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SOME EMBEDDINGS INTO THE MODIFIED MORREY SPACES ASSOCIATED WITH THE DUNKL OPERATOR ON THE REAL LINE

Abstract

On the real line, the Dunkl operators are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. We consider the generalized shift operator, associated with the Dunkl operator

$$\Lambda_\alpha(f)(x) = \frac{d}{dx}f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right).$$

We study some embeddings into the modified Morrey spaces associated with the Dunkl operator on $\mathbb{R}$.

1. Introduction

On the real line, the Dunkl operators are differential-difference operators introduced in 1989 by Dunkl [3] and are denoted by Λ_α, where α is a real parameter $> -1/2$. These operators are associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. The Dunkl kernel E_α is used to define the Dunkl transform $\mathfrak{F}_\alpha$ which was introduced by Dunkl in [4]. Rosler in [14] shows that the Dunkl kernels verify a product formula. This allows us to define the Dunkl translation τ_x, $x \in \mathbb{R}$. As a result, we have the Dunkl convolution.

In the present work, we study some embeddings into the modified Morrey spaces associated with the Dunkl operator on $\mathbb{R}$, so we fix $\alpha \geq -1/2$ and we define the D-Morrey space and modified D-Morrey space using the harmonic analysis associated with the Dunkl operator on $\mathbb{R}$. These operators are associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$.

2. Preliminaries

For a real parameter $\alpha \geq -1/2$, we consider the Dunkl operator, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha(f)(x) = \frac{d}{dx}f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right) \tag{1}$$

Note that $\Lambda_{-1/2} = d/dx$.

For $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the initial value problem :

$$\Lambda_\alpha(f)(x) = \lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R}$$

has a unique solution $E_\alpha(\lambda x)$ called Dunkl kernel [3, 10, 15] and given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)} j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R},$$

where j_α is the normalized Bessel function of the first kind and order α [16], defined by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}, \quad z \in \mathbb{C}.$$

We can write for $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ (see Rösler [14], p. 295)

$$E_\alpha(-i\lambda x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_{-1}^{1} (1-t^2)^{\alpha-1/2} (1-t)\, e^{i\lambda x t}\, dt.$$

Note that $E_{-1/2}(\lambda x) = e^{\lambda x}$.

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1} |x|^{2\alpha+1}\, dx.$$

For every $1 \le p \le \infty$, we denote by $L_p = L_p(d\mu_\alpha)$ the spaces of complex-valued functions f, measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} \equiv \|f\|_{L_{p,\alpha}} = \left(\int_{\mathbb{R}} |f(x)|^p \, d\mu_\alpha(x) \right)^{1/p} < \infty \quad \text{if} \quad p \in [1, \infty),$$

and

$$\|f\|_{L_{\infty,\alpha}} = ess \sup_{x \in \mathbb{R}} |f(x)| \quad \text{if} \quad p = \infty.$$

For $1 \le p < \infty$ we denote by $WL_{p,\alpha}$, the weak $L_{p,\alpha}$ spaces defined as the set of locally integrable functions $f(x)$, $(x) \in \mathbb{R}$ with the finite norm

$$\|f\|_{WL_{p,\alpha}} = \sup_{r>0} r \left(\mu_\alpha \{x \in \mathbb{R} \ : \ |f(x)| > r\}\right)^{1/p}.$$

Note that

$$L_{p,\alpha} \subset WL_{p,\alpha} \quad \text{and} \quad \|f\|_{WL_{p,\alpha}} \le \|f\|_{L_{p,\alpha}} \text{ for all } f \in L_{p,\alpha}.$$

The Dunkl kernel gives rise to an integral transform, called Dunkl transform on $\mathbb{R}$, which was introduced and studied in [6].

The Dunkl transform $\mathcal{F}_\alpha$ of a function $f \in L_{1,\alpha}(\mathbb{R})$, is given by

$$\mathcal{F}_\alpha f(\lambda) := \int_{\mathbb{R}} E_\alpha(-i\lambda x)\, f(x) d\mu_\alpha(x), \quad \lambda \in \mathbb{R}.$$

Here the integral makes sense since $|E_\alpha(ix| \le 1$ for every $x \in \mathbb{R}$ [14], p. 295.

Notation. For all x, y, $z \in \mathbb{R}$, we put

$$W_\alpha(x,y,z) = (1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}) \Delta_\alpha(x,y,z)$$

where

$$\sigma_{x,y,z}=\begin{cases}\frac{x^2+y^2-z^2}{2xy} & \text{if } x,\, y\in\mathbb{R}\setminus 0,\\ 0 & \text{otherwice}\end{cases}$$

and Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x,y,z)=\begin{cases} d_\alpha\frac{([(|x|+|y|)^2-z^2][z^2-(|x|-|y|)^2])^{\alpha-1/2}}{|xyz|^{2\alpha}} & \text{if } |z|\in A_{x,y},\\ 0 & \text{otherwice,}\end{cases}$$

where $d_\alpha=(\Gamma(\alpha+1))^2/(2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2}))$ and $A_{x,y}=[||x|-|y||,|x|+|y|]$.

Properties 1. *(see Rösler [14]) The signed kernel W_α is even and satisfies the following properties*

$$W_\alpha(x,y,z)=W_\alpha(y,x,z)=W_\alpha(-x,z,y),$$

$$W_\alpha(x,y,z)=W_\alpha(-z,y,-x)=W_\alpha(-x,-y,-z)$$

and

$$\int_{\mathbb{R}}|W_\alpha(x,y,z)|\,d\mu_\alpha(z)\le 4.$$

In the sequel we consider the signed measure $\nu_{x,y}$, on $\mathbb{R}$, given by

$$\nu_{x,y}=\begin{cases} W_\alpha(x,y,z)\,d\mu_\alpha(z) & \text{if } x,\, y\in\mathbb{R}\setminus 0,\\ d\delta_x(z) & \text{if } y=0,\\ d\delta_y(z) & \text{if } x=0.\end{cases}$$

Theorem 1. *(see Rösler [14]) (i) Let $\alpha>-1/2$ and $\lambda\in\mathbb{C}$. The Dunkl kernel E_α satisfies the following product formula:*

$$E_\alpha(\lambda x)E_\alpha(\lambda y)=\int_{\mathbb{R}}E_\alpha(\lambda z)\,d\nu_{x,y}(z),\quad x,\, y\in\mathbb{R}.$$

(ii) The measures $\nu_{x,y}$ have the following properties:

$$\operatorname{supp}(\nu_{x,y})=A_{x,y}\cup(-A_{x,y}),\quad \|\nu_{x,y}\|:=\int_{\mathbb{R}}d|\nu_{x,y}|(z)\le 4.$$

Definition 1. *For $x,\, y\in\mathbb{R}$ and f a continuous function on $\mathbb{R}$, we put*

$$\tau_x f(y)=\int_{\mathbb{R}}f(z)\,d\nu_{x,y}(z).$$

The operators τ_x, $x\in\mathbb{R}$, are called Dunkl translation operators on $\mathbb{R}$ and it can be expressed in the following form (see ref. [14])

$$\tau_x f(y)=C_\alpha\int_0^\pi f_e\left(\sqrt{x^2+y^2-2|xy|\cos\theta}\right)\,h_1(x,y,\theta)(\sin\theta)^{2\alpha}\,d\theta$$

$$+C_\alpha\int_0^\pi f_o\left(\sqrt{x^2+y^2-2|xy|\cos\theta}\right)\,h_2(x,y,\theta)(\sin\theta)^{2\alpha}\,d\theta,$$

where $f = f_e + f_o$, f_o and f_e being respectively the odd and the even parts of f, with $C_\alpha = \Gamma(\alpha+1)/(\sqrt{\pi}\,\Gamma(\alpha+1/2))$,

$$h_1(x,y,\theta) = 1 - sgn(xy)\cos\theta \quad\text{and}\quad h_2(x,y,\theta) = \begin{cases} \frac{(x+y)\,[1-sgn(xy)\cos\theta]}{\sqrt{x^2+y^2-2|xy|\cos\theta}} & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

Properties 2. *(see Mourou [9]) (i) The operator τ_x, $x \in \mathbb{R}$, is a continuous linear operator from $\mathcal{E}(\mathbb{R})$ into itself.*

(ii) For all $f \in \mathcal{E}(\mathbb{R})$ and x, $y \in \mathbb{R}$, we have

$$\tau_x f(y) = \tau_y f(x), \quad \tau_0 f(x) = f(x),$$

$$\tau_x \circ \tau_y = \tau_y \circ \tau_x, \quad \Lambda_\alpha \circ \tau_x = \tau_x \circ \Lambda_\alpha.$$

Proposition 1. *(see Soltani [12]) (i) If f is an even positive continuous function, then $\tau_x f$ is positive.*

(ii) For all $x \in \mathbb{R}$ the operator τ_x extends to $L_{p,\alpha}(\mathbb{R})$, $p \geq 1$ and we have for $f \in L_{p,\alpha}(\mathbb{R})$,

$$\|\tau_x f\|_{p,\alpha} \leq 4\|f\|_{p,\alpha}.$$

(ii) For all x, $\lambda \in \mathbb{R}$ and $f \in L_{1,\alpha}(\mathbb{R})$, we have

$$\mathcal{F}_\alpha\,(\tau_x f)\,(\lambda) = E_\alpha(i\lambda x)\,\mathcal{F}_\alpha f(\lambda).$$

Let f and g be two continuous functions on $\mathbb{R}$ with compact support. We define the generalized convolution $*_\alpha$ of f and g by

$$f *_\alpha g(x) := \int_{\mathbb{R}} \tau_x f(-y)\, g(y)\, d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

The generalized convolution $*_\alpha$ is associative and commutative [14]. Note that $*_{-1/2}$ agrees with the standard convolution $*$.

Proposition 2. *(see Soltani [12]) (i) If f is an even positive function and g a positive function with compact support, then $f *_\alpha g$ is positive.*

(ii) Assume that p, q, $r \in [1,+\infty[$ satisfying $1/p + 1/q = 1 + 1/r$ (the Young condition). Then the map $(f,g) \mapsto f *_\alpha g$, defined on $\mathcal{E}_c \times \mathcal{E}_c$, extends to a continuous map from $L_{p,\alpha}(\mathbb{R}) \times L_{q,\alpha}(\mathbb{R})$ to $L_{r,\alpha}(\mathbb{R})$, and we have

$$\|f *_\alpha g\|_{r,\alpha} \leq 4\|f\|_{p,\alpha}\,\|g\|_{q,\alpha}.$$

(ii) For all $f \in L_{1,\alpha}(\mathbb{R})$ and $g \in L_{2,\alpha}(\mathbb{R})$, we have

$$\mathcal{F}_\alpha\,(f *_\alpha g) = (\mathcal{F}_\alpha f)\,\,(\mathcal{F}_\alpha g)\,.$$

Proposition 3. *Let $f \in L_{1,\alpha}(\mathbb{R})$ and $g \in L_{p,\alpha}(\mathbb{R})$, $1 \leq p < \infty$. Then we have*

$$\tau_t\,(f *_\alpha g) = \tau_t f *_\alpha g = f *_\alpha \tau_t g.$$

Let $B(x,t)=\{y\in\mathbb{R}:|y|\in]\max\{0,|x|-t\},|x|+t[\ \}$ and $t>0$. Then $B(0,t)=]-t,t[$ and $\mu_\alpha(]-t,t[)=b_\alpha^{-1}\,t^{2\alpha+2}$, where $b_\alpha=2^{\alpha+1}\,(\alpha+1)\,\Gamma(\alpha+1)$. We now consider the maximal function

$$Mf(x)=\sup_{r>0}\frac{1}{\mu_\alpha B(0,r)}\int_{B(0,r)}\tau_x|f|(y)\,d\mu_\alpha(y).$$

Theorem 2. *[7] 1. If $f\in L_{1,\alpha}(\mathbb{R})$, then for every $\beta>0$*

$$\mu_\alpha\{x\in\mathbb{R}:Mf(x)>\beta\}\le\frac{C}{\beta}\int_{\mathbb{R}}|f(x)|\,d\mu_\alpha(x),$$

where $C>0$ is independent of f.

2. If $f\in L_{p,\alpha}(\mathbb{R})$, $1<p\le\infty$, then $Mf\in L_{p,\alpha}(\mathbb{R})$ and

$$\|Mf\|_{L_{p,\alpha}}\le C_p\|f\|_{L_{p,\alpha}},$$

where $C_p>0$ is independent of f.

Corollary 1. *If $f\in L^{loc}_{1,\alpha}(\mathbb{R})$, then*

$$\lim_{r\to0}\frac{1}{\mu_\alpha B(0,r)}\int_{B(0,r)}\left|\tau_xf(y)-f(x)\right|d\mu_\alpha(y)=0$$

for a. e. $x\in\mathbb{R}$.

Corollary 2. *If $f\in L^{loc}_{1,\alpha}(\mathbb{R})$, then*

$$\lim_{r\to0}\frac{1}{\mu_\alpha B(0,r)}\int_{B(0,r)}\tau_xf(y)d\mu_\alpha(y)=f(x)$$

for a. e. $x\in\mathbb{R}$.

3. Some embeddings into the modified D-Morrey spaces

Definition 2. *Let $1\le p<\infty$. We denote by $WL_{p,\alpha}(\mathbb{R})$ the weak $L_{p,\alpha}(\mathbb{R})$ space defined as the set of locally integrable functions $f(x)$, $x\in\mathbb{R}$ with the finite norms*

$$\|f\|_{WL_{p,\alpha}}=\sup_{r>0}r\,\left(\mu_\alpha\{x\in\mathbb{R}\ :\ |f(x)|>r\}\right)^{1/p}.$$

Note that

$$L_{p,\alpha}(\mathbb{R})\subset WL_{p,\alpha}(\mathbb{R})\quad\text{and}\quad\|f\|_{WL_{p,\alpha}}\le\|f\|_{L_{p,\alpha}}\ \text{for all}\ f\in L_{p,\alpha}.$$

Definition 3. *[1] Let $1\le p<\infty$, $0\le\lambda\le2\alpha+2$. We denote by $L_{p,\lambda,\alpha}(\mathbb{R})$ Morrey space ($\equiv$ D-Morrey space), associated with the Dunkl operator as the set of locally integrable functions $f(x)$, $x\in\mathbb{R}$, with the finite norm*

$$\|f\|_{L_{p,\lambda,\alpha}}=\sup_{t>0,\,x\in\mathbb{R}}\left(t^{-\lambda}\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p}.$$

Definition 4 *Let* $1 \le p < \infty$, $0 \le \lambda \le 2\alpha+2$, $[t]_1 = \min\{1,t\}$. *We denote by* $\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$ *modified Morrey space (≡ modified D-Morrey space), associated with the Dunkl operator as the set of locally integrable functions* $f(x)$, $x \in \mathbb{R}$, *with the finite norm*

$$\|f\|_{L_{p,\lambda,\alpha}} = \sup_{t>0,\, x\in\mathbb{R}} \left([t]_1^{-\lambda} \int_{B(0,t)} \tau_x |f(y)|^p \, d\mu_\alpha(y) \right)^{1/p}.$$

Note that

$$\widetilde{L}_{p,0,\alpha}(\mathbb{R}) = L_{p,0,\alpha}(\mathbb{R}) = L_{p,\alpha}(\mathbb{R}),$$

$$\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}) \subset\!\!\succ L_{p,\alpha}(\mathbb{R}) \quad \text{and} \quad \|f\|_{L_{p,\alpha}} \le \|f\|_{\widetilde{L}_{p,\lambda,\alpha}} \tag{2}$$

and if $\lambda < 0$ or $\lambda > 2\alpha+2$, then $L_{p,\lambda,\alpha}(\mathbb{R}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}$.

Definition 5. *[2] Let* $1 \le p < \infty$, $0 \le \lambda \le 2\alpha+2$. *We denote by* $WL_{p,\lambda,\alpha}(\mathbb{R})$ *the weak D-Morrey space as the set of locally integrable functions* $f(x)$, $x \in \mathbb{R}$ *with finite norm*

$$\|f\|_{WL_{p,\lambda,\alpha}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}} \left(t^{-\lambda} \int_{\{y\in B(0,t):\, \tau_x|f(y)|>r\}} d\mu_\alpha(y) \right)^{1/p}.$$

Definition 6. *[2] Let* $1 \le p < \infty$, $0 \le \lambda \le 2\alpha+2$, $[t]_1 = \min\{1,t\}$. *We denote by* $WL_{p,\lambda,\alpha}(\mathbb{R})$ *the weak D-Morrey space as the set of locally integrable functions* $f(x)$, $x \in \mathbb{R}$ *with finite norm*

$$\|f\|_{WL_{p,\lambda,\alpha}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}} \left([t]_1^{-\lambda} \int_{\{y\in B(0,t):\, \tau_x|f(y)|>r\}} d\mu_\alpha(y) \right)^{1/p}.$$

We note that

$$L_{p,\lambda,\alpha}(\mathbb{R}) \subset WL_{p,\lambda,\alpha}(\mathbb{R}) \ \text{ and } \ \|f\|_{WL_{p,\lambda,\alpha}} \le \|f\|_{L_{p,\lambda,\alpha}}.$$

Lemma 1. *Let* $1 \le p < \infty$, $0 \le \lambda \le n$. *Then*

$$\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}) = L_{p,\lambda,\alpha}(\mathbb{R}) \cap L_p(\mathbb{R}) \subset\!\!\succ L_{p,\lambda,\alpha}(\mathbb{R})$$

and for $f \in \widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$ $\|f\|_{L_{p,\lambda,\alpha}} = \|f\|_{\widetilde{L}_{p,\lambda,\alpha}}$.

Proof. Let $f \in \widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$. Then

$$\|f\|_{L_{p,\alpha}} = \sup_{x\in\mathbb{R},t>0} \left(\int_{B(0,t)} \tau_x |f(y)|^p \, d\mu_\alpha(y) dy \right)^{1/p}$$
$$\le \sup_{x\in\mathbb{R},t>0} \left([t]_1^{-\lambda} \int_{B(0,t)} \tau_x |f(y)|^p \, d\mu_\alpha(y) \right)^{1/p} = \|f\|_{\widetilde{L}_{p,\lambda,\alpha}}$$

and

$$\|f\|_{L_{p,\lambda,\alpha}} = \sup_{x\in\mathbb{R},t>0}\left(t^{-\lambda}\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p}$$
$$\le \max\left\{\sup_{x\in\mathbb{R},0<t\le 1}\left(t^{-\lambda}\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p},\right.$$
$$\left.\sup_{x\in\mathbb{R},t\ge 1}\left(\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p}\right\} = \max\left\{\|f\|_{\widetilde{L}_{p,\lambda,\alpha}}, \|f\|_{L_{p,\alpha}}\right\}.$$

Therefore, $f \in L_{p,\lambda,\alpha}(\mathbb{R}) \cap L_{p,\alpha}(\mathbb{R})$ and the embedding

$$\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}) \subset\!\!\succ L_{p,\lambda,\alpha}(\mathbb{R}) \cap L_{p,\alpha}(\mathbb{R})$$

is valid.

Let $f \in L_{p,\lambda,\alpha}(\mathbb{R}) \cap L_p(\mathbb{R})$. Then

$$\|f\|_{\widetilde{L}_{p,\lambda,\alpha}} = \sup_{x\in\mathbb{R},t>0}\left([t]_1^{-\lambda}\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p}$$
$$= \max\left\{\sup_{x\in\mathbb{R},0<t\le 1}\left(t^{-\lambda}\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p},\right.$$
$$\left.\sup_{x\in\mathbb{R},t>1}\left(\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p}\right\}$$
$$\le \max\left\{\|f\|_{L_{p,\lambda,\alpha}}, \|f\|_{L_{p,\alpha}}\right\}.$$

Therefore, $f \in \widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$ and the embedding $L_{p,\lambda,\alpha}(\mathbb{R}) \cap L_p(\mathbb{R}) \subset\!\!\succ \widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$ is valid.

Thus $\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}) = L_{p,\lambda,\alpha}(\mathbb{R}) \cap L_{p,\alpha}(\mathbb{R}) \subset\!\!\succ L_{p,\lambda,\alpha}(\mathbb{R})$.

Let now $f \in \widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$. Then

$$\|f\|_{L_{p,\lambda,\alpha}} = \sup_{x\in\mathbb{R},t>0}\left(t^{-\lambda}\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p}$$
$$= \sup_{x\in\mathbb{R},t>0}\left(t^{-1}[t]_1\right)^{\frac{\lambda}{p}}\left([t]_1^{-\lambda}\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p}$$
$$= \sup_{x\in\mathbb{R},t>0}\left([t]_1^{-\lambda}\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p} = \|f\|_{\widetilde{L}_{p,\lambda,\alpha}}.$$

It is known [8] that for $1 \le p < \infty$

$$L_{p,2\alpha+2,\alpha}(\mathbb{R}) = L_\infty(\mathbb{R}) \quad \text{and} \quad \|f\|_{L_{p,2\alpha+2,\alpha}} = b_\alpha^{1/p}\,\|f\|_{L_\infty}. \tag{3}$$

From (**??**) and Lemma 1 for $1 \le p < \infty$ we have

$$\widetilde{L}_{p,2\alpha+2,\alpha}(\mathbb{R}) = L_\infty(\mathbb{R}) \cap L_{p,\alpha}(\mathbb{R}). \tag{4}$$

In [8] the following embedding on the D-Morrey spaces was proved.

Lemma 2. *[8] Let $1 \le p < \infty$, $0 \le \lambda < 2\alpha + 2$. Then for $\beta = \frac{2\alpha+2-\lambda}{p}$*

$$L_{p,\lambda,\alpha}(\mathbb{R}) \subset L_{1,2\alpha+2-\beta,\alpha}(\mathbb{R})$$

and

$$\|f\|_{L_{1,2\alpha+2-\beta,\alpha}} \le b_\alpha^{-1/p} \, \|f\|_{L_{p,\lambda,\alpha}} ,$$

where $1/p + 1/p' = 1$.

On the modified D-Morrey spaces the following embedding is valid.

Lemma 3. *Let $1 \le p < \infty$, $0 < \beta < 2\alpha + 2$, $0 \le \lambda < 2\alpha + 2$. Then for $\frac{2\alpha+2-\lambda}{\beta} \le p \le \frac{2\alpha+2}{\beta}$*

$$\widetilde{L}_{p,\lambda,\alpha}(\mathbb{R}) \subset\!\succ \widetilde{L}_{1,2\alpha+2-\beta}(\mathbb{R})$$

and for $f \in \widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$ the following inequality

$$\|f\|_{\widetilde{L}_{1,2\alpha+2-\beta}} \le b_\alpha^{1/p'} \, \|f\|_{\widetilde{L}_{p,\lambda,\alpha}} .$$

is valid.

Proof. Let $0 < \beta < 2\alpha + 2$, $0 \le \lambda < 2\alpha + 2$, $f \in \widetilde{L}_{p,\lambda,\alpha}(\mathbb{R})$ and $\frac{2\alpha+2-\lambda}{\beta} \le p \le \frac{2\alpha+2}{\beta}$. By the Hölder's inequality we have

$$\begin{aligned}
\|f\|_{\widetilde{L}_{1,n-\alpha}} &= \sup_{x\in\mathbb{R},\, t>0} [t]_1^{\beta-2\alpha-2} \int_{B(0,t)} \tau_x |f(y)|^p \, d\mu_\alpha(y) \\
&\le b_\alpha^{1/p'} \sup_{x\in\mathbb{R},\, t>0} \left([t]_1 \, t^{-1}\right)^{-(2\alpha+2)/p'} [t]_1^{\beta-\frac{2\alpha+2-\lambda}{p}} \\
&\times \left([t]_1^{-\lambda} \int_{B(0,t)} \tau_x |f(y)|^p \, d\mu_\alpha(y)\right)^{1/p} \\
&= b_\alpha^{1/p'} \sup_{x\in\mathbb{R},\, t>0} \left([t]_1 \, t^{-1}\right)^{2\alpha+2-\beta} \left([t]_1 \, t^{-1}\right)^{-(2\alpha+2)/p'} [t]_1^{\beta-\frac{2\alpha+2-\lambda}{p}} \\
&\times \left([t]_1^{-\lambda} \int_{B(0,t)} \tau_x |f(y)|^p \, d\mu_\alpha(y)\right)^{1/p} \\
&\le b_\alpha^{1/p'} \, \|f\|_{\widetilde{L}_{p,\lambda,\alpha}} \sup_{t>0} \left([t]_1 \, t^{-1}\right)^{\frac{2\alpha+2}{p}-\beta} [t]_1^{\beta-\frac{2\alpha+2-\lambda}{p}} .
\end{aligned}$$

Note that

$$\sup_{t>0} \left([t]_1 \, t^{-1}\right)^{\frac{2\alpha+2}{p}-\beta} [t]_1^{\beta-\frac{2\alpha+2-\lambda}{p}} = \max\{ \sup_{0<t\le 1} t^{\beta-\frac{2\alpha+2-\lambda}{p}}, \sup_{t>1} t^{\beta-\frac{2\alpha+2}{p}} \} < \infty$$

$$\Longleftrightarrow \frac{2\alpha+2-\lambda}{\beta} \le p \le \frac{2\alpha+2}{\beta}.$$

Therefore $f \in \widetilde{L}_{1,2\alpha+2-\beta}(\mathbb{R})$ and

$$\|f\|_{\widetilde{L}_{1,2\alpha+2-\beta}} \le b_\alpha^{1/p'} \, \|f\|_{\widetilde{L}_{p,\lambda,\alpha}} .$$

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