Mehriban N. OMAROVA

SOME PROPERTIES OF THE FRACTIONAL INTEGRALS ON THE LAGUERRE HYPERGROUP

Abstract

Let $\mathbb{K} = [0, \infty) \times \mathbb{R}$ be the Laguerre hypergroup which is the fundamental manifold of the radial function space for the Heisenberg group. In this paper we obtain pointwise estimates for the fractional integrals in terms of maximal and fractional maximal functions on the Laguerre hypergroup. On the basis of these results the analogue Sobolev's theorem for the fractional integrals on the Laguerre hypergroup is proved.

1. Introduction

In this paper we define fractional maximal function and fractional integrals on Laguerre hypergroup which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group (see, for example [2], [4]-[10]). We study the pointwise and integral estimates for the fractional integrals in terms of maximal functions and fractional maximal functions on the Laguerre hypergroup. On the basis of these results the analogue of Sobolev's theorem for the fractional integrals on the Laguerre hypergroup is proved.

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In Section 3, we give the main results such as Sobolev's theorem for the fractional integrals on the Laguerre hypergroup.

2. Preliminaries

Let m_α be the weighted Lebesgue measure on $\mathbb{K} = [0, \infty) \times \mathbb{R}$, given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}dxdt}{\pi\Gamma(\alpha+1)}, \quad \alpha \geq 0.$$

We denote by $L_p(\mathbb{K}) = L_p(\mathbb{K}; dm_\alpha)$ the spaces of complex-valued functions f, measurable on $\mathbb{K}$ such that

$$\|f\|_{L_p(\mathbb{K})} = \left(\int\limits_{\mathbb{K}} |f(x,t)|^p \, dm_\alpha(x,t) \right)^{1/p} < \infty \quad \text{if} \quad p \in [1, \infty),$$

and

$$\|f\|_{L_\infty(\mathbb{K})} = ess \sup_{(x,t)\in\mathbb{K}} |f(x,t)| \quad \text{if} \quad p = \infty.$$

For $1 \leq p < \infty$ we denote by $WL_p(\mathbb{K})$, the weak $L_p(\mathbb{K})$ spaces defined as the set of locally integrable functions f with the finite norm

$$\|f\|_{WL_p(\mathbb{K})} = \sup_{r>0} r \left(m_\alpha \left\{(x,t) \in \mathbb{K} \ : \ |f(x,t)| > r\right\}\right)^{1/p}.$$

Let $|(x,t)|_{\mathbb{K}} = (x^4 + 4t^2)^{1/4}$ be the homogeneous norm of $(x,t) \in \mathbb{K}$. For $r > 0$ we will denote by $\delta_r(x,t) = (rx, r^2 t)$ the dilation of $(x,t) \in \mathbb{K}$, and by $B_r(x,t)$ the ball centered at (x,t) with radius r, i.e., the set of $B_r(x,t) = \{(y,s) \in \mathbb{K} : |(x-y, t-s)|_{\mathbb{K}} < r\}$, and by B_r the ball $B_r(0,0)$.

We denote by

$$f_r(x,t) = r^{-(2\alpha+4)} f\left(\delta_{\frac{1}{r}}(x,t)\right)$$

the dilated of the function f defined on $\mathbb{K}$ preserving the mean of f with respect to the measure dm_α, in the sense that

$$\int_{\mathbb{K}} f_r(x,t) dm_\alpha(x,t) = \int_{\mathbb{K}} f(x,t) dm_\alpha(x,t), \ \ \forall r > 0 \text{ and } \ f \in L_1(\mathbb{K}).$$

For $(x,t), (y,s) \in \mathbb{K}$ and $\theta \in [0, 2\pi[$, $r \in [0,1]$ let

$$((x,t),(y,s))_{\theta,r} = \left(\left(x^2 + y^2 + 2xyr\cos\theta\right)^{1/2}, t + s + xyr\sin\theta\right).$$

The generalized translation operator $T^{(\alpha)}_{(x,t)}$ defined on the Laguerre hypergroup is given for a suitable function f by

$$T^{(\alpha)}_{(x,t)} f(y,s) = \begin{cases} \frac{1}{2\pi}\int\limits_0^{2\pi} f\left(((x,t),(y,s))_{\theta,1}\right) d\theta, & \text{if } \alpha = 0, \\ \frac{\alpha}{\pi}\int\limits_0^1 \left(\int\limits_0^{2\pi} f\left(((x,t),(y,s))_{\theta,r}\right) d\theta\right) r(1-r^2)^{\alpha-1} dr, & \text{if } \alpha > 0. \end{cases}$$

$$\left\|T^{(\alpha)}_{(x,t)} f(y,s)\right\|_{L_p(\mathbb{K})} \leq \|f\|_{L_p(\mathbb{K})} \tag{1}$$

(see for example [8]).

Let $\Sigma = \Sigma_2$ be the unit sphere in $\mathbb{K}$. We denote by ω_2 the surface area of Σ and by Ω_2 its volume (see [3], [4]). For $\xi = (x,t) \in \mathbb{K}$, consider the transformation given by

$$x = r(\cos\varphi)^{1/2}, \ t = r^2 \sin\varphi,$$

where $-\pi/2 \leq \varphi \leq \pi/2$, $r = |\xi|_{\mathbb{K}}$ and $\xi' = ((\cos\varphi)^{1/2}, \sin\varphi) \in \Sigma$.

The Jacobian of the above transformation is $r^{2\alpha+3}(\cos\varphi)^\alpha$. If f is integrable in $\mathbb{K}$, then

$$\int_{\mathbb{K}} f(x,t)\, dm_\alpha(x,t)$$

$$= \frac{1}{2\pi\Gamma(\alpha+1)} \int\limits_{-\pi/2}^{\pi/2} \int\limits_0^\infty f\left(r(\cos\varphi)^{1/2}, r^2\sin\varphi\right) r^{2\alpha+3} (\cos\varphi)^\alpha dr d\varphi.$$

Since

$$\frac{1}{2\pi\Gamma(\alpha+1)}\int\limits_{-\pi/2}^{\pi/2}(\cos\varphi)^{\alpha}d\varphi=\int\limits_{\Sigma}d\xi',$$

we get

$$\int\limits_{\mathbb{K}}f(x,t)\,dm_{\alpha}(x,t)=\int\limits_{\Sigma}\int\limits_{0}^{\infty}r^{2\alpha+3}f(\delta_r\xi')\,drd\xi'. \qquad (2)$$

Here $d\xi'$ is the surface area element on Σ.

Lemma 1 [3], [4]. *The following equalities are valid*

$$\omega_2=\frac{\Gamma(\frac{\alpha+1}{2})}{2\sqrt{\pi}\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)},\quad \Omega_2=\frac{\Gamma(\frac{\alpha+1}{2})}{4\sqrt{\pi}(\alpha+2)\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}.$$

We define the fractional maximal function on the Laguerre hypergroup by

$$M_{\beta}f(x,t)=\sup_{r>0}\big(m_{\alpha}B_r\big)^{\frac{\beta}{2\alpha+4}-1}\int\limits_{B_r}T^{(\alpha)}_{(x,t)}|f(y,s)|\,dm_{\alpha}(y,s),\quad 0\le\beta<2\alpha+4$$

and the fractional integral by

$$I_{\beta}f(x,t)=\int\limits_{\mathbb{K}}T^{(\alpha)}_{(x,t)}|(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4}f(y,s)dm_{\alpha}(y,s),\quad 0<\beta<2\alpha+4.$$

If $\beta=0$, then $M\equiv M_0$ is the Hardy-Littlewood maximal operator on the Laguerre hypergroup (see [4]).

The following theorem is proved in [4].

Theorem 1. *1. If $f\in L_1(\mathbb{K})$, then $Mf\in WL_1(\mathbb{K})$ and*

$$\|Mf\|_{WL_1(\mathbb{K})}\le A_1\|f\|_{L_1(\mathbb{K})},$$

where $A_1>0$ is independent of f.

2. If $f\in L_p(\mathbb{K})$, $1<p\le\infty$, then $Mf\in L_p(\mathbb{K})$ and

$$\|Mf\|_{L_p(\mathbb{K})}\le A_p\|f\|_{L_p(\mathbb{K})},$$

where $A_p>0$ is independent of f.

Corollary 1. *If $f\in L_{loc}(\mathbb{K})$, then*

$$\lim_{r\to 0}\frac{1}{m_{\alpha}B_r}\int\limits_{B_r}\left|T^{(\alpha)}_{(x,t)}f(y,s)-f(x,t)\right|dm_{\alpha}(y,s)=0$$

for a.e. $(x,t)\in\mathbb{K}$.

3. Estimates of fractional integrals on the Laguerre hypergroup

We first prove a lemma in the following which is being pointwise estimate for fractional integrals $I_\beta f(x,t)$. Such type estimates are given in [1].

Lemma 2. *Let $0<\beta<2\alpha+4$, $1\le p<\frac{\lambda}{\beta}$. Then for any locally summable function f, and for every $r>0$ and $(x,t)\in\mathbb{K}$ the following inequality is valid*

$$I_\beta|f|(x,t)\le C_1\ r^\beta(Mf)(x,t)+C_2\ r^{\beta-\frac{\lambda}{p}}(M_{\frac{\lambda}{p}}f)(x,t), \tag{3}$$

where $C_1=\frac{\Omega_2 2^{2\alpha+4}}{2^\beta-1}$, $C_2=\frac{\Omega_2^{1-\frac{\lambda}{p(2\alpha+4)}}2^{2\alpha+4-\frac{\lambda}{p}}}{1-2^{\beta-\frac{\lambda}{p}}}$.

Proof. For any $r>0$ we have

$$I_\beta|f|(x,t)=\left(\int\limits_{B_r}+\int\limits_{\mathbb{K}\backslash B_r}\right)T^{(\alpha)}_{(x,t)}|f(y,s)|\ |(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4}dm_\alpha(y,s)=$$

$$:=J_1(x,t,r)+J_2(x,t,r).$$

Firstly we estimate $J_1(x,t,r)$. Summarizing on all $k>0$, we have

$$J_1(x,t,r)\le\int\limits_{B_r}T^{(\alpha)}_{(x,t)}\,|f(y,s)|\,|(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4}dm_\alpha(y,s)= \tag{4}$$

$$=\sum_{k=1}^{\infty}\int\limits_{B_{2^{-k+1}r}\backslash B_{2^{-k}r}}T^{(\alpha)}_{(x,t)}\,|f(y,s)|\,|(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4}dm_\alpha(y,s)\le$$

$$\le\sum_{k=1}^{\infty}\left(2^{-k}r\right)^{\beta-2\alpha-4}\int\limits_{B_{2^{-k+1}r}\backslash B_{2^{-k}r}}T^{(\alpha)}_{(x,t)}\,|f(y,s)|\,dm_\alpha(y,s)\le$$

$$\le\Omega_2 r^\beta Mf(x,t)\sum_{k=1}^{\infty}\left(2^{-k}\right)^{\beta-2\alpha-4}\left(2^{-k+1}\right)^{2\alpha+4}=$$

$$=\Omega_2 2^{2\alpha+4}r^\beta Mf(x,t)\sum_{k=1}^{\infty}2^{-k\beta}\le C_1 r^\beta Mf(x,t).$$

Therefore

$$J_1(x,t,r)\le C_1 r^\beta Mf(x,t). \tag{5}$$

Secondly, we estimate $J_2(x,t,r)$.

$$J_2(x,t,r)=\int\limits_{\mathbb{K}\backslash B_r}T^{(\alpha)}_{(x,t)}|f(y,s)|\ |(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4}dm_\alpha(y,s)\le$$

$$\le\sum_{k=0}^{\infty}\int\limits_{B_{2^{k+1}r}\backslash B_{2^k r}}T^{(\alpha)}_{(x,t)}|f(y,s)|\ |(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4}dm_\alpha(y,s)\le$$

$$\leq \sum_{k=0}^{\infty}(2^k r)^{\beta-2\alpha-4} \int\limits_{B_{2^{k+1}r}\setminus B_{2^k r}} T^{(\alpha)}_{(x,t)}|f(y,s)|dm_\alpha(y,s) \leq$$

$$\leq \Omega_2^{1-\frac{\lambda}{p(2\alpha+4)}} M_{\frac{\lambda}{p}} f(x,t) \sum_{k=0}^{\infty}(2^k r)^{\beta-2\alpha-4}(2^{k+1}r)^{2\alpha+4-\frac{\lambda}{p}} \leq C_2\, r^{\beta-\frac{\lambda}{p}}\, M_{\frac{\lambda}{p}} f(x,t),$$

where $\beta-\frac{\lambda}{p}<0$.

Therefore

$$J_2(x,t,r) \leq C_2 r^{\beta-\frac{\lambda}{p}} M_{\frac{\lambda}{p}} f(x,t). \tag{6}$$

Then from (5) and (6) we get the inequality (3). Therefore the proof of Lemma 2 is completed.

Theorem 2. *Let* $0<\beta<\lambda$, $1<p<\frac{\lambda}{\beta}$, $1\leq r\leq\infty$, *and* $\frac{1}{q}=\frac{1}{p}-\frac{\beta}{\lambda}+\frac{\beta p}{\lambda r}$. *Then for any* $f\in L_p(\mathbb{K})$ *and* $M_{\frac{\lambda}{p}} f\in L_r(\mathbb{K})$ *the following estimation is valid:*

$$\|I_\beta f\|_{L_q(\mathbb{K})} \leq (C_1+C_2) A_p^{1-\frac{\beta p}{\lambda}}\, \|M_{\frac{\lambda}{p}} f\|^{\frac{\beta p}{\lambda}}_{L_r(\mathbb{K})}\, \|f\|^{1-\frac{\beta p}{\lambda}}_{L_p(\mathbb{K})}. \tag{7}$$

Proof. Taking

$$r=r(x,t)=\left(\frac{M_{\frac{\lambda}{p}} f(x,t)}{Mf(x,t)}\right)^{\frac{p}{\lambda}},$$

in (3) for every $(x,t)\in\mathbb{K}$ we have

$$I_\beta|f|(x,t) \leq (C_1+C_2)\left(M_{\frac{\lambda}{p}} f(x,t)\right)^{\frac{\beta p}{\lambda}} (Mf(x,t))^{1-\frac{\beta p}{\lambda}}. \tag{8}$$

Integrating on $\mathbb{K}$ and applying Hölder's inequality to inequality (8) we get

$$\int\limits_{\mathbb{K}} I_\beta|f|(x,t)^q dm_\alpha(x,t) \leq$$

$$\leq (C_1+C_2)^q \int\limits_{\mathbb{K}} \left(M_{\frac{\lambda}{p}} f(x,t)\right)^{\frac{\beta pq}{\lambda}} (Mf(x,t))^{q-\frac{\beta pq}{\lambda}} dm_\alpha(x,t)$$

$$\leq (C_1+C_2)^q \left(\int\limits_{\mathbb{K}} \left(M_{\frac{\lambda}{p}} f(x,t)\right)^{\frac{\beta pqs'}{\lambda}} dm_\alpha(x,t)\right)^{1/s'} \times$$

$$\times \left(\int\limits_{\mathbb{K}} (Mf(x,t))^{(q-\frac{\beta pq}{\lambda})s} dm_\alpha(x,t)\right)^{1/s},$$

where $(q-\frac{\beta pq}{\lambda})s=p$, $s'=\frac{s}{s-1}=\frac{\lambda r}{\beta pq}$, $\frac{1}{q}=\frac{1}{p}-\frac{\beta}{\lambda}+\frac{\beta p}{\lambda r}$.

Then we have

$$\left(\int\limits_{\mathbb{K}} |I_\beta f(x,t)|^q\, dm_\alpha(x,t)\right)^{1/q} \leq$$

$$\leq (C_1 + C_2) \left(\int_{\mathbb{K}} (Mf(x,t))^p \, dm_\alpha(x,t) \right)^{1/sq} \times$$

$$\times \left(\int_{\mathbb{K}} \left(M_{\frac{\lambda}{p}} f(x,t) \right)^r dm_\alpha(x,t) \right)^{\frac{\beta p}{\lambda r}} \leq$$

$$\leq (C_1 + C_2) A_p^{\frac{p}{sq}} \left(\int_{\mathbb{K}} |f(x,t)|^p dm_\alpha(x,t) \right)^{1/sq} \times$$

$$\times \left(\int_{\mathbb{K}} \left(M_{\frac{\lambda}{p}} f(x,t) \right)^r dm_\alpha(x,t) \right)^{\frac{\beta p}{\lambda r}}$$

and therefore

$$\|I_\beta f\|_{L_q(\mathbb{K})} \leq (C_1 + C_2) A_p^{\frac{p}{sq}} \; \|f\|_{L_p(\mathbb{K})}^{\frac{p}{sq}} \; \left\| M_{\frac{\lambda}{p}} f \right\|_{L_r(\mathbb{K})}^{\frac{\beta p}{\lambda}}$$

$$\leq (C_1 + C_2) A_p{}^{1-\frac{\beta p}{\lambda}} \, \|f\|_{L_p(\mathbb{K})}^{1-\frac{\beta p}{\lambda}} \left\| M_{\frac{\lambda}{p}} f \right\|_{L_r(\mathbb{K})}^{\frac{\beta p}{\lambda}}.$$

Thus the proof of Theorem 2. is completed.

By using Lemma 2 and Theorems 1. and 2. it can be easily proved that the following Hardy-Littlewood-Sobolev theorem for fractional integrals on the Laguerre hypergroup is valid.

Theorem 3. *Let $0 < \beta < 2\alpha + 4$ and $1 \leq p < \frac{2\alpha+4}{\beta}$.*

1) If $1 < p < \frac{2\alpha+4}{\beta}$, $f \in L_p(\mathbb{K})$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+4}$, then $I_\beta f \in L_q(\mathbb{K})$ and

$$\|I_\beta f\|_{L_q(\mathbb{K})} \leq (C_1 + C_2) \, A_p^{\frac{p}{q}} \; \|f\|_{L_p(\mathbb{K})}.$$

2) If $f \in L_1(\mathbb{K})$ and $1 - \frac{1}{q} = \frac{\beta}{2\alpha+4}$, then $I_\beta f \in WL_q(\mathbb{K})$ and

$$\|I_\beta f\|_{WL_q(\mathbb{K})} \leq q(q-1)^{1/q-1} C_1^{1/q} (C_2)^{1-1/q} A_1 \|f\|_{L_1(\mathbb{K})}.$$

Proof. i) For $r = \infty$, $\lambda = 2\alpha + 4$ from (1) and (7), we have

$$\|I_\beta f\|_{L_q(\mathbb{K})} \leq (C_1 + C_2) \, A_p{}^{\frac{p}{q}} \; \|M_{\frac{2\alpha+4}{p}} f\|_{L_\infty(\mathbb{K})}^{1-\frac{p}{q}} \;\; \|f\|_{L_p(\mathbb{K})}^{\frac{p}{q}}$$

$$\leq (C_1 + C_2) \, A_p{}^{\frac{p}{q}} \; \operatorname*{ess\,sup}_{(x,t)\in\mathbb{K}} \|T_{(x,t)}^{(\alpha)} |f|\|_{L_p(\mathbb{K})}^{1-\frac{p}{q}} \; \|f\|_{L_p(\mathbb{K})}^{\frac{p}{q}} \leq$$

$$\leq (C_1 + C_2) \, A_p^{\frac{p}{q}} \;\; \|f\|_{L_p(\mathbb{K})}.$$

ii) From (3) for $p=1$ and $\lambda=2\alpha+4$ we obtain

$$I_\beta|f|(x,t)\le C_1 r^\beta Mf(x,t)+C_2 r^{\beta-2\alpha-4}M_{2\alpha+4}f(x,t)$$
$$\le q(q-1)^{1/q-1}C_1^{1/q}C_2^{1-1/q}\left(Mf(x,t)\right)^{\frac{1}{q}}\ \left(M_{2\alpha+4}f(x,t)\right)^{1-\frac{1}{q}}$$
$$\le q(q-1)^{1/q-1}C_1^{1/q}C_2^{1-1/q}\left(Mf(x,t)\right)^{\frac{1}{q}}\ \|f\|_{L_1(\mathbb{K})}^{1-\frac{1}{q}}.$$

Then applying Theorem 1. we have

$$\mu_\alpha\left\{(x,t)\in\mathbb{K}: I_\beta|f|(x,t)>t\right\}^{1/q}$$
$$\le\mu_\alpha\left\{(x,t)\in\mathbb{K}: Mf(x,t)>q^{-q}(q-1)^{q-1}C_1^{-1}C_2^{1-q}t^q\|f\|_{L_1(\mathbb{K})}^{1-q}\right\}^{1/q}$$
$$\le q(q-1)^{1/q-1}C_1^{1/q}C_2^{1-1/q}\frac{A_1}{t}\|f\|_{L_1(\mathbb{K})}.$$

Therefore the proof of the theorem is completed.

References

[1]. Adams D.R., Hedberg L.I. *Function Spaces and Potential Theory.* Springer-Verlag Berlin Heidelberg, 1996, 369 p.

[2]. Assal M. and Abdallah H.B. *Generalized Besov type spaces on the Laguerre hypergroup.* Ann. Math. Blaise Pascal, 2005, 12 No 1, pp. 117–145.

[3]. Guliyev V.S. *Polar coordinates in Laguerre hypergroup.* Khazar Journal of Mathematics, 2006, 2, No 3, pp.3-11.

[4]. Guliyev V.S., Assal M. *On maximal function on the Laguerre hypergroup.* Fractional Calculus and Applied Analysis, 2006, 9, No 3, pp.307-318.

[5]. Guliyev V.S. and Omarova M. *On fractional maximal function and fractional integral on the Laguerre hypergroup.* Journal of Mathematical Analysis and Applications, 2008, 340, Issue 2, pp.1058-1068.

[6]. Guliyev V.S. and Omarova M. (L_p, L_q) *boundedness of the fractional maximal operator on the Laguerre hypergroup.* (submitted in Integral Transforms and Special Functions, 2008, pp.1-10.)

[7]. Nessibi M.M. and Sifi M., *Laguerre hypergroup and limit theorem*, in B.P. Komrakov, I.S. Krasilshchik, G.L. Litvinov and A.B. Sossinsky (eds.), Lie Groups and Lie Algebra - Their Representations, Generalizations and Applications, Kluwer Acad. Publ. Dordrecht, 1998.

[8]. Nessibi M.M. and Trimeche K. *Inversion of the Radon Transform on the Laguerre Hypergroup by using Generalized Wavelets.* J. Math. Anal. Appl. 1997, 208, No 2, pp.337–363.

[9]. Stempak K. *Mean summability methods for Laguerre series.* Trans. Amer. Math. Soc. 1990, 322, No 2, pp.671–690.

[10]. Stempak K. *Almost everywhere summability of Laguerre series*, Studia Math. 1991, 100, No 2, pp.129-147.

Mehriban N. Omarova
Baku State University
23, Z.Khalilov str., Az1148, Baku, Azerbaijan
E-mail: mehriban_omarova@yahoo.com

Received June 12, 2008; Revised September 25, 2008.