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ON SOME PROOF METHOD OF SOLVABILITY OF ELLIPTIC EQUATIONS WITH SMALL BMO SEMI-NORM OF COEFFICIENTS

Abstract

In this paper we consider the strong solvability of the Dirichlet problem in the Sobolev spaces for elliptic operators.

The paper is devoted to strong solvability in Sobolev spaces of the Dirichlet problem

$$\begin{cases} Lu = f \quad in \;\; x \in E_n \\ \lim\limits_{x\to\infty} u(x) = 0 \end{cases} \tag{1}$$

for elliptic operators $L = \sum\limits_{i,j=1}^{n} a_{ij}(x) \dfrac{\partial^2}{\partial x_i \partial x_j}$ with discontinuous coefficients $a_{ij}(x)$, where the right hand side $f \in L_p(E_n)$, $n \geq 3$, the coefficients $a_{ij}(x)$ are bounded measurable functions satisfying the additional condition of smallness of BMO norm, and $\exists \mu \in (0,1]$ for $\forall \xi \in E_n$

$$\mu |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \mu^{-1} |\xi|^2 . \tag{2}$$

The papers of Ladyzhenskaya O.A. and Uraltseva N.N. [1], Ivanov A.V. [2], Agmon S., Douglis A., Nirebberg L. [3] in the case of continuous coefficients, of Talenti J [4], Alkhutov Yu.A. and Mamedov I.T. [5], Zhikov V.V. and Sirajidunov M.M. [6] in the case of discontinuous coefficients of the operator with small scatter of eigen numbers of the matrix of coefficients (when $p = 2$ it is expressed by the Cordes condition) are devoted to the problems of proof of $W^{2,p}$ a priori estimates for elliptic and parabolic operators. In the papers of Vitanza C. [7.9], Palagachev D. [9], Krylov N.V. [10], Byon, Sung Sig [11], Kim Kyeong-Hun and Krylov N.V. [12], Doyoon Kim [13] . $W^{2,p}$ a priori estimates were proved for elliptic operators with coefficients from the class BMO.

The goal of our paper is to prove strong solvability of the problem (1) in a Sobolev space. Provided sufficient smallness of BMO norm of the coefficients $\|a\|_{BMO} = \sum\limits_{i,j=1}^{n} \|a_{ij}\|_{BMO}$ a priori estimation

$$\|u\|_{2,p} \leq C(n, p, \mu, D, \|a\|_{BMO}) \|Lu\|_p , \quad p \in (1, n/2) \tag{3}$$

is proved for the functions $u \in W^{2,p}(E_n)$. Applying the obtained estimation we prove strong solvability of the problem (1) in the space $W^{2,p}(E_n)$ for any $f \in L_p(E_n)$.

The proof method essentially uses the Calderon-Zigmund theorem on boundedness of a singular integral and the theorem on boundedness of a commutator integral.By applying such methods we succeed to get a prime proof of a one valued strong global solvability of Dirichlet's homogeneous problem in E_n, and on the coefficients of the operator the VMO condition is not required, instead of this sufficient smallness of their BMO norm is required.

By $\|u\|_p$ we denote a Lebesgue norm of the function $u = u(x)$ in the Lebesgue space $L_p(E_n)$. Determine the space $W^{2,p}(E_n)$ as a closure of a class of the functions $u \in C^\infty(E_n)$ by the norm $\|u\|_{2,p} = \|u\|_{p^*} + \sum\limits_{|\alpha \leq 2|} \|D^\alpha u\|_p$, where $p^* = pn/(n-2p)$ for $p \in (1, n/2]$, $p^* = \infty$ and $\|u\|_{p^*} := \sup\limits_D |u|$ for $p \geq n/2$. Denote by $\overset{\cdot}{W}{}^{2,p}(E_n)$ a subspace of $W^{2,p}(E_n)$ obtained by completion of a class of functions $u \in C^\infty(E_n)$, $\lim\limits_{x\to\infty} u(x) = 0$ by the norm $\|u\|_{2,p}$. We also denote

$$\|f\|_{BMO} = \sup_{B \subset E_n} (1/|B|) \int\limits_B |f - f_B|\, dx,$$

semi-norm of the function $f(x)$ in the space BMO, where $|B|$ denotes Lebesgue measure of an arbitrary ball $B \subset E_n$. Assume

$$\|f\|_{BMO,r} = \sup_{B \subset E_n} (1/|B|) \int\limits_B |f - f_B|\, dx,$$

where the upper bound is taken over all the balls B of radius $\leq r$. Inclusion $f \in VMO$ means $\lim\limits_{r\to 0} \|f\|_{BMO,r} = 0$.

In the proof of the main result we essentially use the following theorem.

Theorem 1. *Let* $p \in (1,\infty)$, $a_{ik} \in C^\infty(E_n)$, $f \in C^\infty(E_n)$. $u(x) \in W^{2,p}(E_n)$ $\left(u \in \dot{W}^{2,p}(E_n^+)\right)$, *and a semi-norm of the coefficients* $\|a\|_{BMO}$ *of the operator* L *be sufficiently small. Then, estimation* (3) *is valid.*

Proof. Let $F \in W^{2,p'}(E_n)$, we denote by $\psi \in \dot{W}^{2,p}(E_n)$ a solution of the boundary value problem:

$$L^*\left(\psi|\psi|^{p-2}\right) = F_{x_k x_l} \ \ in \ \ E_n\left(E_n^+\right); \quad k,l \in [1,n], \tag{4}$$

where $L^* v = \sum\limits_{i,j=1}^{n} \dfrac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(x)\, v)$ (For existence of the solution of problem (5) see: f.e. in [1]). Then

$$\int\limits_{E_n} \psi|\psi|^{p-2} L\eta dx = \int\limits_{E_n} F\eta_{x_k x_l} dx, \quad \forall \eta \in \dot{W}^{2,p}(E_n). \tag{5}$$

Assume $\eta(x) = \int\limits_{E_n} G_y(x)\,\psi(y)\,dy$ in the identity (5), where $G_y(x)$ is a Levi function with singularity at the point $y \in E_n$:

$$G_y(x) = \frac{1}{\sqrt{\det a(y)}} \left[\sum_{i,j=1}^{n} A_{ij}(y)(x_i - y_i)(x_j - y_j)\right]^{\frac{2-n}{2}},$$

where $A_{ik}(y)$ are the elements of the matrix inverse to $\|a_{ik}(y)\|$. We have

$$\int_{E_n} \psi(x) |\psi(x)|^{p-2} L_x \left(\int_{E_n} G_y(x) \psi(y)\, dy \right) = \int_{E_n} F(x) \left(\int_{E_n} G_y(x) \psi(y)\, dy \right)_{x_k x_l} dx$$

Applying the differentiation formula of integrals with weak singularity [14] by $LG_y(x) = -\delta_y(x)$, where $\delta_y(x)$ is a delta function with singularity at the point $y \in E_n$ for the left hand side of the previous equality we have:

$$\int_{E_n} \psi(x) |\psi(x)|^{p-2} L_y \left(\int_{E_n} G_y(x) \psi(y)\, dy \right) dx +$$

$$+ \int_{E_n} \psi(x) |\psi(x)|^{p-2} (L_x - L_y) \left(\int_{E_n} G_y(x) \psi(y)\, dy \right) =$$

$$= -C_0 \int_{E_n} |\psi(x)|^p dx + \int_{E_n} \psi(x) |\psi(x)|^{p-2} \left(\int_{E_n} (L_x - L_y) G_y(x) \psi(y)\, dy \right) dx$$

Therefore

$$C \int_{E_n} |\psi(x)|^p dx \le - \int_{E_n} F(x) \left(\int_{E_n} [G_y(x)]_{x_k x_l} \psi(y)\, dy \right) dx +$$

$$+ \int_{E_n} |\psi(x)|^{p-1} \left| \int_{E_n} (L_x - L_y) G_y(x) \psi(y)\, dy \right| dx, \tag{6}$$

where $L_x = \sum_{i,j=1}^{n} a_{ij}(x) \dfrac{\partial^2}{\partial x_i \partial x_j}$, $L_y = \sum_{i,j=1}^{n} a_{ij}(y) \dfrac{\partial^2}{\partial x_i \partial x_j}$.

Denote the integrals in the right hand side of (6) by i_1 and i_2. By the Calderon-Zigmund estimation [15],

$$\left\| \int_{E_n} \psi(y) (G_y(x))_{x_k x_l} dy \right\|_p \le C \|\psi\|_p ,$$

then

$$i_1 \le \|F\|_{p'} \left\| \int_{E_n} \psi(y) (G_y(x))_{x_k x_l} dy \right\|_p \le C \|F\|_{p'} \|\psi\|_p . \tag{7}$$

For estimating i_2 we'll use that $a_{ij} \in BMO$. Then by the theorem on boundedness of commutator integral [16] we have:

$$i_2 = \int_{E_n} \psi(x) |\psi(x)|^{p-2} \left| \int_{E_n} (L_x - L_y) G_y(x) \psi(y)\, dy \right| dx =$$

$$= \int_{E_n} \left| \int_{E_n} \sum_{i,j=1}^{n} (a_{ij}(x) - a_{ij}(y)) \frac{\partial^2}{\partial x_i \partial x_j} G_y(x) \psi(y)\, dy \right| \psi(x) |\psi(x)|^{p-2} dx \le$$

$$\le \|\psi\|_p^{p-1} \left\| \int_{E_n} \left[\sum_{i,j=1}^{n} (a_{ij}(x) - a_{ij}(y)) \frac{\partial^2}{\partial x_i \partial x_j} G_y(x) \right] \psi(y)\, dy \right\|_p \le$$

$$\le C \|\psi\|_p^p \|a\|_{BMO} \tag{8}$$

As a result, from (6) and estimations (7) – (8) we get

$$\|\psi\|_p^p \le C \left(\|a\|_{BMO} \|\psi\|_p^p + \|\psi\|_p \|F\|_{p'} \right).$$

Hence, if $\|a\|_{BMO}$ is such that $1 - C\|a\|_{BMO} > 0$

$$\|\psi\|_p^p (1 - C\|a\|_{BMO}) \le C \|\psi\|_p \|F\|_{p'} \ ,$$

whence

$$\|\psi\|_p \le C \|F\|_{p'}^{1/(p-1)}. \tag{9}$$

Let $\psi_{kl} \in \dot{W}^{2,p}(E_n)$ be a solution of the equation

$$L^* \left(\psi_{kl} |\psi_{kl}|^{p-2} \right) = \varphi^{kl}_{x_k x_l} \ , \quad \psi_{kl} \in \dot{W}^{2,p}(E_n); \quad k,l = 1,2,...n$$

Using the estimation (9) for any function $\varphi^{kl} \in C^\infty(E_n)$, $\varphi^{k,l} \in L_{p'}(E_n)$ we have

$$\left| \int_{E_n} u(x) \varphi^{kl}_{x_k x_l} dx \right| = \left| \int_{E_n} u(x) L^* \left(\psi_{kl} |\psi_{kl}|^{p-2} \right) dx \right| =$$

$$= \left| \int_{E_n} \psi_{kl} |\psi_{kl}|^{p-2} Lu(x)\, dx \right| \le \|\psi_{kl}\|_p^{p-1} \|Lu\|_p \ .$$

Now, let's use the estimation (9) :

$$\left| \int_{E_n} u(x) \varphi^{kl}_{x_k x_l}(x)\, dx \right| \le C \left\| \varphi^{kl} \right\|_{p'} \|Lu\|_p \ ,$$

whence we get

$$\|u_{x_l x_k}\|_p \le C \|Lu\|_p; \quad k,l = 1,2,...,n \tag{10}$$

To complete the proof of theorem 1 we prove the inequality

$$\|u\|_{p^*} \le C \|Lu\|_p \tag{11}$$

Reasoning as above for $\Phi \in L_{(p^*)'}(E_n)$ by we denote $\psi(x)$ as a solution of the problem:

$$L^* \left(\psi |\psi|^{p-2} \right) = \Phi \ \text{ in } \ E_n \ \left(E_n^+ \right), \quad \psi \in \dot{W}^{2,p}(E_n);$$

i.e. in the right hand of the equation (4) the expression $F_{x_k x_l}$ is replaced by the function Φ. Then

$$\int\limits_{E_n} \psi \left|\psi\right|^{p-2} L\eta dx = \int\limits_{E_n} \Phi\eta dx, \quad \forall\eta \in \overset{\cdot}{W}^{2,p} \left(E_n\right).$$

and assuming $\eta\left(x\right) = \int\limits_{E_n} G_y\left(x\right)\psi\left(y\right)dy$ we'll have

$$\int\limits_{E_n} \psi\left(x\right)\left|\psi\left(x\right)\right|^{p-2} L_x \left(\int\limits_{E_n} G_y\left(x\right)\psi\left(y\right)dy\right) = \int\limits_{E_n} \Phi\left(x\right)\left(\int\limits_{E_n} G_y\left(x\right)\psi\left(y\right)dy\right)$$

whence

$$C\int\limits_{E_n} \left|\psi\left(x\right)\right|^p dx \leq \int\limits_{E_n} \left|\Phi\left(x\right)\right| \left|\int\limits_{E_n} G_y\left(x\right)\psi\left(y\right)dy\right| dx +$$

$$+\int\limits_{E_n} \left|\psi\left(x\right)\right|^{p-1} \left|\int\limits_{E_n} \left(L_x - L_y\right) G_y\left(x\right)\psi\left(y\right)dy\right| dx := I_1 + i_2$$

For I_1 on the base of the Holder inequality and Hardy-Sobolev estimation [11] for the potentials, we have:

$$I_1 \leq \left\|\Phi\right\|_{(p^*)'} \left\|\int\limits_{E_n} G_y\left(x\right)\psi\left(y\right)dy\right\|_{p^*} \leq C\left\|\Phi\right\|_{(p^*)'}\left\|\psi\right\|_p$$

By the estimation (8) for i_2 and previous inequality we'll obtain

$$\left\|\psi\right\|_p^p \leq C\left(\left\|a\right\|_{BMO}\left\|\psi\right\|_p^p + \left\|\psi\right\|_p\left\|\Phi\right\|_{(p^*)'}\right)$$

wherefrom if $\left\|a\right\|_{BMO}$ is sufficiently small, then

$$\left\|\psi\right\|_p \leq C\left(\left\|\Phi\right\|_{(p^*)'}\right)^{1/(p-1)}$$

By the given estimation we have:

$$\left|\int\limits_{E_n} u\left(x\right)\Phi\left(x\right)dx\right| = \left|\int\limits_{E_n} u\left(x\right)L^*\left(\psi\left(\psi\right)^{p-2}\right)dx\right| = \left|\int\limits_{E_n} \psi\left|\psi\right|^{p-2} Lu\left(x\right)dx\right| \leq$$

$$\leq \left\|\psi\right\|_p^{p-1}\left\|Lu\right\|_p \leq C\left\|\Phi\right\|_{(p^*)'}\left\|Lu\right\|_p$$

whence by the duality we get the estimation (11).

Theorem 1 is proved.

In the following theorems 2 and 3, the estimation (3) for the solution of problem (1), obtained in theorem 1, is applied to existence of the solution of elliptic equations in the whole of Euclidean space E_n^+.

Theorem 2. *Let $p \in (1, \infty)$ be an L be an operator of the form* (1) *with bounded measurable coefficients for which a semi-norm $\|a\|_{BMO}$ is sufficiently small. Then for any $f \in L_p(E_n)$ the problem* (1) *has a strong solution from the space $\dot{W}^{2,p}(E_n)$ and a priori estimation* (3) *is valid for its solution.*

Proof. To prove the existence of the solution of the problem (1) we consider a smooth approximation $a_{ij}^h(x)$ of the functions $a_{ij}(x)$. Let $u^h(x); h \in (0, h_0)$ be a family of classic solutions of the problem

$$L^h u^h = f^h \quad in \quad E_n, \quad \lim_{x \to \infty} u^h(x) = 0.$$

By the estimation (3) for the functions $u^h(x)$ we have:

$$\left\|u^h\right\|_{2,p} \leq C \left\|f^h\right\|_p .$$

We can assume $\left\|f^h\right\|_p \leq 2\|f\|_p$. Then $\left\|u^h\right\|_{2,p} \leq C$, therefore $\exists u^{h_k} \to u$ weakly in $W^{2,p}(E_n)$ for some $u \in W^{2,p}(E_n)$. Tending $h_k \to 0$ in the equality

$$\int_{E_n} \varphi L^{h_k} u^{h_k} dx = \int_{E_n} \varphi f^{h_k} dx, \quad \forall \varphi \in C^\infty$$

we get

$$\int_{E_n} \varphi L u dx = \int_{E_n} \varphi f dx, \quad \forall \varphi \in C^\infty(E_n).$$

Then

$$Lu = f \quad p.v. \;\; E_n .$$

It follows from the convergence $u^{h_k} \to u$ that $\lim\limits_{k \to \infty} \inf \left\|u^{h_k}\right\|_{2,p} \geq \|u\|_{2,p}$, that

$$\|u\|_{2,p} \leq C \|f\|_p$$

The following theorem is well-known (see [9-13]), we get its proof by our methods.

Theorem 3. *Let $p \in (1, \infty)$ be bounded measurable coefficients $a_{ij}(x) \in BMO$, then the problem* (1) *is solvable in the space $\dot{W}^{2,p}(E_n)$ and for the solution $u(x)$ the estimation*

$$\|u\|_{2,p} \leq C \|f\|_p \tag{12}$$

is valid.

Proof. Cover the domain E_n by balls $\{B_v\}$ of radius $r > 0$ of finite multiplicity. Let $\{\eta_v\}$ be expansion of a unit subjected to this covering i.e. $\eta_v \in C_0^\infty(B_v)$, sup p $\eta_v \in B_v$ and $\sum\limits_v \eta_v \equiv 1$ in E_n . Then we select r so sufficiently small that the estimation (3) be valid. Then

$$\|u\|_{2,p;D} \leq \sum_v \|u_v\|_{2,p;B_v} \leq C \sum_v \|Lu_v\|_{p;B_v} \leq$$

$$\leq C \left[\sum_v \left(\|Lu\|_{p;B_v} + (1/r) \|u_x\|_{p;B_v} + (1/r^2) \|u\|_{p;B_v} \right) \right] \leq$$

$$\leq C(r,n,p)\left(\|Lu\|_p + \|u\|_p\right),$$

where $u_v = u\eta_v$. Notice that in the last inequality we used interpolational inequality [11] :

$$\|u_x\|_p \leq \varepsilon \|u\|_{2,p} + C(\varepsilon)\|u\|_p .$$

As a result, the estimation

$$\|u\|_{2,p} \leq C\left(\|Lu\|_p + \|u\|_p\right)$$

is true.

In this inequality we assume $u(\lambda x)$ instead of $u(x)$ and get the estimation

$$\|u\|_{2,p} \leq C\left(\|Lu\|_p + \lambda^{-2}\|u\|_p\right), \tag{13}$$

whence tending $\lambda \to \infty$ we get

$$\|u\|_{2,p} \leq C\|Lu\|_p$$

Further, all the reasonings of theorem 2 related with smooth approximation are repeated.

Theorem 3 is proved.

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