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ON THE THEOREM ON CONTINUATION OF FUNCTIONS BEYOND THE DOMAIN OF DEFINITION

Abstract

In the paper, we construct a space of functions $f = f(x)$ with a finite set of difference-differential characteristics in the domain

$$G \subset E_n = E_{n_1} \times ... \times E_{n_s} \quad (n = n_1 + ... + n_s) \quad (1 \le s \le n),$$

satisfying the "σ - semi-horn" condition.

We find the conditions by fulfilling of which the theorems on continuation of functions (from the constructed spaces) beyond the domain of definition $G \subset E_n$, on the E_n, with preservation of appropriate smoothness properties are proved.

1. Some denotation and definition.

1.1. Given a fixed "positive" vector

$$r = (r_1; ...; r_s), \tag{1.1}$$

with coordinate-vectors

$$r_k = (r_{k,1}; ...; r_{k,n_k}) \quad (k = 1, 2, ..., s), \tag{1.2}$$

i.e. the coordinates

$$r_{r,j} > 0 \, (j = 1, 2, ..., n_k) \quad (k = 1, 2, ..., s). \tag{1.3}$$

Now, let

$$[r] = ([r_1]; ...; [r_s]) \tag{1.4}$$

be an "integral nonnegative" vector with coordinate vectors

$$[r_k] = ([r_{k,1}], ..., [r_{k,n_k}]) \;\; (k = 1, 2, -, s), \tag{1.5}$$

where $[r_{r,j}]$ is an entire part of appropriate coordinate $r_{k,j}$ from (1.1)-(1.3) for $j = 1, 2, .., n_k$ $(k = 1, 2, ..., s)$.

Obviously,

$$0 \le r_{kj} - [r_{kij}] < 1 \;\; (j = 1, 2.., n_k) \tag{1.6}$$

for all $k = 1, 2, ..., s$.

By means of the set Q of all possible vectors $i = (i_1, .., i_s)$ with coordinates $i_k \subset \{0, 1, .., n_k\}$ $(k = 1, 2, ..., s)$, to each "positive" vector from (1.1)-(1.3) we assign a collection of vectors

$$r^i = \left(r_1^{i_1}; ...; r_s^{i_s} \right) \;\; (i \, (i_1, ..., i_s) \in Q) \tag{1.7}$$

with coordinate vectors

$$r_k^{i_k} = \begin{cases} 0,..,0, r_{k,i_k}, 0,..,0 & for \quad i_k \neq 0, \\ (0,...,0) & for \quad i_k = 0 \end{cases} \tag{1.8}$$

for all $k = 1, 2, ..., s$.

We similarly define the collection of vectors

$$\left[r_k^{i_k}\right] = \begin{cases} (0,..,0, [r_{k,i_k}], 0,..,0) & for \quad i_k \neq 0, \\ (0,...,0) & for \quad i_k = 0 \end{cases} \tag{1.9}$$

for all $k = 1, 2, ..., s$.

1.2. Recall that

$$\|f\|_{L_p(G)} = \|f\|_{p,G} = \left(\int\limits_G |f(x)|^p \, dx \right)^{\frac{1}{p}} \tag{1.10}$$

for $1 \leq p < \infty$

$$\|f\|_{L_\infty(G)} = \|f\|_{\infty,G} = vrai \sup_{x \in G} |f(x)| \tag{1.11}$$

for $p = \infty$.

Let

$$\Delta_{k,j}^1 (t_{k,j}) f(..., x_{k,j,...}) = f(..., x_{k,j} + t_{k,j} ...) - f(..., x_{k,j,...}) \tag{1.12}$$

be a finite difference of the function $f = f(x)$ of first order in the direction of the variable $x_{k,j}$ with step $t_{k,j}$ for each $j = 1, 2, .., n_k$ $(k = 1, 2, ..., s)$.

We denote by (for each $i = (i_i, .., i_s) \in Q$)

$$e^i = \sup pi \tag{1.13}$$

a set of indices of non-zero coordinates of the vector $i = (i_1, .., i_s) \in Q$, consequently,

$$e^i \subset \{1, 2, ..., s\}. \tag{1.14}$$

Note that (for each $i \in Q$)

$$r_k^{i_k} - \left[r_k^{i_k}\right] = \begin{cases} (0,..,0, r_{k,i_k} - [r_{k,i_k}], 0,..,0) & for \quad i_k \neq 0, \\ (0,...,0) & for \quad i_k = 0 \end{cases} \tag{1.15}$$

$$(k = 1, 2, .., s).$$

Consequently, (for each $i \in Q$)

$$0 \leq r_{k,i_k} - [r_{k,i_k}] < 1 \left(k \in e^i\right). \tag{1.16}$$

Let

$$e_*^j = \left\{k \in e^i r_{k,i_k} - [r_{k,i_k}] > \infty\right\}. \tag{1.17}$$

This says that

$$r_{k,i_k} > 0 \text{ are non integer } \left(k \in e^i\right) \tag{1.18}$$

for each $i = (i_1, .., i_s) \in Q$.

Let

$$\omega = \left(\underbrace{1,...,1}_{n_1};...;\underbrace{1,...,1}_{n_s} \right) = (\omega_1;..;\omega_s)\,. \tag{1.19}$$

Denote by

$$\omega_*^i = \left(\omega_{*1}^{i_1};...;\omega_{*s}^{i_s}\right) (i = (i_1,..,i_s) \in Q) \tag{1.20}$$

with coordinate-vectors

$$\omega_{*k}^{i_k} = \begin{cases} (0,..,0,1,0,..,0) & for \ \ k \in e_*^i, \\ (0,...,0) & for \ \ k \in e_s \setminus e_*^i. \end{cases} \tag{1.21}$$

Introduce the denotation

$$\Delta^{\omega_*^i}(t)\, D^{[r^{i*}]} f(x) = \left\{ \prod_{k\in e_*^i} \Delta_{k,i_k}^1 (t_{k,i_k}) \right\} D^{[r^{i*}]} f(x)\,. \tag{1.22}$$

Let $G \subset E_n$, then we assume

$$\Delta^{\omega_*^i}(t;G)\, D^{[r^i]} f(x) = \Delta^{\omega_*^i}(t)\, D^{[r^i]} f(x)\,, \tag{1.23}$$

if multiple differences of the function $f = f(x)$ are constructed on vertices of a polygonal wholly lying on domain $G \subset E_n$, otherwise it is assumed

$$\Delta^{\omega_*^i}(t;G)\, D^{[r^i]} f(x) = 0. \tag{1.24}$$

Introduce a semi-norm by the equality

$$\|f\|_{L_p^{<r^i>}}(G;s) = \left\{ \int\limits_{E_{|\omega_*^i|}} \left\| \frac{\Delta^{\omega_*^i}(t;G)\, D^{[r^i]} f}{\prod\limits_{k\in e_*^i} |t_{k,i_k}|^{r_{k,i_k} - [r_{k,i_k}]}} \right\|_{L_p(G)}^p \frac{dt}{t} \right\}^{\frac{1}{p}} \tag{1.25}$$

For $1 \le p < \infty$, in the case $p = \infty$

$$\|f\|_{L_p^{<r^i>}(G;s)} = vrai \sup_{t\in E_{1\omega_{*1}^i}} \int\limits_{E_{|\omega_*^i|}} \left\| \frac{\Delta^{\omega_*^i}(t;G)\, D^{[r^i]} f}{\prod\limits_{k\in e_*^i} |t_{k,i_k}|^{r_{k,i_k} - [r_{k,i_k}]}} \right\|_{L_p(G)}, \tag{1.26}$$

where

$$\frac{dt}{t} = \prod_{k\in e_*^i} \frac{dt_{k,i_k}}{t_{k,i_k}}. \tag{1.27}$$

Definition. *The space*

$$W_p^{<r>}(G;s) = \bigcap_{(i=(i_1,..,i_s)\in Q)} L_p^{<r^i>}(G;s) \tag{1.28}$$

is the closure of the set of sufficiently smooth, finite in E_n function $f = f(x)$ by the norm

$$\|f\|_{W_p^{<r>}(G;s)} = \sum_{(i=(i_1,..,i_s)\in Q)} \|f\|_{L_p^{<r>}(G;s)} < \infty, \tag{1.29}$$

where the sum is taken over all possible vectors $i = (i_1, .., i_s) \in Q$ with coordinate vectors $i_k \in \{0, 1, ..., n_k\}$ $(k = 1, .., s)$ whose amount equals

$$|Q| = \prod_{k=1}^{s} (1 + n_k). \tag{1.30}$$

The constructed space (1.28) of differentiable functions of many variables depend on the parameter $s\,(1 \le s \le)$. In the case $s = 1$, these spaces is a generalization of the known $W_p^{(r_1...,r_n)}$ S.L. Sobolev - L.N.Slobodetskiy spaces, but in the case $s = n$ it is a generalization of the known spaces $S_p^r W(G)$ of differentiable functions with dominant mixed difference - differential characteristics investigated in the papers of S.M. Nikolskii, P.I. Lizorkin, A.D. Jabrailov and others.

1.3. Let $\delta_k = (\delta_1; ...; \delta_s)$ be a vector with coordinate - vectors

$$\delta_k = (\delta_{k,1}, ..., \delta_{k,n_k}) \quad (k = 1, 2, .., s),$$

more exactly

$$\delta_{k,j} = +1, \text{ if } \delta_{k,j} = -1\,(j = 1, 2, ..., n_k)\,(k = 1, 2, ..., s).$$

1.4. Let $\sigma = (\sigma_1; ...; \sigma_s)$ be "a positive vector" with coordinate-vector

$$\sigma_k = (\sigma_{k,1}; ...; \sigma_{k,n_k})\,(k = 1, 2, .., s),$$

more exactly $\sigma_{k,j} > 0\,(j = 1, 2, ..., n_k)$.

Denote by

$$R_\sigma(\sigma; h) = \bigcup_{0<v\le h} \left\{ \begin{array}{l} y \in E_n; C^*_{k,j} \le \frac{y_{k,j}\delta_{k,j}}{v^{\sigma_{k,j}}} \le C^{**}_{k,j} \\ j = 1, 2, ..., n_k \quad (k = 1, 2, .., s) \end{array} \right\} \tag{1.31}$$

the points set $y \in E_n$ whose coordinates are subjected to restrictions

$$C^*_{k,j} \le \frac{y_{k,j}\delta_{k,j}}{v^{\sigma_{k,j}}} \le C^{**}_{k,j} j = 1, 2, ..., n_k\,(0 < v \le h)\,(k = 1, 2, .., s)$$

for all $k = 1, 2, .., s$, where $C^*_{k,j} C^{**}_{k,j}\,(j = 1, 2, ..., n_k)$ are the fixed constants.

The points set (1.31) determines some "semi-horn" with a vertex at the origin of coordinates.

Then for $x \in E_n$, the set

$$x + R_\delta(\sigma; h) \tag{1.32}$$

is said to be a "σ - semi-horn" with a vertex at the point $x \in E_n$. For $s = 1$, this "σ - semi-horn" is a "σ - semi-horn" with a vertex at the point $x \in E_n$, determined first in the papers of O.V. Besov. In the case $s = n$, this semi-horn (1.32) is an n - dimensional rectangle with a vertex at the point $x \in E_n$, with faces parallel to coordinate axes, given in V.P. Il'in papers.

For fixed $h > 0$ and $\sigma = (\sigma_1; ...; \sigma_s)$, the amount of "$\sigma$ - semi-horns" with a vertex at the point $x = E_n$ equals 2^n. Therewith if we fix the vector $\delta = (\delta_1; ...; \delta_s)$ as well, then there exsts a unique "σ - semi-horn" with a vertex at the point $x \in E_n$.

Definition. *The subdomain $\Omega \in G$ is said to be a subdomain satisfying the "σ - semi-horn" condition, if there exists a "σ - semi-horn" with a vertex at the point $x \in E_n$ for which*

$$x + R_\sigma(\sigma; h) \subset G \tag{1.33}$$

for all $x \in \Omega$.

Definition. *The domain $G \in E_n$ is said to be a domain satisfying the "σ - semi-horn" condition, of there exists a finite collection of subdomains*

$$\Omega_1, \Omega_2, ..., \Omega_N \subset G, \tag{1.34}$$

satisfying the "σ - semi-horn" condition and covering the domain G, i.e. such that

$$\bigcup_{\mu=1}^{N} \Omega_\mu = G. \tag{1.35}$$

2. Basic results.

Theorem. *1) Let*

$$f \in W_p^{<r>}(\sigma; s), \tag{2.1}$$

where $1 < p < \infty$, the vector $r = (r_1; ...; r_s)$ with coordinates $r_k = (r_{k,1}, ..., r_{k,n_k})$ $(k < 1, 2, ..., s)$ is "positive", i.e.

$$r_{k,j} > 0\,(j = 1, 2, ..., n_k)\,(k = 1, 2, .., s)$$

2) Let a domain $G \in E_n$ satisfy the "σ - semi-horn" condition, therewith $\sigma = (\sigma_1; ...; \sigma_s)$ with coordinate vectors $\sigma_k = (\sigma_{k,1}, ..., \sigma_{k,n_k})\,(k = 1, ..., s)$, and assumed that $\sigma_{k,j} > 0\,(j = 1, 2, .., n_k)$ for all $k = 1, 2, ..., s$.

3) Let an integer non-negative vector $\nu = (\nu_1; ...; \nu_s)$ with coordinate-vectors $\nu = \left(\nu_{k,1}, ..., \nu_{k_{n_k}}\right)$ $(k = 1, 2, ..., s)$ be such that

$$\chi_{k,i_k} = r_{k,i_k}\sigma_{k,i_k} - (\nu_k, \sigma_k) > 0 \tag{2.2}$$

$$(k = 1, 2, ..., i_s)$$

for each $i = (i_1; ...; i_s) \in Q$, where

$$(\nu_k, \sigma_k) = \sum_{j=1}^{n} \nu_{k,j}\sigma_{k,j}\,(k = 1, 2, ..., i_s).$$

Then there exists an generalized derivative

$$D^v f \in L_p(G) \tag{2.3}$$

and one can construct the function $\widetilde{f_v} = f_v(x)$ determined on the whole of E_n such that

$$\left.\widetilde{f_v}\right|_G = D^v f(x), \tag{2.4}$$

and it holds the integral inequality

$$\left\| \widetilde{f}_v \right\|_{L_p(E_n)} (E_n) \leq C \left\| f \right\|_{W_p^{<r>}(G;s)}, \tag{2.5}$$

where C is a constant independent of the function $f = f(x)$.

Proof. Assume that the function $f = f(x)$ belonging to the space (2.1) is sufficiently smooth, and it holds the integral representation

$$D^v f(x) = \sum_{(i=(i_1,..,i_s)\in Q)} B_{i,\delta} f(x), \tag{2.6}$$

where integral operators $B_{i,\delta} f(x)$ are determined by the equalities

$$B_{i,\delta} f(x) = C_i \left(\prod_{k\in e_s - e_*^i} h_k^{-\alpha_{k,0}} \right) \times$$

$$\times \int_{\overrightarrow{0}} \prod_{k\in e_*^i} \frac{dv_k}{\vartheta_k^{1+\alpha_{k,i_k}}} \int_{E|\omega_*^i|} dz \int_{oE_n} \left\{ \Delta^{\omega_*^i}(z) D^{[r^i]} f(x+y) \right\} \underline{\Phi}_{i,\delta}(...) \, dy. \tag{2.7}$$

Here

$$\alpha_{k,0} = |\sigma_k| + (v_k, \sigma_k) \text{ for } i_k = 0,$$

$$\alpha_{k,i_k} = |\sigma_k| + (v_k, \sigma_k) - [r_{k,i_k}] \sigma_{k,i_k} + \sigma_{k,i_k} \qquad \text{for } i_k \neq 0 \tag{2.8}$$

where $|\sigma_k| = \sigma_{k,1} + ... + \sigma_{k,n_k}$ $(k - 1, 2, ..., s)$ and the main denotation from (1.1)-(1.20) are preserved. In integral operators (2.7), the kernels

$$\underline{\Phi}_{i,\delta} = \underline{\Phi}_{i,\delta}(...)$$

are sufficiently smooth and finite (see[1]).

The "σ - semi-horn" $x + R_\delta(\sigma; h)$ with a vertex at the point $x \in G$ is a support of integral representation (2.6) of the function $f = f(x)$ in domain $G \in E_n$.

Now, let

$$\Omega_1, \Omega_2, ..., \Omega_N \subset G \tag{2.9}$$

be subdomains satisfying the "σ - semi-horn" condition and covering the domain G, i.e. such that

$$\bigcup_{\mu=1}^{N} \Omega_\mu = G, \tag{2.10}$$

therewith

$$\delta^\mu = (\delta_1^\mu, ..., \delta_s^\mu) \, (\mu = 1, 2, ..., N)$$

is a collection of vectors with corresponding coordinate vectors

$$\delta_k^\mu = \left(\delta_{k,1}^\mu, ..., \delta_{k,n_k}^\mu \right) (k = 1, 2, ..., s),$$

for which

$$\delta_{k,j}^\mu = +1 \text{ or } \delta_{k,j}^\mu = -1 \, (j = 1, 2, ..., n_k),$$

moreover,

$$\Omega_\mu + R_{\delta^\mu}(\sigma; h) \subset G \quad (\mu = 1, 2, ..., N). \tag{2.11}$$

A collection of auxiliary functions

$$\widetilde{f}_{\nu,\mu} = \widetilde{f}_{\nu,\mu}(x) \quad (\mu = 1, 2, ..., N), \tag{2.12}$$

coinciding on appropriate

$$\Omega_\mu + R_{\delta^\mu}(\sigma; h)$$

with the functions $D^v f(x)$ are determined by the equalities

$$\widetilde{f}_{\nu\mu}(x) = \sum_{i=(i_1,..,i_s)\in Q} B^*_{i,\delta^\mu} f(x) \tag{2.13}$$

for all $\mu = 1, 2, ..., N$, the integral operators standing at the right side of (2.13) are given in the form:

$$B^*_{i,\delta^\mu} f(x) = C_i \left(\prod_{k\in e_s - e^i_*} h_k^{-\alpha_{k,0}} \right) \times$$

$$\times \int_{\vec{0}}^{\vec{h}} \prod_{k\in e^i_*} \frac{d\vartheta_k}{\vartheta_k^{1+\alpha_{k,i_k}}} \int_{E|\omega^i_*|} dz \int_{oE_n} \left\{ \Delta^{\omega^i_*}(z; \Omega_\mu + R_{\delta^\mu}) D^{[r^i]} f(x+y) \right\} \underline{\Phi}_{i,\delta}(...)\, dy \tag{2.14}$$

for all $\mu = 1, 2, ..., N$, moreover (in the case of zero vector $i \in Q$) instead of the function

$$D^{[r^i]} f(x+y) = f(x+y)$$

the function

$$\chi(\Omega_\mu + R_{\delta^\mu}) f(x+y) \tag{2.15}$$

is under the integral for each $\mu = 1, 2, ..., N$.

We construct the function $\widetilde{f}_v = \widetilde{f}_v(x)$ by the equality

$$\widetilde{f}_v(x) \sum_{\mu=1}^{N} \eta_\mu(x) \widetilde{f}_{v\mu}(x). \tag{2.16}$$

In (2.16), the collection of functions

$$\eta_\mu = \eta_\mu(x) \quad \mu = 1, 2, ..., N \tag{2.17}$$

determine expansion of a unit (see[2]) in domain $G \in E$ by the covering

$$\{\Omega_\mu\}_{\mu=1,2,...,N}.$$

It follows from (2.16) that

$$\left\| \widetilde{f}_\nu \right\|_{L_p(E_n)} \leq C \sum_{\mu-1}^{N} \xi \left\| \widetilde{f}_{\nu,\mu} \right\|_{L_p(E_n)}. \tag{2.18}$$

We can see that

$$\left\| \widetilde{f}_{v,\mu} \right\|_{L_p(E_n)} \leq C \sum_{(i=(i_1,..,i_s)\in Q)} \prod_{k=1}^{s} h_k^{\chi_{k,i_k}} \|f\|_{L_p^{<r>}(\Omega_\mu+\Omega_\mu;s)} .$$

The basic inequality

$$\left\| \widetilde{f}_{\nu\mu} \right\|_{L_p(E_n)} \leq C \sum_{\mu-1}^{N} \sum_{(i=(i_1,..,i_s)\in Q)} \left(\prod_{k=1}^{s} h_k^{\chi_{k,i_k}} \right) \|f\|_{L_p^{<r^i>}(\Omega_\mu+R_{\delta\mu};s)} \leq \tag{2.20}$$

$$\leq C \sum_{i=(i_1,..,i_s)\in Q} \left(\prod_{k=1}^{s} h_k^{\chi_{k,i_k}} \right) \|f\|_{L_p^{<r^i>}(G;s)} ,$$

where χ_{k,i_k} (for $i_k \neq 0$) are defined by equalities (22) and (for $i_k = 0$)

$$\chi_{k,0} = -(v_k, \sigma_k) \quad (k = 1, 2, ..., s) \tag{2.21}$$

for each $i = (i_1, .., i_s) \in Q$ follows from the two inequalities (2.18) and (2.19).

This proves inequalities (2.5) of the theorem.

References

[1]. F.G. Maksudov., A.J. Jabrailov. *Methods of integral representations in space theory.* Baku, "Elm" 2000. (Russian).

[2]. O.V. Besov., B.L. Il'in., S.M. Nikolskiy. *Integral representations of functions and imbedding theorem.* Moscow, "Nauka", 1975. (Russian).

[3]. A.J. Jabrailov., S.M.Najafova. *Theorems on continuation of functions beyond the domain.* Baku, Proc. IMM AN Azerb. XI, pp. 454-5908. (Russian).

[4]. Nikolskiy - *Approximation of functions of many variables and imbedding theorems.* Moscow, "Nauka" 1972. (Russian).

[5]. S.L. Sobolev. *Some applications of functional analysis in mathematical physics.* L. LTU, 1950. (Russian).

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