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ON A PERIODIC BOUNDARY VALUE PROBLEM FOR AN EQUATION OF STRATIFIED LIQUID VIBRATIONS

Abstract

In the paper we investigate a periodic boundary value problem for an equation of stratified liquid vibrations.At first we prove a theorem on uniqueness of classic solution. Further, by the method of separation of variables we construct classic solution in an obvious form.

While investigating the problems of small vibrations of exponentially stratified liquid there arises an equation

$$\frac{\partial^2}{\partial t^2}\Delta_3 u+\omega_0^2\Delta_2 u=0,$$

that is similar in many respects to S.L. Sobolev's known equation [1]. Here Δ_3 is a Laplacatian with respect to variables x_1, x_2, x_3, Δ_2 is a Laplacian with respect to variables x_1, x_2 and ω_0 is a real number parameter. In the present paper we consider an one-dimensional variant of this equation.

Let's consider the following boundary value problem:

$$u_{ttxx}(x,t)+\alpha^2 u_{xx}(x,t)-\beta^2 u(x,t)=f(x,t),(x,t)\in D_T=$$

$$=\{(x,t):0\le x\le 1,\ 0\le t\le T\} \tag{1}$$

$$u(0,t)=u(1,t),\quad u_x(0,t)=u_x(1,t),\quad 0\le t\le T, \tag{2}$$

$$u(x,0)+\delta u(x,T)=\varphi(x),\quad u_t(x,0)+\delta u_t(x,T)=\psi(x),\quad 0\le x\le 1, \tag{3}$$

where $\alpha\neq 0,\quad \beta\neq 0,\quad \delta$ are the given numbers, $f(x,t),\ \varphi(x),\psi(x)$ are the given functions, $u(x,t)$ is a desired function.

Definition. *Under the classic solution of problem (1)-(3) we understand a function $u(x,t)$ continuous in closed domain D_T together with all its derivatives contained in equation (1) and satisfying all the conditions of (1)-(3) in the ordinary sense.*

Theorem 1. *If $\beta\neq 0,\quad \delta\neq\pm 1$, the problem (1)-(3) may have at most one classic solution.*

Proof. The proof of this theorem is carried out by the following scheme [2]. Assume that there exist two classic solutions $u_1(x,t)$ and $u_2(x,t)$ of the problem under consideration and consider the difference

$$\nu(x,t)=u_1(x,t)-u_2(x,t).$$

Obviously, the function $v(x,t)$ satisfies the homogeneous equation

$$\nu_{ttxx}(x,t)+\alpha^2\nu_{xx}(x,t)-\beta^2\nu(x,t)=0, \quad (0\le x\le 1,\ \ 0\le t\le T) \tag{4}$$

and the conditions

$$\nu(0,t)=\nu(1,t),\ \nu_x(0,t)=\nu_x(1,t), \quad 0\le t\le T, \tag{5}$$

$$\nu(x,0)+\delta\nu(x,T)=0, \quad \nu_t(x,0)+\delta\nu_t(x,T)=0, \quad 0\le x\le 1. \tag{6}$$

Prove that the function $\nu(x,t)$ identically equates zero.

Multiply the both hand sides of equation (4) by the function $2\nu_t(x,t)$ and integrate the obtained equality with respect to x from 0 to 1:

$$2\int_0^1 \nu_{ttxx}(x,t)\,\nu_t(x,t)\,dx+2\alpha^2\int_0^1 \nu_{xx}(x,t)\,\nu_t(x,t)\,dx-$$

$$-2\beta^2\int_0^1 \nu(x,t)\,\nu_t(x,t)\,dx=0. \tag{7}$$

Obviously,

$$2\int_0^1 \nu_{ttxx}(x,t)\,\nu_t(x,t)\,dx=2\left(\nu_{ttx}(1,t)\,\nu_t(1,t)-\nu_{ttx}(0,t)\,\nu_t(0,t)\right)-$$

$$-2\int_0^1 \nu_{ttx}(x,t)\,\nu_{tx}(x,t)\,dx=-\frac{d}{dt}\int_0^1 \nu_{tx}^2(x,t)\,dx;$$

$$2\int_0^1 \nu_{xx}(x,t)\,\nu_t(x,t)\,dx=2\left(\nu_x(1,t)\,\nu_t(1,t)-\nu_x(0,t)\,\nu_t(0,t)\right)-$$

$$-2\int_0^1 \nu_x(x,t)\,\nu_{tx}(x,t)\,dx=-\frac{d}{dt}\int_0^1 \nu_x^2(x,t)\,dx;$$

$$2\int_0^1 \nu(x,t)\,\nu_t(x,t)\,dx=\frac{d}{dt}\int_0^1 \nu^2(x,t)\,dx.$$

Then from (7) $\forall t\in[0,T]$ we have:

$$\frac{d}{dt}\int_0^1 \nu_{tx}^2(x,t)\,dx+\alpha^2\frac{d}{dt}\int_0^1 \nu_x^2(x,t)\,dx+\beta^2\frac{d}{dt}\int_0^1 \nu^2(x,t)\,dx=0,$$

or

$$y(t)=\int_0^1 \nu_{tx}^2(x,t)\,dx+\alpha^2\int_0^1 \nu_x^2(x,t)\,dx+\beta^2\frac{d}{dt}\int_0^1 \nu^2(x,t)\,dx=C,$$

where $C = const$.

Hence, allowing for (6) we get:

$$y(0) - \delta^2 y(T) = \int_0^1 \left(\nu_{tx}^2(x,0) - \delta^2 \nu_{tx}^2(x,T)\right) dx +$$

$$+\alpha^2 \int_0^1 \left(\nu_x^2(x,0) - \delta^2 \nu_x^2(x,T)\right) dx + \beta^2 \int_0^1 \left(\nu^2(x,0) - \delta^2 \nu^2(x,T)\right) dx =$$

$$= \int_0^1 (\nu_{tx}(x,0) - \delta\nu_{tx}(x,T))(\nu_{tx}(x,0) + \delta\nu_{tx}(x,T))\, dx +$$

$$+\alpha^2 \int_0^1 (\nu_x(x,0) - \delta\nu_x(x,T))(\nu_x(x,0) + \delta\nu_x(x,T))\, dx +$$

$$+\beta^2 \int_0^1 (\nu(x,0) - \delta\nu(x,T))(\nu(x,0) + \delta\nu(x,T))\, dx = 0.$$

Thus:

$$y(0) - \delta^2 y(T) = C\left(1 - \delta^2\right) = 0.$$

Since $\delta \neq \pm 1$, then $C = 0$. Consequently for any $t \in [0,T]$ we have:

$$\int_0^1 \nu_{tx}^2(x,t)\, dx + \alpha^2 \int_0^1 \nu_x^2(x,t)\, dx + \beta^2 \int_0^1 \nu^2(x,t)\, dx = 0,$$

wherefrom we conclude that,

$$\nu_{tx}(x,t) \equiv 0, \quad \nu_x(x,t) \equiv 0, \quad \nu(x,t) \equiv 0.$$

Thereby we prove that $\nu(x,t) \equiv 0$ for any $(x,t) \in D_T$.

Thus, if there exist two classic solutions $u_1(x,t)$ and $u_2(x,t)$ of problem (1)-(3), then $u_1(x,t) \equiv u_2(x,t)$. Hence, it follows that if a classic solution of problem (1)-(3) exists, it is unique. The theorem is proved. Obviously, fulfilment of agreement conditons

$$\varphi(0) = \varphi(1), \quad \varphi'(0) = \varphi'(1), \quad \psi(0) = \psi(1), \quad \psi'(0) = \psi'(1)$$

is a necessary condition of the existence of classic solution.

Now, let's consider a spectral problem:

$$X''(x) + \lambda^2 X(x) = 0, 0 \leq x \leq 1, \tag{8}$$

$$X(0) = X(1), X'(0) = X'(1). \tag{9}$$

It is known [3] that the eigen numbers of problem (8),(9) consist of the numbers $\lambda_k = 2\pi k \quad (k = 0, 1, 2, ...)$, moreover for $k \geq 1$ to each eigen value λ_k there correspond two linear independent eigen functions $\cos\lambda_k x$, $\sin\lambda_k x$; besides the system

$$1, \quad \cos\lambda_1 x, \quad \sin\lambda_1 x, ..., \ \cos\lambda_k x, \ \sin\lambda_k x...$$

forms in $L_2(0,1)$ an orthogonal basis.

We'll look for the classic solution of problem (1)-(3) in the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_{1,k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} u_{2,k}(t)\sin\lambda_k x, \tag{10}$$

where

$$u_{1,k}(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx \quad (k = 0, 1, 2, ...),$$

$$u_{2,k}(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx \quad (k = 1, 2, ...).$$

Applying the Fourier method, from (1)-(3) we get:

$$\beta^2 u_{1,0}(t) = -f_{1,0}(t),$$

$$\lambda_k^2 u''_{i,k}(t) + \left(\alpha^2\lambda_k^2 + \beta^2\right) u_{i,k}(t) = -f_{i,k}(t), \ (i = 1, 2; \ k = 1, 2, ...), \ 0 \leq t \leq T, \tag{11}$$

$$u_{i,k}(0) + \delta u_{i,k}(T) = \varphi_{i,k}, \ u'_{i,k}(0) + \delta u'_{i,k}(T) = \psi_{i,k} \ \ (i = 1, 2; k = 1, 2, ...), \tag{12}$$

where

$$f_{1,k}(t) = 2\int_0^1 f(x,t)\cos\lambda_k x \ dx \ \ (k = 0, 1, 2, ...),$$

$$f_{2,k}(t) = 2\int_0^1 f(x,t)\sin\lambda_k x \ dx \qquad (k = 1, 2, ...),$$

$$\varphi_{1,k} = 2\int_0^1 \varphi(x)\cos\lambda_k x \ dx \ \ (k = 0, 1, 2, ...),$$

$$\varphi_{2,k} = 2\int_0^1 \varphi(x)\sin\lambda_k x \ dx \qquad (k = 1, 2, ...),$$

$$\psi_{1,k} = 2\int_0^1 \psi(x)\cos\lambda_k x \ dx \ \ (k = 0, 1, 2, ...),$$

$$\psi_{2,k} = 2\int_0^1 \psi(x)\sin\lambda_k x \ dx \ \ (k = 1, 2, ...).$$

Since the roots of the characteristic equation

$$\mu_{i,k}^2 + \alpha_k^2 = 0,$$

corresponding to (9) is determined by the formula

$$\mu_{i,j,k} = (-1)^j \alpha_k \sqrt{-1}, \quad i = 1,2; \quad j = 1,2; \quad k = 1,2,...,$$

where

$$\alpha_k^2 = \frac{\alpha^2 \lambda_k^2 + \beta^2}{\lambda_k^2}$$

then a general solution of (11) by the Lagrange method is of the form:

$$u_{i,k}(t) = C_{i1,k}(t) \cos \alpha_k t + C_{i2,k}(t) \sin \alpha_k t, \quad (i = 1,2; \quad k = 1,2,...), \tag{13}$$

where $C_{i1,k}(t)$ and $C_{i2,k}(t)$ are the unknown functions.

Further, for definition of functions $C_{i1,k}(t)$ and $C_{i2,k}(t)$ we have the system:

$$\begin{cases} C'_{i1,k}(t) \cos \alpha_k t + C'_{i2,k}(t) \sin \alpha_k t = 0, \\ C'_{i1,k}(t) \sin \alpha_k t - C'_{i2,k}(t) \cos \alpha_k t = \dfrac{1}{\alpha_k \lambda_k^2} f_{i,k}(t) \qquad (i = 1,2). \end{cases}$$

Hence we find:

$$C'_{i1,k}(t) = \frac{1}{\alpha_k \lambda_k^2} f_{i,k}(t) \sin \alpha_k t,$$

$$C'_{i2,k}(t) = -\frac{1}{\alpha_k \lambda_k^2} f_{i,k}(t) \cos \alpha_k t, \ i = 1,2.$$

Integrating from 0 to t we have:

$$C_{i1,k}(t) = \frac{1}{\alpha_k \lambda_k^2} \int_0^t f_{i,k}(\tau) \sin \alpha_k \tau d\tau + C_{i1,k},$$

$$C_{i2,k}(t) = -\frac{1}{\alpha_k \lambda_k^2} \int_0^t f_{i,k}(\tau) \cos \alpha_k \tau d\tau + C_{i2,k}, \ i = 1,2, \tag{14}$$

where $C_{i1,k}$ and $C_{i2,k}$ are arbitrary constants.

Substituting (14) into (13) we get:

$$u_{i,k}(t) = C_{i1,k} \cos \alpha_k t + C_{i2,k} \sin \alpha_k t +$$

$$+ \frac{1}{\alpha_k \lambda_k^2} \int_0^t f_{i,k}(\tau) \sin \alpha_k (t - \tau) d\tau, \ i = 1,2. \tag{15}$$

From (13) we find:

$$u'_{i,k}(t) = -\alpha_k C_{i1k}(t) \sin \alpha_k t + \alpha_k C_{i2,k}(t) \cos \alpha_k t +$$

$$+\frac{1}{\lambda_k^2}\int_0^t f_{ik}(\tau)\cos\alpha_k(t-\tau)\,d\tau,\quad i=1,2. \tag{16}$$

Now, using conditions (12) we determine $C_{i1,k}$ and $C_{i2,k}$ $(i=1,2)$:

$$\begin{cases} (1+\delta\cos\alpha_k T)\,C_{i1,k}+C_{i2,k}\delta\sin\alpha_k T=q_{i1,k}(T), \\ -C_{i1,k}\alpha_k\delta\sin\alpha_k T+\alpha_k(1+\delta\cos\alpha_k T)\,C_{i2,k}=q_{i2,k}(T)\quad (i=1,2), \end{cases} \tag{17}$$

where

$$q_{i1,k}(T)=\varphi_{i,k}-\frac{\delta}{\alpha_k\lambda_k^2}\int_0^T f_{i,k}(\tau)\sin\alpha_k(T-\tau)\,d\tau,$$

$$q_{i1,k}(T)=\psi_k-\frac{\delta}{\lambda_k^2}\int_0^T f_{i,k}(\tau)\cos\alpha_k(T-\tau)\,d\tau\qquad (i=1,2). \tag{18}$$

Obviously,

$$\Delta_k(T)\equiv\begin{vmatrix} 1+\delta\cos\alpha_k T & \delta\sin\alpha_k T \\ -\alpha_k\delta\sin\alpha_k T & \alpha_k(1+\delta\cos\alpha_k T) \end{vmatrix}=\left(1+2\delta\cos\alpha_k T+\delta^2\right)\alpha_k,$$

$$\Delta_{i1,k}(T)\equiv\begin{vmatrix} q_{i1,k}(T) & \delta\sin\alpha_k T \\ q_{i2,k}(T) & \alpha_k(1+\delta\cos\alpha_k T) \end{vmatrix}=$$

$$=\alpha_k(1+\delta\cos\alpha_k T)\,q_{i1,k}(T)-\delta q_{i2,k}(T)\sin\alpha_k T,$$

$$\Delta_{i2,k}(T)\equiv\begin{vmatrix} 1+\delta\cos\alpha_k T & q_{i1,k}(T) \\ -\alpha_k\delta\sin\alpha T & q_{i2,k}(T) \end{vmatrix}=$$

$$=(1+\delta\cos\alpha_k T)\,q_{i2,k}(T)+\alpha_k q_{i1,k}(T)\sin\alpha_k T\quad (i=1,2).$$

Hence we have:

$$\begin{cases} C_{i1,k}=\dfrac{1}{\alpha_k\rho_k(T)}\left[\alpha_k(1+\delta\cos\alpha_k T)\,q_{i1,k}(T)-\delta q_{i2,k}(T)\sin\alpha_k T\right], \\ C_{i2,k}=\dfrac{1}{\alpha_k\rho_k(T)}\left[(1+\delta\cos\alpha_k T)\,q_{i2,k}(T)+\alpha_k\delta q_{i1,k}(T)\sin\alpha_k T\right],\ i=1,2 \end{cases} \tag{19}$$

where $\rho_k(T)=1+2\delta\cos\alpha_k T+\delta^2$.

Substituting the values of (19) into (15) we get:

$$u_{i,k}(t)=\frac{1}{\alpha_k\rho_k(T)}\left\{\left[\alpha_k(1+\delta\cos\alpha_k T)\,q_{i1,k}(T)-\delta q_{i2,k}(T)\sin\alpha_k T\right]\cos\alpha_k t+\right.$$

$$\left.+\left[(1+\delta\cos\alpha_k T)\,q_{i2,k}(T)+\alpha_k\delta q_{i1,k}(T)\sin\alpha_k T\right]\sin\alpha_k t\right\}+$$

$$+\frac{1}{\alpha_k\lambda_k^2}\int_0^t f_{i,k}(\tau)\sin\alpha_k(t-\tau)\,d\tau,\ \ (\ i=1,2),$$

or

$$u_{i,k}(t)=\frac{1}{\alpha_k\rho_k(T)}\left\{\alpha_k\left[\cos\alpha_k t+\delta\cos\alpha_k(T-t)\right]q_{i1,k}(T)+\right.$$

$$\left.+\left[\sin\alpha_k t-\delta\sin\alpha_k(T-t)\right]q_{i2,k}(T)\right\}+\frac{1}{\alpha_k\lambda_k^2}\times$$

$$\times\int_0^t f_k(\tau)\sin\alpha_k(t-\tau)\,d\tau\quad(\ i=1,2).$$

Hence, allowing for (16) we find:

$$u_{i,k}(t)=\frac{1}{\alpha_k\rho_k(T)}\left\{\alpha_k\left(\cos\alpha_k t+\delta\cos\alpha_k(T-t)\right)\varphi_{i,k}-\right.$$

$$+\left(\sin\alpha_k t-\delta\sin\alpha_k(T-t)\right)\psi_{i,k}-$$

$$\left.-\frac{\delta}{\lambda_k^2}\int_0^T f_{i,k}(\tau)\left(\sin\alpha_k(T+t-\tau)+\delta\sin\alpha_k(t-\tau)\right)d\tau\right\}+$$

$$+\frac{1}{\alpha_k\lambda_k^2}\int_0^t f_{i,k}(\tau)\sin\alpha_k(t-\tau)\,d\tau\quad(i=1,2).$$

Thus, solving problem (11),(12) we find:

$$u_{i,k}(t)=\frac{1}{\alpha_k\rho_k(T)}\left\{\alpha_k\left(\cos\alpha_k t+\delta\cos\alpha_k(T-t)\right)\varphi_{i,k}+\right.$$

$$+\left(\sin\alpha_k t-\delta\sin\alpha_k(T-t)\right)\psi_{i,k}-$$

$$\left.-\frac{\delta}{\lambda_k^2}\int_0^T f_{i,k}(\tau)\left(\sin\alpha_k(T+t-\tau)-\delta\sin\alpha_k(t-\tau)\right)d\tau\right\}+$$

$$+\frac{1}{\alpha_k\lambda_k^2}\int_0^t f_{i,k}(\tau)\sin\alpha_k(t-\tau)\,d\tau\quad(i=1,2;\quad k=1,2,...),\tag{20}$$

Obviously

$$u'_{i,k}(t)=\frac{1}{\rho_k(T)}\left\{\alpha_k\left(-\sin\alpha_k t+\delta\sin\alpha_k(T-t)\right)\varphi_{i,k}+\right.$$

$$+\left(\cos\alpha_k t+\delta\cos\alpha_k(T-t)\right)\psi_{i,k}-$$

$$\left.-\frac{\delta}{\lambda_k^2}\int_0^T f_{i,k}(\tau)\left(\cos\alpha_k(T+t-\tau)+\delta\cos\alpha_k(t-\tau)\right)d\tau\right\}+$$

$$+\frac{1}{\lambda_k^2}\int_0^t f_{i,k}(\tau)\cos\alpha_k(t-\tau)\,d\tau \quad (i=1,2; \quad k=1,2,...), \tag{21}$$

$$u''_{i,k}(t)=\frac{1}{\lambda_k^2}f_{i,k}(t)-\frac{\alpha_k}{\rho_k(T)}\left\{\alpha_k(\cos\alpha_k t+\delta\cos\alpha_k(T-t))\varphi_{i,k}+\right.$$
$$+(\sin\alpha_k t-\delta\sin\alpha_k(T-t))\psi_{i,k}-$$
$$\left.-\frac{\delta}{\lambda_k^2}\int_0^T f_{i,k}(\tau)(\sin\alpha_k(T+t-\tau)+\delta\sin\alpha_k(t-\tau))\,d\tau\right\}-$$
$$-\frac{\alpha_k}{\lambda_k^2}\int_0^t f_{i,k}(\tau)\sin\alpha_k(t-\tau)\,d\tau \quad (i=1,2; \quad k=1,2,...), \tag{22}$$

Theorem 2. *Let*

1. $\varphi(x)\in C^2[0,1]$, $\varphi^{(3)}(x)\in L_2(0,1)$ *and* $\varphi(0)=\varphi(1)$, $\varphi'(0)=\varphi'(1)$, $\varphi''(0)=\varphi''(1)$.

2. $\psi(x)\in C^2[0,1]$, $\psi^{(3)}(x)\in L_2(0,1)$ *and* $\psi(0)=\varphi(1)$, $\psi'(0)=\psi'(1)$, $\psi''(0)=\psi''(1)$.

3. $f(x,t)\in C(D_T)$ *and* $f_x(x,t)\in L_2(D_T)$ *and* $f(0,t)=f(1,t)$ *for any* $t\in[0;T]$.

4. $\delta\neq\pm1$.

Then the function

$$u(x,t)=-\frac{2}{\beta^2}\int_0^1 f(x,t)\,dx+\sum_{k=1}^{\infty}\left\{\frac{1}{\alpha_k\rho_k(T)}\left[\alpha_k(\cos\alpha_k t+\delta\cos\alpha_k(T-t))\varphi_{1,k}+\right.\right.$$
$$+(\sin\alpha_k t-\delta\sin\alpha_k(T-t))\psi_{1,k}-$$
$$-\left.\left(\frac{\delta}{\lambda_k^2}\int_0^T f_{1,k}(\tau)(\sin\alpha_k(T+t-\tau)-\delta\sin\alpha_k(t-\tau))\,d\tau\right]-\right.$$
$$\left.-\frac{1}{\alpha_k\lambda_k^2}\int_0^t f_{1,k}(\tau)\sin\alpha_k(t-\tau)\,d\tau\right\}\cos\lambda_k x+\sum_{k=1}^{\infty}\times$$
$$\times\left\{\frac{1}{\alpha_k\rho_k(T)}\left[\alpha_k(\cos\alpha_k t+\delta\cos\alpha_k(T-t))\varphi_{2,k}+\right.\right.$$
$$+(\sin\alpha_k t-\delta\sin\alpha_k(T-t))\psi_{2,k}-$$
$$-\left.\left(\frac{\delta}{\lambda_k^2}\int_0^T f_{2,k}(\tau)(\sin\alpha_k(T+t-\tau)-\delta\sin\alpha_k(t-\tau))\,d\tau\right]-\right.$$
$$\left.-\frac{1}{\alpha_k\lambda_k^2}\int_0^t f_{2,k}(\tau)\sin\alpha_k(t-\tau)\,d\tau\right\}\sin\lambda_k x \tag{23}$$

is a classic solution of problem (1)-(3).

Proof. Obviously

$$\alpha^2\lambda_k^2 < \alpha^2\lambda_k^2 + \beta^2 < \left(\alpha^2+\beta^2\right)\lambda_k^2, \quad \alpha < \alpha_k < \alpha+\beta,$$

$$\left|\rho_k\left(T\right)\right| = \left|1+2\delta\cos\alpha_k T + \delta^2\right| \geq 1+\delta^2 - 2\left|\delta\right| \equiv \rho.$$

Taking into account these relations, from (18),(19) and (20) we find:

$$\left|u_{i,k}\left(t\right)\right| \leq \frac{1}{\rho}\left(1+\left|\delta\right|\right)\left|\varphi_{i,k}\right| + \frac{1}{\rho\alpha}\left(1+\left|\delta\right|\right)\left|\psi_{i,k}\right| + \frac{1}{\alpha}\times$$

$$\times\left(1+\frac{1}{\rho}\left|\delta\right|\left(1+\left|\delta\right|\right)\right)\sqrt{T}\lambda_k^{-2}\left(\int\limits_0^T\left|f_{i,k}\left(\tau\right)\right|^2 d\tau\right)^{\frac{1}{2}} \quad (i=1,2),$$

$$\left|u'_{i,k}\left(t\right)\right| \leq \frac{\alpha+\beta}{\rho}\left(1+\left|\delta\right|\right)\left|\varphi_{i,k}\right| + \frac{1}{\rho}\left(1+\left|\delta\right|\right)\left|\psi_{i,k}\right| +$$

$$+\left(1+\frac{\left|\delta\right|}{\rho}\left(1+\left|\delta\right|\right)\right)\sqrt{T}\lambda_k^{-2}\left(\int\limits_0^T\left|f_{i,k}\left(\tau\right)\right|^2 d\tau\right)^{\frac{1}{2}} \quad (i=1,2),$$

$$\left|u''_{i,k}\left(t\right)\right| \leq \lambda_k^{-2}\left|f_{i,k}\left(t\right)\right| + \frac{\left(\alpha+\beta\right)^2}{\rho}\left(1+\left|\delta\right|\right)\left|\varphi_{i,k}\right| +$$

$$+\frac{\alpha+\beta}{\rho}\left(1+\left|\delta\right|\right)\left|\psi_{i,k}\right| + \left(\alpha+\beta\right)\left(1+\frac{\left|\delta\right|}{\rho}\left(1+\left|\delta\right|\right)\right)\times$$

$$\times\sqrt{T}\lambda_k^{-2}\left(\int\limits_0^T\left|f_k\left(\tau\right)\right|^2 d\tau\right)^{\frac{1}{2}} \qquad (i=1,2).$$

Hence we have:

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|u_{i,k}\left(t\right)\right|\right)^2\right)^{\frac{1}{2}} \leq \frac{\sqrt{3}}{\rho}\left(1+\left|\delta\right|\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{i,k}\right|\right)^2\right)^{\frac{1}{2}} +$$

$$+\frac{\sqrt{3}}{\rho\alpha}\left(1+\left|\delta\right|\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\psi_{i,k}\right|\right)^2\right)^{\frac{1}{2}} +$$

$$+\frac{\sqrt{3}}{\alpha}\left(1+\frac{1}{\rho}\left|\delta\right|\left(1+\left|\delta\right|\right)\right)\sqrt{T}\left(\int\limits_0^T\sum_{k=1}^{\infty}\left|\lambda_k\left(f_{i,k}\tau\right)\right|^2 d\tau\right)^{\frac{1}{2}} \leq$$

$$\leq \frac{\sqrt{3}}{\rho}\left(1+\left|\delta\right|\right)\left\|\varphi^{(3)}\left(x\right)\right\|_{L_2(0,1)} +$$

$$+\frac{\sqrt{3}}{\rho\alpha}\left(1+\left|\delta\right|\right)\left\|\psi^{(3)}\left(x\right)\right\|_{L_2(0,1)} + \frac{\sqrt{3}}{\alpha}\left(1+\frac{1}{\rho}\left|\delta\right|\left(1+\left|\delta\right|\right)\right)\times$$

$$\times\sqrt{T}\left\|f_x(x,t)\right\|_{L_2(D_T)} \quad i=1,2, \tag{24}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|u'_{i,k}(t)\right|\right)^2\right)^{\frac{1}{2}} \leq \frac{\sqrt{3}(\alpha+\beta)}{\rho}(1+|\delta|)\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{i,k}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+\frac{\sqrt{3}}{\rho}(1+|\delta|)\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\psi_{i,k}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{3}\left(1+\frac{|\delta|}{\rho}(1+|\delta|)\right)\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\left|f_{i,k}(\tau)\right|\right)^2 d\tau\right)^{\frac{1}{2}} \leq$$

$$\leq \frac{\sqrt{3}(\alpha+\beta)}{\rho}(1+|\delta|)\left\|\varphi^{(3)}(x)\right\|_{L_2(0,1)}+$$

$$+\frac{\sqrt{3}}{\rho}(1+|\delta|)\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)}+\sqrt{3}\left(1+\frac{|\delta|}{\rho}(1+|\delta|)\right)\times$$

$$\times\sqrt{T}\left\|f_x(x,t)\right\|_{L_2(D_T)} \quad i=1,2, \tag{25}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|u''_{i,k}(t)\right|\right)^2\right)^{\frac{1}{2}} \leq 2\left(\sum_{k=1}^{\infty}\left(\lambda_k\left|f_{i,k}(t)\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+\frac{2(\alpha+\beta)^2}{\rho}(1+|\delta|)\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{i,k}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+\frac{2(\alpha+\beta)}{\rho}(1+|\delta|)\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\psi_{i,k}\right|\right)^2\right)^{\frac{1}{2}}+2(\alpha+\beta)\times$$

$$\times\left(1+\frac{|\delta|}{\rho}(1+|\delta|)\right)\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left|\lambda_k\left(f_{i,k}(\tau)\right)\right|^2 d\tau\right)^{\frac{1}{2}} \leq$$

$$\leq 2\left\|\left\|f_x(x,t)\right\|_{L_2(0,1)}\right\|_{C[0,T]}+\frac{2(\alpha+\beta)^2}{\rho}(1+|\delta|)\left\|\varphi^{(3)}(x)\right\|_{L_2(0,1)}+$$

$$+\frac{2(\alpha+\beta)}{\rho}(1+|\delta|)\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)}+$$

$$+2(\alpha+\beta)\left(1+\frac{|\delta|}{\rho}(1+|\delta|)\right)\sqrt{T}\left\|f_x(x,t)\right\|_{L_2(D_T)} \quad (i=1,2). \tag{26}$$

Obviously

$$|u(x,t)| \leq \left|u_{1,0}(t)\right|+\left(\sum_{k=1}^{\infty}\lambda_k^{-6}\right)^{\frac{1}{2}}\times$$

$$\times\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|u_{1,k}(t)\right|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|u_{2,k}(t)\right|\right)^2\right)^{\frac{1}{2}}\right], \tag{27}$$

$$|u_t(x,t)| \le \left|u'_{1,0}(t)\right| + \left(\sum_{k=1}^{\infty} \lambda_k^{-6}\right)^{\frac{1}{2}} \times$$

$$\times \left[\left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \left|u'_{1,k}(t)\right| \right)^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \left|u'_{2,k}(t)\right| \right)^2 \right)^{\frac{1}{2}} \right], \tag{28}$$

$$|u_{tt}(x,t)| \le \left|u''_{1,0}(t)\right| + \left(\sum_{k=1}^{\infty} \lambda_k^{-6}\right)^{\frac{1}{2}} \times$$

$$\times \left[\left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \left|u''_{1,k}(t)\right| \right)^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \left|u''_{2,k}(t)\right| \right)^2 \right)^{\frac{1}{2}} \right], \tag{29}$$

$$|u_{xx}(x,t)| \le \left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \times$$

$$\times \left[\left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \left|u_{1,k}(t)\right| \right)^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \left|u_{2,k}(t)\right| \right)^2 \right)^{\frac{1}{2}} \right], \tag{30}$$

$$|u_{ttxx}(x,t)| \le \left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \times$$

$$\times \left[\left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \left|u''_{1,k}(t)\right| \right)^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \left|u''_{2,k}(t)\right| \right)^2 \right)^{\frac{1}{2}} \right]. \tag{31}$$

Allowing for (24)-(26) it follows from (27)-(31) that the functions $u(x,t)$, $u_t(x,t)$, $u_{tt}(x,t)$, $u_{xx}(x,t)$, $u_{ttxx}(x,t)$ are contiuous in D_T. By immediate verification we can easily see that the function $u(x,t)$ satisfies equation (1) and conditions (2),(3) in the ordinary sense. The theorem is proved.

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