boundary layer solutions. Multiply all η_j by the smoothing functions and redenote the obtained new functions, again by $\eta_j; j = 0, 1, ..., n + m - 1$.

By lemma 1 the functions $W_i; j = 0, 1, ..., n$ together with all their derivatives vanish for $t = x$, and in particular for $t = x = 0$. Therefore, it follows from (14) that the functions $\omega_k^{(0)}(\xi, x); k = 1, 2, ..., n + m$ vanish for $x = 0$. So, we get from (29), (36), (37), (48) that all the functions $\eta_s; s = 0, 1, ..., n+m-1$ vanish for $x = 0$. Therefore from, (4), (8), (10) we get that the sum $W + \eta$ aside from (22), (19) satisfies the boundary condition

$$(W + \eta)|_{x=0} = 0. \tag{49}$$

The constructed sum $W + \eta$, generally speaking, doesn't satisfy homogeneous boundary conditions on S_2. In this connection, a boundary layer type function should be constructed near the boundary S_2. The boundary layer functions near the boundary S_2 are constructed as boundary layer functions near the boundary S_1. Therefore, for boundary layer functions near the boundary S_2 we note the followings.

At first we write a new decomposition of the operator $L_{\varepsilon,2}$ of the operator L_ε near the boundary S_2, for that we make change of variables $1 - t = \varepsilon y, x = x$. A boundary layer function ψ near the boundary S_2 is found in the form

$$\psi = \varepsilon^{m+1}(\psi_0 + \varepsilon \psi_1 + ... + \varepsilon^{n+m-1}\psi_{n+m-1}), \tag{50}$$

as a solution of the equation

$$L_{\varepsilon,2}(W+\eta+\psi)-L_{\varepsilon,2}(W+\eta)=0(\varepsilon^{n+2m+1}). \tag{51}$$

The equations for $\psi_0,\psi_1,...,\psi_{n+m-1}$ that are obtained from (51) by substitution into it a new expansion of $W+\eta$ in powers of ε in the coordinates (y,x) have the same forms with the equations obtained for $\eta_0,\eta_1,...,\eta_{n+m-1}$.

Boundary conditions for the equations whose solutions will be the functions $\psi_0,\psi_1,...,\psi_{n+m-1}$ are found from the requirement that the sum $W+\eta+\psi$ should satisfy the following boundary conditions:

$$\frac{\partial^{m+k}}{\partial t^{m+k}}(W+\eta+\psi)\mid_{t=1}=0;k=1,2,...,m. \tag{52}$$

We can represent the boundary conditions found form (52) by the following formula:

$$\frac{\partial^{m+k}\psi_s}{\partial y^{m+k}}\mid_{y=0}=(-1)^{m+k-1}\frac{\partial^{m+k}W_{s+1-k}}{\partial t^{m+k}}\mid_{t=1}; \quad k=1,2,...,m;s=0,1,...,n+m-1, \tag{53}$$

where the functions W_r for $r<0$ or $r>n$ should be considered identity zeros.

The following statement is proved similar to the proof of lemma 2.

Lemma 3. *The boundary layer type functions near the boundary S_2 are determined by the formula*

$$\psi_s=\sum_{i=1}^{m}\left[b_{s0}^{(i)}(x)+b_{s1}^{(i)}(x)y+...+b_{ss}^{(i)}(x)y^s\right]e^{\lambda_i y};s=0,1,..,p+2, \tag{54}$$

where the coefficients $b_{sj}^{(i)}(x)$ are expressed by the function

$$\frac{\partial^{m+1+k}W_r(1,x)}{\partial t^{m+1+k_1}\partial x^{k_2}};k=k_1+k_2;k_1=0,1,.., \quad m-1;r=0,1,..,n;k_1+k_2+r=s. \tag{55}$$

It should be noted that in the formula for $\psi_j;j=p+3,p+4,...,n+m-1$ in addition to the terms in (54) there will be additional terms of the form

$$P_j^{(1)}(\omega_0^{(1)},\omega_1^{(1)},...,\omega_{j-1}^{(1)})e^{(k_r\lambda_r+k_s\lambda_s)y}, \tag{56}$$

where $\omega_0^{(1)}=W_0(1,x)$ is independent of y, and the remaining functions $\omega_k^{(1)}$ are determined by the formula

$$\omega_k^{(1)}=\sum_{i+j=k}\frac{(-1)^i}{i!}\frac{\partial^i W_j(1,x)}{\partial t^i}y^i;k=1,2,..,n+2m. \tag{57}$$

Multiplying all the functions ψ_s by the smoothing function, for the obtained functions we leave previous denotation $\psi_s;s=0,1,..,n+m-1$..

Since the function ψ vanishes for $t=0$ at the expense of smoothing function, it follows from (19), (22) that the sum $W+\eta+\psi$ alongside with conditions (52) satisfies the following boundary conditions as well:

$$(W+\eta+\psi)\,|_{t=0}=\varepsilon^{n+1}\varphi_\varepsilon(x),$$
$$\frac{\partial^k}{\partial t^k}(W+\eta+\psi)\,|_{t=0}=0;k=1,2,..,m. \tag{58}$$

Following (49) and (50) we have that if the function ψ_j will vanish for $x=0$, i.e.

$$\psi_j\,|_{x=0}=0;j=0,1,...,n+m-1, \tag{59}$$

the sum $W+\eta+\psi$ aside from (52), (58) will satisfy the boundary condition

$$(W+\eta+\psi)\,|_{x=0}=0. \tag{60}$$

It follows from (54)-(57) that in order to fulfill the conditions (59) it suffices that the functions W_r satisfy the following conditions:

$$\frac{\partial^{m+1+k}W_r(1,0)}{\partial t^{m+1+k_1}\partial x^{k_2}}=0;k=k_1+k_2;r=0,1,..,n;$$
$$k_1+k_2+r=n+m-1. \tag{61}$$

Assume that the function $f(t,x)$ satisfies the following condition at the corner point $t=1,x=0$:

$$\frac{\partial^k f(1,0)}{\partial t^{k_1}\partial x^{k_2}}=0;k=k_1+k_2;k=0,1,..,n+2m-1. \tag{62}$$

Then, using the formula (27) we can show that the conditions (61) for the function W_0 will be fulfilled. Hence it follows that the right hand side of the equation for W_1 (of equation (7) for $j=1$) vanishes at the corner point $t=1,x=0$ together with its own derivatives. Therefore, the conditions (61) for the function W_1 are fulfilled. Continuing the process, we have that if (62) is valid, then conditions (61) are fulfilled for all W_r.

Thus, the constructed sum $W+\eta+\psi$ satisfies the boundary conditions (58), (52), (60). But this sum may not satisfy the second boundary condition from (1) for $x=1$. Therefore a boundary layer type function V should be constructed near S_3 so that the function V provide fulfilment of the boundary condition

$$(W+\eta+\psi+V)\,|_{x=1}=0. \tag{63}$$

We can somehow simplify the left hand side of (63). Considering the fact that the function $W_i;i=0,1,..,n$ together with its derivatives vanishes for $t=x=1$, it follows form (54)-(57) that

$$\psi_s\,|_{x=1}=0;s=0,1,...,n+m-1. \tag{64}$$

Further following (29), (36), (37), (48), (14) we can affirm that if the function $f(t,x)$ satisfies the following conditions at the corner point $t=0, x=1$:

$$\frac{\partial^k f(0,1)}{\partial t^{k_1}\partial x^{k_2}}=0; k=k_1+k_2; k=0,1,..,n+m, \tag{65}$$

the function η_s will vanish for $x=1$, i.e.

$$\eta_s\,|_{x=1}\,=0; s=0,1,..,n+m-1. \tag{66}$$

So, considering (66), (10) and (64), (50) we can represent the equality (63) in the form

$$(W+V)\,|_{x=1}\,=0. \tag{67}$$

While constructing the function V it should be taken into account that it must satisfy the equality

$$L_{\varepsilon,3}(W+\eta+\psi+V)-L_{\varepsilon,3}(W+\eta+\psi)=0(\varepsilon^{n+1}), \tag{68}$$

and also the function V while adding to the sum $W+\eta+\psi$ wouldn't violate provided boundary conditions (58), (52), (60). In (68) $L_{\varepsilon,3}$ denotes a new decomposition of the operator L_ε near S_3 that should be determined.

Local coordinates near the boundary S_3 are introduced in the following way: $t=t, 1-x=\varepsilon\tau$. A decomposition of the operator L_ε in the coordinates (t,τ) is of the form:

$$\begin{gathered}L_{\varepsilon,3}\equiv\varepsilon^{-1}\left\{-\left[\frac{\partial}{\partial\tau}\left(\frac{\partial U}{\partial\tau}\right)^p+\frac{\partial^2 U}{\partial\tau^2}+\frac{\partial U}{\partial\tau}\right]+\right.\\ \left.+\varepsilon\left[\frac{\partial U}{\partial t}+aU-f(t,x)\right]+(-1)^m\varepsilon^{2m+1}\frac{\partial^{2m+1}U}{\partial t^{2m+1}}\right\}.\end{gathered} \tag{69}$$

We look for a boundary layer function V in the form

$$V=V_0(t,\tau)+\varepsilon V_1(t,\tau)+...+\varepsilon^{n+1}V_{n+1}(t,\tau). \tag{70}$$

A new expansion of the sum $W+\eta+\psi$ in powers of ε in the coordinates (t,τ) is of the form

$$\widetilde{W}=W+\eta+\psi=\sum_{j=0}^{n+1}\varepsilon^j\Omega_j+0(\varepsilon^{n+2}), \tag{71}$$

where $\Omega_0=W_0(t,1)$ is independent of τ, $\Omega_k=\sigma_k+h_{k-1}^{(0)}$ for $k=1,2,..,m;$ $\Omega_l=$ $=\sigma_l+h_{l-1}^{(0)}+h_{l-m-1}^{(1)}$ for $l=m+1,m+2,..n+1$. The functions $\sigma_k, h_k^{(0)}, h_k^{(1)}$ are determined by the formula:

$$\sigma_k(t,\tau)=\sum_{i+j=k}\frac{(-1)^i}{i!}\frac{\partial^i W_j(t,1)}{\partial x^i}\tau^i; k=1,2,..,n+1;$$

$$h_k^{(0)}(\xi,\tau)=\sum_{i+j=k}\frac{(-1)^i}{i!}\frac{\partial^i \eta_j(\xi,1)}{\partial x^i}\tau^i;\left(\xi=\frac{t}{\varepsilon}\right); k=0,1,..,n,$$

$$h_k^{(1)}(y,\tau)=\sum_{i+j=k}\frac{(-1)^i}{i!}\frac{\partial^i\psi_j(y,1)}{\partial x^i}\tau^i;\left(y=\frac{1-t}{\varepsilon}\right);k=0,1,..,n-m.$$

Obviously, we can represent the equation (68) in the form

$$\varepsilon^{-1}\left\{-\frac{\partial}{\partial\tau}\left[\left(\frac{\partial\widetilde{W}}{\partial\tau}+\frac{\partial V}{\partial\tau}\right)^p-\left(\frac{\partial V}{\partial\tau}\right)^p\right]-\frac{\partial^2 V}{\partial\tau^2}-\frac{\partial V}{\partial\tau}+\right.$$
$$\left.+\left(\frac{\partial V}{\partial t}+aV\right)+(-1)^m\varepsilon^{2m+1}\frac{\partial^{2m+1}V}{\partial t^{2m+1}}\right.=0(\varepsilon^{n+1}). \tag{72}$$

Substituting the expressions for $V,\widetilde{W}$ from (70), (71) into (72) and expanding the nonlinear terms in the left hand side of (72) in powers of small parameter we get the following equations whose solutions are the functions $V_0,V_1,...,V_{n+1}$:

$$\frac{\partial}{\partial\tau}\left(\frac{\partial V_0}{\partial\tau}\right)^p+\frac{\partial^2 V_0}{\partial\tau^2}+\frac{\partial V_0}{\partial\tau}=0, \tag{73}$$

$$p\frac{\partial}{\partial\tau}\left[\left(\frac{\partial V_0}{\partial\tau}\right)^{p-1}\frac{\partial V_j}{\partial\tau}\right]+\frac{\partial^2 V_j}{\partial\tau^2}+\frac{\partial V_j}{\partial\tau}=\Phi_j;j=1,2,..,n+1, \tag{74}$$

where Φ_j are the known functions that polynomially depend on the first and second derivatives of the function $V_0,V_1,...,V_{j-1}$; $\Omega_1,\Omega_2,...,\Omega_{j-1}$, and this dependence is such that when the functions $V_0,V_1,...,V_{j-1}$ and their derivatives vanish, the functions Φ_j also vanish. We can give to the equation (74) the following form

$$\frac{\partial}{\partial\tau}\left\{\left[p\left(\frac{\partial V_0}{\partial\tau}\right)^{p-1}+1\right]\frac{\partial V_j}{\partial\tau}\right\}+\frac{\partial V_j}{\partial\tau}=\Phi_j;j=1,2,..,n+1\ . \tag{75}$$

From (67) and from the fact that $V_j;j=0,1,..,n+1$ should be boundary layer type functions we get the following conditions for the equations (73), (75):

$$V_j\mid_{\tau=0}=\varphi_j(t),\ \lim_{\tau\to+\infty}V_j=0, \tag{76}$$

where $\varphi_j(t)=-W_j(t,1)$ for $j=0,1,..,n+1$ and $\varphi_{n+1}\equiv 0$.

In the paper [8] the following statement is proved.

Lemma 4. *Let $\varphi_0(t)\in C^k[0,1]$. Then for each fixed $t\in[0,1]$ the problem (73), (76) (for $j=0$) has a unique solution and $V(t,\tau)$ with respect to τ is infinitely differentiable, and with respect to t has continuous derivatives up to k-th order, inclusively. Therefore the following estimations of the form*

$$\left|\frac{\partial^i V_0(t,\tau)}{\partial t^{i_1}\partial\tau^{i_2}}\right|\leq C\exp(-\tau);i=i_1+i_2;i_1=0,1,..,k$$

are valid uniformly with respect to $t\in[0,T]$.

The construction of remaining functions $V_1, V_2, ..., V_{n+1}$ as solutions of linear problems (75), (76) (for $j = 1, 2, .., n+1$) is based on the theorem whose proof is given in [8]. We notice only the formula for the functions $V_j; j = 1, 2, .., n+1$

$$V_j(t,\tau) = \left\{ \varphi_j(t) - \int\limits_0^\tau g^{-1}(t,z) \left[\int\limits_z^{+\infty} \Phi_j(t,\xi)d\xi \right] \times \right.$$
$$\left. \times \exp\left[\int\limits_0^\xi g^{-1}(t,\xi)d\xi \right] dz \right\} \exp\left[-\int\limits_z^\tau g^{-1}(t,\xi)d\xi \right], \tag{77}$$

where $g(t,\tau) = p\left(\frac{\partial V_0}{\partial \tau}\right)^{p-1} + 1$, and the estimation

$$\left| \frac{\partial^i V_j}{\partial t^{i_1} \partial \tau^{i_2}} \right| \leq C(a_0 + a_1\tau + .. + a_j\tau^j) \exp(-\tau);$$
$$i = i_1 + i_2; i = 0, 1, .., 2m + 2n + 2 - 2j.$$

is true.

Multiply all the functions $V_j; j = 1, 2, .., n+1$ by a smoothing multiplier and for the new obtained functions leave previous denotation.

So, we constructed the sum $\widetilde{U} = W + \eta + \psi + V$ that satisfies the condition (63). Since the function V vanishes for $x = 0$ at the expense of smoothing multiplier, then it follows from (60) that in addition to (63) this sum satisfies the following boundary condition as well

$$(W + \eta + \psi + V)\,|_{x=0} = 0. \tag{78}$$

Following (58), (52) we have that if the functions V_j will vanish together with their derivatives with respect t for $t = 0$ and $t = 1$

$$\frac{\partial^k V_j}{\partial t^k}\,|_{t=0} = 0; k = 0, 1, 2,..,m; \frac{\partial^{m+k} V_j}{\partial t^{m+k}}\,|_{t=1} = 0; k = 1, 2,..,m, \tag{79}$$

the function $\widetilde{U}$ will satisfy the following boundary conditions as well:

$$\widetilde{U}\,|_{t=0} = \varepsilon^{n+1}\varphi_\varepsilon(x), \frac{\partial^k \widetilde{U}}{\partial t^k}\,|_{t=0} = 0; \frac{\partial^{m+k} \widetilde{U}}{\partial t^{m+k}}\,|_{t=1} = 0; k = 0, 1, 2,..,m. \tag{80}$$

Using the obvious form of the function $V_0(t,\tau)$ we can see that in order to satisfy the conditions (79) for $j = 0$, it suffices the following conditions

$$\varphi_0^{(r)}(0) = 0; r = 1, 2,..,m; \varphi_0^{(s)}(1) = 0; s = 1, 2,..,2m. \tag{81}$$

be fulfilled.

Since $\varphi_0(t) = -W_0(0,1)$, the validity of the first condition from (81) follows from (27) and (62). The validity of the second condition from (81) is obtained from the fact that by lemma 1 the function $W_0(t,x)$ vanishes together with all its derivatives for $t = x$ and particularly, for $t = x = 1$.

Thus, the conditions (79) are fulfilled for the function $V_0(t,\tau)$. It follows from (77) that if the conditions

$$\varphi_j^{(k)}(0)=0,\ \frac{\partial^k\Phi_j(t,\tau)}{\partial t^k}\,|_{t=0}=0;k=0,1,..,m; \tag{82}$$

$$\varphi_j^{(m+k)}(1)=0,\ \frac{\partial^{m+k}\Phi_j(t,\tau)}{\partial t^{m+k}}\,|_{t=1}=0;k=1,2,..,m; \tag{83}$$

will be fulfilled, then (79) will be valid for the functions $V_j; j=1,2,..,n+1$, as well.

Validity of the conditions (82), (83) for the functions $\varphi_j(t)=-W_j(0,1); j=1,2,..,n$ are obtained by similar reasonings that were carried out above for $\varphi_0(t)$. Further, it was noted that the functions $\Phi_j; j=1,2,..,n+1$ depend on $V_0,V_1,..,V_{j-1}$ and their derivatives, so that when all these functions together with derivatives vanish, the functions Φ_j also vanish. For example, the function $\Phi_1(t,\tau)$ being the right hand side of the equation for V_1 has the form

$$\Phi_1=\frac{\partial V_0}{\partial t}+aV_0-p\frac{\partial}{\partial\tau}\left[\left(\frac{\partial V_0}{\partial\tau}\right)^{p-1}\frac{\partial\Omega_1}{\partial\tau}\right].$$

It follows from the fulfilment of the condition (79) for V_0 that the function Φ_1 satisfies the conditions (82), (83). Continuing similar reasonings and considering the property of the functions $\Phi_j(t,\tau)$ we get that the conditions (79) are fulfilled for all the functions $V_j; j=1,2,..,n+1$.

Thus, the constructed function $\widetilde{U}$ satisfies the boundary conditions (63), (78), (80). We introduce the denotation

$$U-\widetilde{U}=z \tag{84}$$

and call the function z a remainder term. From (84), (4), (10), (50), (70) we get the following asymptotic expansion in small parameter of the solution of the problem (1)-(3):

$$U=\sum_{i=0}^{n}\varepsilon^iW_i+\sum_{s=0}^{n+m-1}\varepsilon^{1+s}\eta_s+\sum_{s=0}^{n+m-1}\varepsilon^{1+m+s}\psi_s+\sum_{j=0}^{n+1}\varepsilon^jV_j+z. \tag{85}$$

Now, the remainder term should be estimated. The following lemme is valid.

Lemma 5. *For the remainder term z in (85) the estimation*

$$\varepsilon^{2m}\int_0^1\left(\frac{\partial^m z}{\partial t^m}\,|_{t=1}\right)^2dx+\varepsilon^p\iint_D\left(\frac{\partial z}{\partial x}\right)^{p+1}dtdx+ \\ +\varepsilon\iint_D\left(\frac{\partial z}{\partial x}\right)^2dtdx+C_1\iint_D z^2dtdx\le C_2\varepsilon^{2(n+1)}, \tag{86}$$

is valid, where $C_1>0, C_2>0$ are constants independent of ε.

Proof. Summing up (5), (11), (51), (68) we have that $\widetilde{U}$ satisfies the equation

$$L_\varepsilon\widetilde{U}=0(\varepsilon^{n+1}). \tag{87}$$

Subtracting (87) from (1) we get

$$(-1)^m \varepsilon^{2m} \frac{\partial^{2m+1} z}{\partial t^{2m+1}} - \varepsilon^p \frac{\partial}{\partial x}\left[\left(\frac{\partial U}{\partial x}\right)^p - \left(\frac{\partial \widetilde{U}}{\partial x}\right)^p\right] - \\ -\varepsilon \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x} + az = 0(\varepsilon^{n+1}). \tag{88}$$

It follows from (2), (3), (63), (78), (80), (84) that z satisfies the following boundary conditions:

$$z\,|_{t=0} = -\varepsilon^{n+1}\varphi_\varepsilon(x), \frac{\partial^k z}{\partial t^k}\,|_{t=0} = 0; \\ \frac{\partial^{m+k} z}{\partial t^{m+k}}\,|_{t=1} = 0; k = 1, 2, .., m, \tag{89}$$

$$z\,|_{x=0} = z\,|_{x=1} = 0, \tag{90}$$

and the function $\varphi_\varepsilon(x)$ is determined by the formula (23) and

$$\varphi_\varepsilon(0) = \varphi_\varepsilon(1) = 0. \tag{91}$$

When obtaining a uniform estimation for z, inhomogeneity of the first boundary condition in (89) creates some difficulty. In this connection we consider the auxiliary function

$$\psi_\varepsilon(t,x) = \varepsilon^{n+1}\left[t^{m+1}(1-t)^{2m+1}x(1-x) - \varphi_\varepsilon(x)\right], \tag{92}$$

that also satisfies the boundary conditions (89), (90).

Representing the remainder term z in the form

$$z = \psi_\varepsilon + z_1, \tag{93}$$

at first we get a uniform estimation for z_1, and then for z. Obviously the function z_1 will satisfy the homogeneous boundary conditions:

$$\frac{\partial^k z_1}{\partial t^k}\,|_{t=0} = 0; k = 0, 1, .., m; \frac{\partial^{m+k} z_1}{\partial t^{m+k}}\,|_{t=1} = 0; k = 1, 2, .., m, \tag{94}$$

$$z_1\,|_{x=0} = z_1\,|_{x=1} = 0. \tag{95}$$

Substituting the expression of z from (93) into (88), considering (92), after some transformations we get the equation

$$(-1)^m \varepsilon^{2m} \frac{\partial^{2m+k} z_1}{\partial t^{2m+1}} - \varepsilon^p \frac{\partial}{\partial x} \times$$
$$\times \left[\left(\frac{\partial z_1 + \widetilde{U} + \psi_\varepsilon}{\partial x} \right)^p - \left(\frac{\partial \left(\widetilde{U} + \psi_\varepsilon \right)}{\partial x} \right)^p \right] -$$
$$-\varepsilon \frac{\partial}{\partial x} \left[\left(\frac{\partial \left(\widetilde{U} + \psi_\varepsilon \right)}{\partial x} \right)^p - \left(\frac{\partial \widetilde{U}}{\partial x} \right)^p \right] - \tag{96}$$
$$-\varepsilon \frac{\partial^2 z_1}{\partial x^2} + \frac{\partial z_1}{\partial x} + \frac{\partial z_1}{\partial t} + a z_1 = 0(\varepsilon^{n+1}).$$

Multiplying the both hand sides of (96) by z_1 and integrating by parts the both parts of the obtained equality allowing for boundary conditions (94), (95), after certain transformations we get validity of estimations (86) for z_1. The validity of the estimations (86) for z follows from (92), (93) and from the estimation for z_1. The lemma 5 is proved.

Combining the results obtained above we arrive at the following statement.

Theorem. *Assume $f(t,x) \in C^{2m+2n+2}(D)$ and the conditions (25) (for $s = 2m+2n+2$), (62), (65) are fulfilled. Then for the solution of the problem (1)-(3) it is valid asymptotic representation (85), where the functions W_i are determined by the first iteration process, η_s, ψ_s, V_j are the boundary layer type functions near the boundaries S_1, S_2, S_3, that are determined by corresponding iteration processes, z is a remainder term and the estimation (86) is valid for it.*

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