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APPROXIMATION OF FUNCTIONS BY FABER-LAURENT RATIONAL FUNCTIONS ON CLOSED QUASICONFORMAL CURVES

Abstract

The present paper deals with the problem on approximation of arbitrary continuous functions given on a finite Jordon curve Γ *by Faber - Laurent rational functions.*

1. Introduction and main result

In the paper [1] approximation of function by partial sums of a series by Faber polynomials was studied for some domains with piecewise-smooth boundaries. By F.D.Lesley, V.S.Vinge and S.E.Warschawski [10] the obtained results were extended to the domains whose boundaries is a Jordan rectifiable curve of the same order length of arch and span. In V.V.Andrievskii's paper [2] the similar problems were studied for continua whose boundaries may be non-Jordan and inrectifiable in any of its part.

Similar problems were also studied in integral metric. The problems of approximation by Faber polynomials and Faber-Laurent rational functions in integral metric were investigated in the papers [4], [7], [8], [9]. More extensive knowledge about them can be found in [5, pp. 40-57] and [14, pp. 52-236].

The problem on approximation of continuous functions given on Jordan rectifiable curves of the same order of length of arc and chord by Faber - Laurent rational functions is studied in [13]. In the present paper the similar problems are considered for quasiconformal curves that may be rectifiable at any of its part.

Let Γ be an arbitrary bounded Jordan curve with two-component complements $G = C\Gamma = G_1 \cup G_2$ $(0 \in G_1,\ \infty \in G_2)$. Let's consider the function $w = \Phi_i(z)$ $(i = 1, 2)$ that conformally and invalently map G_1 and G_2 onto the exterior of a unit circle with normalization $\Phi_1(0) = \infty$, $\lim\limits_{z\to 0} z\Phi_1(z) > 0$, $\Phi_2(\infty) = \infty$ and $\lim\limits_{z\to\infty} \Phi_2(z)/z > 0$.

By U_1 and U_2 we denote the interior and exterior of a unit circle. We denote the boundary of a unit circle by T.

The function inverse to $\Phi_i(z)$ we denote by $z = \Psi_i(w)$ $(i = 1, 2)$. Because of Caratheodory's theorem, see for example [12 , pp. 44] the functions Φ_1 and Φ_2 (Ψ_1 and Ψ_2) have continuous extensions to $\Gamma(T)$.

We'll also use the symbol $A \preceq B$ that means $A \le CB$, where $C = const > 0$ is independent on A and B, and $A \asymp B$ if simultaneously $A \preceq B$ and $B \preceq A$.

In the present paper we'll be under in the case when Γ is a quasiconfomal curve. The convenient geometrical quasiconformality of the curve is the following (see [11, p. 100]).

Let's consider a Jordan curve Γ and two arbitrary points z_1 and, z_2 on it. By $\Gamma(z_1, z_2)$ we denote one of the two curves (with less diameter) on which the points z_1 and z_2 divide the curve Γ.

The feasibility of the relation

$$diam\,\Gamma(z_1, z_2) \preceq |z_1 - z_2|$$

is the necessary and sufficient condition for the quasconformality of the curve Γ. As P.P.Belinskii's example shows (see [3, p. 42]) in quasiconformal curve may be unrectifiable at any of its pats.

We denote by $C(\Gamma)$ the set of all continuous (complex - valued) functions on Γ.

The main result of the given paper is the following theorem

Theorem 1. *Let Γ be a closed quasiconformal curve, $f(z) \in C(\Gamma)$, $0 < \alpha \leq 1$. Then*

$$|f(z) - R_n(f,z)| \preceq E_n(f,\Gamma)\left[(n+1)^{1-\alpha} + \ln(n+2)\right]\ln(n+2),$$

where $R_n(f,z)$-is the Faber-Laurent rational function of degree n of f. $E_n(f,\Gamma)$ is the best uniform approximation of the function $f(z)$ on Γ by rational functions of degree n of f.

2. Auxiliary results

If $f \in C(\Gamma)$, then we associate its generalized Faber - Laurent series

$$f(z) \sim \sum_{k=0}^{\infty} a_k(f) F_k(z) + \sum_{k=0}^{\infty} (-\widetilde{a}_k(f)) \widetilde{F}_k(1/z), \tag{1}$$

where the coefficients $a_k(f)$. and $\widetilde{a}_k(f)$ are defined by

$$a_k(f) := \frac{1}{2\pi i}\int\limits_T \frac{f[\Psi_2(w)]}{w^{k+1}}dw, \quad k = 0, 1, 2, \cdots. \tag{2}$$

and

$$\widetilde{a}_k(f) := \frac{1}{2\pi i}\int\limits_T \frac{f[\Psi_1(w)]}{w^{k+1}}dw, \quad k = 0, 1, 2, \cdots. \tag{3}$$

We call the coefficients $a_k(f)$ and $\widetilde{a}_k(f)$ the Faber - Laurent coefficients of $f \in C(\Gamma)$. The polynomial $F_k(z)$ is called the Faber polynomial of degree k for the curve Γ :

$$F_k(z) = \frac{1}{2\pi i}\int\limits_{\Gamma_R} \frac{[\Phi_2(\zeta)]^k}{\zeta - z}d\zeta, \tag{4}$$

where

$$\Gamma_R := \{\zeta \in G_2 : |\Phi_2(\zeta)| = R\}.$$

The rational function $\widetilde{F}_k(1/z)$ is said to be the Faber principle part of degree k for the curve Γ :

$$\widetilde{F}_k(1/z) = -\frac{1}{2\pi i}\int\limits_{\widetilde{\Gamma}_R} \frac{[\Phi_1(\zeta)]^k}{\zeta - z}d\zeta \tag{5}$$

where

$$\widetilde{\Gamma}_R := \{\zeta \in G_1 : |\Phi_1(\zeta)| = R\}.$$

Let us formulate one result of V.V. Andrievskii (see [2, lemma 1]) in insignificantly changed form.

Lemma 1. *Let Γ be a closed quasiconformal curve, $0 < \alpha \leq 1$, $z \in \Gamma$. Then*

$$\int\limits_{\widetilde{\Gamma}_{1+\frac{1}{n}}} \frac{\left|d\widetilde{\zeta}\right|}{\left|\widetilde{\zeta} - z\right|} \leq (n+1)^{1-\alpha} + \ln(n+2),$$

and

$$\int\limits_{\Gamma_{1+\frac{1}{n}}} \frac{\left|d\widetilde{\zeta}\right|}{\left|\widetilde{\zeta} - z\right|} \leq (n+1)^{1-\alpha} + \ln(n+2),$$

where

$$\widetilde{\Gamma}_{1+\frac{1}{n}} = \left\{\widetilde{\zeta} : \left|\Phi_1\left(\widetilde{\zeta}\right)\right| = 1 + \frac{1}{n}\right\},$$

$$\Gamma_{1+\frac{1}{n}} = \left\{\widetilde{\zeta} : \left|\Phi_2\left(\widetilde{\zeta}\right)\right| = 1 + \frac{1}{n}\right\}.$$

Definition 1. *Let $f \in C(\Gamma)$ and $a_k(f)$, $\widetilde{a}_k(f)$ be its Faber - Laurent coefficients. Then the rational function*

$$R_n(f,z) := \sum_{k=0}^{n} a_k(f) F_k(z) + \sum_{k=0}^{n} -\widetilde{a}_k(f) \widetilde{F}_k(1/z)$$

is called the Faber - Laurent rational function of degree n of f.

3. Proof of new result

By the paper [2] there exists a finite number of quasiconformal arcs Γ_j $\left(j = \overline{1,k}\right)$ covering the curve Γ.

Let $\Gamma^{(j)}_{1+\frac{1}{n}} = \Gamma_{1+\frac{1}{n}}(\Gamma_j)$, $j = \overline{1,k}$ $\Big($here $\Gamma_{1+\frac{1}{n}}(\Gamma_j)$, $j = \overline{1,k}$ is the level line of the arc $\Gamma_j, j = \overline{1,k}\Big)$. From the arcs lying on $\Gamma^{(j)}_{1+\frac{1}{n}} \cap G_i$ $\left(j = \overline{1,k},\ \ i = 1,2\right)$ compose a closed curve $\Gamma^{(i)}$, $i = 1,2$. Obviously, $\Gamma^{(1)} \subset ext\widetilde{\Gamma}_{1+\frac{1}{n}}$, $\Gamma^{(2)} \subset int\Gamma_{1+\frac{1}{n}}$.

Thus, using lemma 1 after simple calculations we get

$$\int\limits_{\Gamma^{(i)}} \frac{|d\zeta|}{|\zeta - z|} = \sum_{j=1}^{k} \int\limits_{\Gamma^{(j)}_{1+\frac{1}{n}} \cap \Gamma^{(i)}} \frac{|d\zeta|}{|\zeta - z|} \leq (n+1)^{1-\alpha} + \ln(n+2), \quad i = 1,2. \tag{6}$$

By (4), (5) the following representations are valid for Faber polynomials $F_k(z)$ and rational functions $\widetilde{F}_k(1/z)$

$$F_k(z) = \frac{1}{2\pi i} \int\limits_{\Gamma^{(2)}} \frac{[\Phi_2(\zeta)]^k}{\zeta - z} d\zeta \tag{7}$$

$$\widetilde{\Gamma}_k(1/z) = -\frac{1}{2\pi i}\int\limits_{\Gamma^{(1)}} \frac{[\Phi_1(\zeta)]^k}{\zeta - z}\, d\zeta. \qquad (8)$$

Consequently, by (6) and (7) we have

$$|F_k(z)| \le \frac{(1+1/n)^n}{2\pi}\int\limits_{\Gamma^{(2)}} \frac{|d\zeta|}{|\zeta - z|} \le (n+1)^{1-\alpha} + \ln(n+2).$$

Similar to previous one by (6) and (8) the following inequality is true

$$\left|\widetilde{F}_k(1/z)\right| \le (n+1)^{1-\alpha} + \ln(n+2).$$

If the coefficients $a_k(f)$, $\widetilde{a}_k(f)$ are defined by formulae (2), (3) then as shown in the paper [1] the following inequalities hold

$$\left|\sum_{k=0}^{n} a_k(f)\, w^k\right| \le \ln(n+2)\max_{z\in\Gamma}|f(z)|, \quad |w| = 1, \qquad (9)$$

$$\left|\sum_{k=0}^{n} \widetilde{a}_k(f)\, w^k\right| \le \ln(n+2)\max_{z\in\Gamma}|f(z)|, \quad |w| = 1, \qquad (10)$$

According to well known S.N.Bernshteyn theorem (see example; [15, p.26]), (9), (10) from these conditions we have the evaluations following

$$\left|\sum_{k=0}^{n} a_k(f)\,[\Phi_2(z)]^k\right| \le \ln(n+2)\max_{z\in\Gamma}|f(z)|, \quad z\in\Gamma^{(2)}, \qquad (11)$$

$$\left|\sum_{k=0}^{n} \widetilde{a}_k(f)\,[\Phi_1(z)]^k\right| \le \ln(n+2)\max_{z\in\Gamma}|f(z)|, \quad z\in\Gamma^{(1)}, \qquad (12)$$

Using the inequalities (6), (11), (12) and taking into account the relations (7), (8) we have

$$\left|\sum_{k=0}^{n} a_k(f)\,F_k(z) + \sum_{k=0}^{n}(-\widetilde{a}_k(f))\,\widetilde{F}_k(1/z)\right| =$$

$$= \left|\frac{1}{2\pi i}\int\limits_{\Gamma^{(2)}} \frac{d\zeta}{\zeta - z}\sum_{k=0}^{n} a_k(f)\,[\Phi_2(\zeta)]^k + \right.$$

$$\left. + \frac{1}{2\pi i}\int\limits_{\Gamma^{(1)}} \frac{d\zeta}{\zeta - z}\sum_{k=0}^{n} \widetilde{a}_k(f)\,[\Phi_1(\zeta)]^k\right| \le$$

$$\le \frac{1}{2\pi}\left|\int\limits_{\Gamma^{(2)}} \frac{d\zeta}{\zeta - z}\sum_{k=0}^{n} a_k(f)\,[\Phi_2(\zeta)]^k\right| +$$

$$+\frac{1}{2\pi}\left|\int\limits_{\Gamma^{(1)}}\frac{d\zeta}{\zeta-z}\sum_{k=0}^{n}\widetilde{a}_k\left(f\right)\left[\Phi_1\left(\zeta\right)\right]^k\right|\leq$$

$$\leq\left[\left(n+1\right)^{1-\alpha}+\ln\left(n+2\right)\right]\ln\left(n+2\right)\max_{z\in\Gamma}\left|f\left(z\right)\right|.$$

Let $R_n\left(z\right)$ be a rational function of the n. th degree of the best uniform approximation of the function $f\left(z\right)$ on Γ. Obviously, $R_n\left(R_n;z\right)=R_n\left(z\right)$. Then

$$\left|f\left(z\right)-R_n\left(f,z\right)\right|\leq\left|f\left(z\right)-R_n\left(z\right)\right|+\left|R_n\left(f-R_n;z\right)\right|\preceq$$

$$\preceq E_n\left(f;\Gamma\right)\left[\left(n+1\right)^{1-\alpha}+\ln\left(n+2\right)\right]\ln\left(n+2\right).$$

That is, the theorem 1 is proved.

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