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BLOW-UP SOLUTIONS TO NONLNEAR PARABOLIC EQUATIONS

Abstract

In the paper we consider infinitely growing for finite time solutions to nonlinear parabolic equations. In certain conditions such equations describe the processes of electronic and ionic heat conduction in plasm, diffusion of neutrons, alpha-particles and etc.

Investigations of infinite solutions or aggravation regimes occupy a special place in the theory of nonlinear equations.

One of the principal concepts in the theory of nonlinear evolutionary equations are notion on so called eigen functions of nonlinear dissipative medium.

It is known well that even in the very simple cases of nonlinearity, depening on the critical exponent, the solutions of a nonlinear parabolic equation may infinitely grow for finite time ("blow-up") i.e. there exists such $T > 0$ that

$$\|u(x,t)\|_{L_\infty(R^n)} \to \infty, \qquad t \to T < \infty.$$

For example, in the paper [1] in the simpliest case of nonlinearity the existence of infinitely growing solution for finite time is proved. In the paper [2] it is shown that depending on the critical exponent any non-negative solution blows-up for finite time.

Such results were obtained also in the papers [3] and appropriate theorems are called Fujita and Fujita-Hayakawa type theorems as well.

One can find more detailed review in the papers [6],[4],[5].

We consider the equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{d}{dx_j}\left(\left|\frac{\partial u}{\partial x_i}\right|^{m-2}\frac{\partial u}{\partial x_i}\right) + f(x,t,u) \tag{1}$$

in the bounded domain $\Omega \subset R^n$, $n \geq 2$ with nonsmooth boundary, i.e. the boundary $\partial\Omega$ contains conical points with opening of an angle $\omega \in (0,\pi)$. Denote $\Pi_{a,b} = \{(x,t) : x \in \Omega, a < t < b\}, \Gamma_{a,b} = \{(x,t) : x \in \partial\Omega, a < t < b\}, \Pi_a = \Pi_{a,\infty}, \Gamma_a = \Gamma_{a,\infty}$.

The functions $f(x,t,u), \dfrac{\partial f(x,t,u)}{\partial u}$ are continuous with respect to u and uniform in $\overline{\Pi}_0 \times \{u : |u| \leq M\}$ for any $M < \infty$, $f\ (x,t,0) \equiv 0$, $\left.\dfrac{\partial f}{\partial u}\right|_{u=0} \equiv 0$. Besides, the function f is measurable with respect to all arguments and doesn't decrease with respect to u. We consider Dirichlet's boundary condition

$$u = 0, \quad x \in \partial\Omega \tag{2}$$

and initial condition

$$u|_{t=0} = \varphi(x) \tag{3}$$

in some domain $\Pi_{0,a}$ where $\varphi(x)$ is a smooth function. As a solution we understand generalized solution in standard definition from the Sobolev space $W_m^1(\Pi_{0,\acute{a}})$ for all $a < \acute{a}$. The solution of the problem (1)-(3) exists either in Π_0, or

$$\lim_{t\to T-0}\max_{\Omega}|u(x,t)| = +\infty \tag{4}$$

for some $T = const$.

In the papers [6] and [7] where the principal part of the equation (1) is linear, it is shown that if $f(x,t,u) = |u|^{q-1}u, q = const > 1, \varphi \geq 0, \varphi(x) \not\equiv 0, u(x,t)$ is a solution to Neumann or Dircihlet problem, then it holds (4). The existence of the solution for some φ is shown in [7].

In the given paper, it is proved that if $u(x,t)$ is a solution of the problem (1)-(3) in Π_0, $f(x,t,u) = |u|^{q-1}u, q > 1$ and

$$\lim_{t\to+\infty} u(x,t) = 0 \tag{5}$$

then

$$|u| < Ce^{\alpha t}, \alpha = const > 0, \tag{6}$$

where α is independent of u.

It is also proved that if $f(x,t,u) = |u|^{q-1}u, u(x,t)$ is a solution of the problem (1)-(3) where $\varphi(x) \geq 0$ and "is not very small", it holds (4), i.e."blow-up". Otherwise, if $(|\varphi(x)|)$ is small, it is valid (5),(6).

Sufficient conditions are established on $f(x,t,u)$ for which any solution of the problem (1)-(3) for $\varphi \geq 0, \varphi(x) \neq 0$ has "a blow-up" (without restriction of smallness on $\varphi(x)$).

Let's formulate some auxiliary results from [8],[9]. To this end we determine p harmonic operator $L_p u = div\left(|\nabla u|^{p-2}\nabla u\right), p > 1$.

Lemma 1. ([8]). *There exists a positive eigen value of the spectral problem for the operator L_p, to which an eigen function positive in Ω corresponds.*

Lemma 2. ([9]). *Let $u, v \in W_p^1(\Omega), u \leq v$ on $\partial\Omega$ and*

$$\int_{\Omega} L_p(u)\,\eta_{xi}dx \int_{\Omega} L_p(v)\,\eta_{xi}dx$$

for any $\eta \in \overset{\circ}{W}{}_p^1(\Omega)$, with $\eta \geq 0$. Then $u \leq v$ in all the domain Ω.

Determine generalized solution of the problem (1)-(3) in $\Pi_{a,b}$ with boundary $\Gamma_{a,b}$. This function $u(x,t) \in W_p^1(\Pi_{a,b}), u(x,t) \in L_\infty(\Pi_{a,b})$ is such that

$$\int_{\Pi_{a,b}} \psi\frac{\partial u}{\partial t}dxdt + \sum_{i,j=1}^{n}\int_{\Pi_{a,b}}\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxdt =$$

$$= \int_{\Pi_{a,b}} f(x,t,u)\,\psi(x,t)\,dxdt, \tag{7}$$

where $\psi(x,t)$ is an arbitrary function from $W_p^1(\Pi_{a,b})$, $\psi|_{\Gamma_{a,b}} = 0$, $0 < a < b$ are any numbers.

Let $u_0(x) > 0$ be an eigen function of the spectral problem for the operator L_p corresponding to $\lambda = \lambda_1 > 0$, and $\int\limits_\Omega u_0(x)\,dx = 1$.

Assume that the following condition is fulfilled

$$(L_p u_0, u) \geq (L_p\ u, u_0)\,. \qquad (A)$$

Condition (A) means that

$$\int\limits_\Omega \left(\left| \frac{\partial u_0}{\partial x_i} \right|^{p-2} - \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \right) \frac{\partial u}{\partial x_i} \frac{\partial u_0}{\partial x_i} dx \geq 0.$$

Theorem 1. *Let $f(x,t,u) \geq a_0 |u|^{\sigma-1} u$ for $(x,t) \in \Pi_0, u \geq 0$ where $\sigma = const > 1, a_0 = const > 0$. There exists such $k = const > 0$ that if $u(x,0) \geq 0$, $\int\limits_\Omega u(x,0)\, u_0 dx \geq k$ and condition (A) is fulfilled, then $\lim\limits_{t\to T-0} \max\limits_\Omega u(x,t) = \infty$, where $T = const > 0$.*

Proof. Let's assume the contrary. Then $u(x,t)$ is a solution of the equation (1) in Π_0 and condition (2) is fulfilled on Γ_0. By lemma 2 $u(x,t) > 0$ in Π_0. Assume in (7) $\psi = \varepsilon^{-1} u_0(x)\,,\ b = a + \varepsilon,\ a > 0,\ \varepsilon > 0$, where $u_0(x) > 0$ in Ω is an eigen function of the spectral problem for the operator L_p, corresponding to the eigen value $\lambda_1 > 0$. Such an eigen value and eigen function exist by lemma 1.

As the result we have

$$\varepsilon^{-1} \left[\int\limits_\Omega u_0(x)\, u(x, a+\varepsilon)\, dx - \int\limits_\Omega u_0(x)\, u(x,a)\, dx \right] +$$

$$+\varepsilon^{-1} \lambda_1 \int\limits_{\Pi_{a.a+\varepsilon}} u(x,t)\, u_0(x)\, dxdt = \varepsilon^{-1} \times$$

$$\times \int\limits_{\Pi_{a.a+\varepsilon}} u_0 f(x,t,u)\, dxdt_1 + \varepsilon^{-1} \int\limits_{\Pi_{a.a+\varepsilon}} \left(\left| \frac{\partial u_0}{\partial x_i} \right|^{p-2} - \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \right) \frac{\partial u}{\partial x_i} \frac{\partial u_0}{\partial x_i} dxdt.$$

In this equation, we tend ε to zero and get that for almost all $t > 0$

$$\frac{\partial}{\partial t} \int\limits_\Omega u_0(x)\, u(x,t)\, dx = -\lambda_1 \int\limits_\Omega u_0(x)\, u(x,t)\, dx + \int\limits_\Omega u_0 f(x,t,u)\, dx.$$

Hence, having denoted

$$g(t) = \int\limits_\Omega u_0(x)\, u(x,t)\, dx$$

we have $g'(t) \geq -\lambda_1 g + a_0 \int\limits_\Omega u_0(x)\, u(x,t)\, dx$. By the Holder inequality

$$\left(\int\limits_\Omega u(x,t)\, u_0 dx \right)^\sigma \leq \left[\left(\int\limits_\Omega u^\sigma u_0 dx \right)^{\frac{1}{\sigma}} \left(\int\limits_\Omega u_0 dx \right)^{\frac{\sigma-1}{\sigma}} \right]^\sigma \leq C_1 \int\limits_\Omega u^\sigma u_0 dx.$$

Then

$$g'(t) \geq -\lambda_1 g(t) + C g^{\sigma}(t), \; C = const > 0. \tag{8}$$

If $g(0) > C_2 = \left(\frac{\lambda_1}{C}\right)^{1/\sigma}$, then from (8) we get $\lim\limits_{t\to T-0} g(t) = +\infty$. It means that $\lim\limits_{t\to T-0} \max\limits_{\Omega} u(x,t) = \infty$. The theorem is proved.

Thus, equation (1) has no solutions in Π_0 satisfying the boundary condition (2), if $u(x,0) \geq 0$ is not very small. Now, show that if $|u(x,0)|$ is small, the solution of the problem (1),(2),(3) exists on all the domain Π_0 .

Theorem 2. *Let $|f(x,t,u)| \leq (C_1 + C_2 t^m)|u|^{\sigma}, \sigma > 1$. There exists $\delta > 0$ such that if $|\varphi(x)| \leq \delta$, then the solution of the problem (1)-(3) exists in Π_0 and $|u(x,t)| \leq Ce^{-\alpha t}, \alpha = const > 0$ independent of n.*

Proof. Let $\overline{\Omega} \subset B_R$, where $B_R = \{x : |x| \leq R\}$. Let $\upsilon > 0$ in B_R is an eigen function corresponding to positive eigen value λ_1 of the boundary value problem

$$L_m u + \lambda u = 0, \; x \in \Omega, \; u = 0, \; x \in \partial\Omega. \tag{9}$$

Let's consider the function $V(x,t) = \varepsilon e^{-\lambda_1 t/2} \upsilon(x)$. We have

$$V_t - L_m V - f(x,t,V) = \frac{1}{2}\varepsilon\lambda_1 e^{-\lambda_1 t/2}\upsilon(x) - (C_1 + C_2 t^m)\varepsilon^{\sigma} e^{-\lambda_1 t/2}\upsilon^{\sigma} \geq 0,$$

$$(x,t) \in \Pi_0 \text{ and } V > 0, \; (x,t) \in \Gamma_0 \tag{10}$$

if $\varepsilon > 0$ is sufficiently small. The inequality (10) is understood in the weak sense (see[10]).

It follows from (10) and lemma 2 that $|u| \leq V \leq Ce^{-\lambda_1 t/2}$, if $|\varphi(x)| \leq \delta = \varepsilon \min\limits_{\Omega} \upsilon(x)$. Now, let's define a class of functions K consisting of $\omega(x,t)$ continuous in $\overline{\Pi}_{-\infty,+\infty}$ equal to zero for $t \leq T$, and such that $|\omega| \leq Ke^{-ht}$. K is a subset of Banach space of continuous in $\overline{\Pi}_{-\infty,+\infty}$ functions with the norm $\|\omega\| = \sup\limits_{\Pi_{-\infty,+\infty}} |\omega e^{ht}|$.

Let $\theta(t) \in C^{\infty}(R^1), \; \theta(t) = 0, \; t \leq T, \; \theta(t) = 1, \; t > T+1$. Determine the operator H on K, having assumed $H\omega = \theta(t) z, \; \omega \in K$. By the estimation obtained above, H transfers K into K, if T is sufficiently great. The operator H is completely continuous. It follows from the obtained estimation and theorem on Holder property of solutions of parabolic problems in $\Pi_{-a,a}$ for any a ([10]). It follows from Leray-Schauder theorem that the operator H has a fixed point z.

This indicates the existence of the solution.

The theorem is proved.

It follows from theorem 2 that if $u(x,0) \geq 0, \; |u(x,0)| \leq \delta$, the solution of the problem (1)-(3) exists in Π_0 and is positive in Π_0 by lemma 2.

Write sufficient conditions under which all non-negative solutions of the problem (1)-(3) have "blow-up", i.e.

$$\lim_{t\to T-0} \max_{\Omega} u(x,t) = +\infty, \tag{11}$$

where $T = const > 0$.

Theorem 3. *Let $f(x,t,u) \geq Ce^{\lambda_1 \sigma t} u^{\sigma}$ for $(x,t) \in \Pi_0, u \geq 0, \; \sigma = const > 1, \lambda_1$ be positive eigen value of the problem (9) in Ω, to which positive eigen function* *in* Ω *corresponds. If* $u(x,0) \geq 0,\ u(x,0) \not\equiv 0,$ *where* $u(x,t)$ *is a solution of the problem (1)-(3), it holds (11).*

Proof. Similar to the establishment of the inequality (8) we get

$$g'(t) \geq -\lambda_1\, g + Ce^{\lambda_1 \sigma t} g^{\sigma}(t)\,, \qquad (12)$$

where, $g(t) = \int\limits_{\Omega} u_0(x)\, u(x,t)\, dx.$

Let $g(t) = \psi(t)\, e^{-\lambda_1 \cdot t}$. It follows from (12) that $\psi' \geq C\psi^{\sigma}$. Hence $\psi(t) \to +\infty$ as $t \to T-0$ and consequently $\max\limits_{\overline{\Omega}} u(x,t)$ also tends to infinity.

The theorem is proved.

Using theorem 3 we can prove the following property of the solutions to equation (1).

Corollary 1. *Let* $f(x,t,u) \geq Ce^{\lambda_1 \sigma t} u^{\sigma}$ *for* $(x,t) \in \Pi_0, u \geq 0,$ *where* $\sigma =$ *const* $> 1, \lambda_1$ *is a positive eigen value of the problem (9) to which positive in* Ω *eigen function corresponds. Then there are positive solutions of equation (1) in* Π_0.

Remark. The above-mentioned results are also true for the case of more wide class of equations of the following form

$$\frac{\partial u}{\partial t} \sum_{i=1}^{n} \frac{d}{dx_i} a_i(x,u,u_x) + f(x,t,u) \qquad (13)$$

under certain conditions on the coefficients $a_i(x,u,u_x)$.

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