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ON COMPLETENESS OF ELEMENTARY SOLUTIONS OF A FOURTH ORDER OPERATOR-DIFFERENTIAL EQUATION ON A FINITE SEGMENT

Abstract

In the paper we find conditions providing completeness of elementary solutions in the space of generalized solutions of operator-differential equation of fourth order on finite segment.

Let H be a separable Hilbert space, A be a positive-definite self-adjoint operator in H, and H_θ be a space of Hilbert scales generated by the operator A, i.e. $H_\theta = D\left(A^\theta\right)$, $(x,y)_\theta = \left(A^\theta x, A^\theta y\right)$, $x, y \in D\left(A^\theta\right)$, $\theta \geq 0$. For $\theta = 0$ we assume that $H_0 = H$.

Let's consider the following boundary value problem

$$P\left(d/dt\right)u\left(t\right) = \frac{d^4u}{dt^4} + A^4u + \sum_{j=1}^{4} A_j \frac{d^{4-j}u}{dt^{4-j}} = 0,\ t \in (0,1) \tag{1}$$

$$u^{(j)}\left(0\right) = \varphi_j, \quad u^{(j)}\left(1\right) = \psi_j,\ j = 0,1, \tag{2}$$

where the vector-function $u\left(t\right)$ with values from H, φ_j, ψ_j $(j = 0,1)$ are the known vectors from H, the derivatives are understood in the sense of distributions theory [1], A_j $(j = \overline{1,4})$ are linear, generally speaking, unbounded operators in H.

Let's define the following Hilbert spaces [1]. Let $a, b \in R = (-\infty, \infty)$, $a < b$ and

$$L_2\left((a,b);H\right) = \left\{ f : \|f\|_{L_2((a,b);H)} = \left(\int_a^b \|f\left(t\right)\|^2 dt \right)^{\frac{1}{2}} < \infty \right\},$$

and

$$W_2^2\left((a,b);H\right) = \left\{u : u'' \in L_2\left((a,b):H\right),\ A^2u \in: L_2\left((a,b);H\right)\right\}$$

with the norm

$$\|u\|_{W_2^2((a,b);H)} = \left(\left\|u''\right\|^2_{L_2((a,b);H)} + \left\|A^2u\right\|^2_{L_2((a,b);H)} \right)^{\frac{1}{2}}$$

Further, we denote by

$$\overset{\circ}{W}{}_2^2\left((a,b);H\right) = \left\{ u : u \in W_2^2\left((a,b);H\right), u^{(j)}\left(a\right) = u^{(j)}\left(b\right) = 0, j = 0,1 \right\}$$

Let $D\left([a,b];H_4\right)$ be a linear set of vector-functions $u\left(t\right)$ with values in H_4 and possesing compact carriers in the segment $[a,b]$.

This set is everywhere dense in the space $W_2^2\left((a,b);H\right)$ [1]. The linear set

$$\overset{\circ}{D}\left([a,b];H_4\right) = \left\{ u : u \in D\left((a,b);H\right), u^{(j)}\left(a\right) = u^{(j)}\left(b\right) = 0, j = 0,1 \right\}$$

is determined in the similar way.

It follows from the theorem on traces [1,p.29] that the set $\overset{\circ}{D}([a,b];H_4)$ is everywhere dense in the space $\overset{\circ}{W}{}_2^2((a,b);H)$.

The following lemma is specifically proved in the paper [2]:

Lemma 1. *Let the following conditions be satisfied:*

1) A is a positive-definite self-adjoint operator with completely continuons inverse $A^{-1}=C$.

2) $B_j=A_jA^{-j}$ $(j=1,2)$ *and* $B_j=A^{-2}A_jA^{2-j}$ $(j=3,4)$ *are the bounded operators in* H.

Then the bilinear functional $P(u,g)=(P(d/dt)u,g)_{L_2((0,1);H)}$ *continues by continuity from the space* $D([0,1];H_4)\oplus\overset{\circ}{D}([0,1];H_4)$ *to the space* $W_2^2((0,1);H)+\overset{\circ}{W}{}_2^2((0,1);H)$ *as a bilinear functional, acting in the following way:*

$$P(u,g)=(u,g)_{W_2^2((0,1);H)}+P_1(u,g), \tag{3}$$

where

$$(u,g)_{W_2^2((0,1);H)}=\left(u'',g''\right)_{L_2((0,1);H)}+\left(A^2u,A^2g\right)_{L_2((0,1);H)}$$

and

$$P_1(u,g)=\sum_{j=1}^{2}\left(A_ju^{(2-j)},g''\right)_{L_2((0,1);H)}+\sum_{j=3}^{4}\left(A_ju^{4-j},g\right)_{L_2((0,1);H)} \tag{4}$$

Definition 1. *If the vector-function* $u\in W_2^2((0,1);H)$ *satisfies the equality (3) for all* $g\in\overset{\circ}{W}{}_2^2((0,1);H)$ *and* $\lim\limits_{t\to0}\left\|u^{(j)}(t)-\varphi_j\right\|_{2-j-\frac12}=0$, $\lim\limits_{t\to1}\left\|u^{(j)}(t)-\psi_j\right\|_{2-j-\frac12}=0$, $j=0,1$, *then* $u(t)$ *is said to be a generalized solution of problem (1), (2).*

In the paper [2] the following theorem is also proved:

Theorem 1 [2]. *Let the conditions 1) and 2) from lemma 1 be fulfilled and it hold the inequality*

$$\alpha=\sum_{j=1}^{4}m_j\left\|B_j\right\|<1, \tag{5}$$

where $m_1=m_3=\dfrac{1}{\sqrt2}$, $m_2=\dfrac12$, $m_4=1$. *Then for any* $\varphi_j\in H_{2-j-\frac12}$ *and* $\psi_j\in H_{2-j-\frac12}$ $(j=0,1)$ *there exists a unique generalized solution and for any* $g\in\overset{\circ}{W}{}_2^2((0,1);H)$ *it holds the inequality*

$$\operatorname{Re}P(g,g)\ge(1-\alpha)\left\|g\right\|^2_{W_2^2((0,1);H)}. \tag{6}$$

In the present paper under some additional conditions we'll prove the four-fold completeness of a system of chains of eigen and adjoint vectors responding to the boundary value problem 1), 2) corresponding to the operator pencil

$$P(\lambda)=\lambda^4E+A^4+\sum_{j=1}^{4}A_j\lambda^{4-j}, \tag{7}$$

and also completeness of elementary solutions of homogeneous equation $P(d/dt)u = 0$ in the space of generalized solutions.

Under another conditions the similar problems were studied for instance, in the papers [3,4,8,9].

Definition 2. *If for some λ_0 the equation $P(\lambda_0)\varphi_0 = 0$ has a non-zero solution, the number λ is said to be an eigen value of the operator pencil $P(\lambda)$, and φ_0 an eigen vector of the pencil $P(\lambda)$, responding to the eigenvalue λ_0. if the vectors φ_0, φ_1,...,φ_m satisfy the equations*

$$\sum_{j=0}^{k} \frac{1}{j!} P^{(j)}(\lambda_0)\varphi_{k-j}, \quad k = \overline{0,m},$$

then φ_0, φ_1,...,φ_m is said to be a chain of eigen and adjoint elements of the operator pencil $P(\lambda)$, responding to the eigen vector φ_0.

Definition 3. *If $\{\varphi_0, \varphi_1, ..., \varphi_m\}$ is a chain of eigen and adjoint vectors of the pencil $P(\lambda)$ responding to the eigenvalue λ_0, the vector-functions*

$$u_h(t) = e^{\lambda_0 t}\left(\varphi_h + \frac{t}{1!}\varphi_{h-1} + ... + \frac{t^h}{h!}\varphi_0\right), \quad h = \overline{0,m}$$

satisfy the equation $P(d/dt)\, u(t) = 0$ and are said to be elementary solutions responding to the eigenvalue λ_0 [5].

If λ_0 are eigenvalues, the elementary solutions have the following traces

$$\varphi_h^{(\nu)} = \left.\frac{d^\nu}{dt^\nu} u_h(t)\right|_{t=0}, \ \psi_h^{(\nu)} = \left.\frac{d^\nu}{dt^\nu} u_h(t)\right|_{t=1}, \ h = \overline{0,m}, \ \nu = 0,1,$$

that are said to be derivative chains.

By means of derivative chains $\varphi_h^{(\nu)}$ and $\psi_h^{(\nu)}$ $(\nu = 0,1)$ we determine the vectors $\widetilde{\varphi}_h = \left(\varphi_h^{(0)}, \varphi_h^{(1)}, \psi_h^{(0)}, \psi_h^{(1)}\right) \in H^4$, $h = \overline{0,m}$.

By $K(\Pi)$ we denote all positive vectors $\widetilde{\varphi}_h$ responding to all eigen values and eigen vectors of the pencil $P(\lambda)$.

Definition 4. *The system $K(\Pi)$ is said to be four-fold complete in the traces space if the system $K(\Pi)$ is complete in the space*

$$\widetilde{H} = \left(\overset{1}{\underset{i=0}{\oplus}} H_{2-i-\frac{1}{2}}\right) \oplus \left(\overset{1}{\underset{i=0}{\oplus}} H_{2-i-\frac{1}{2}}\right).$$

Lemma 2. *Let the conditions 1) and 2) be satisfied. In order the system $K(\Pi)$ be four-fold complete in the traces space, it is necessary and sufficient that for any vectors $\chi \in H_{2-i-\frac{1}{2}}$ and $\theta_i \in H_{2-i-\frac{1}{2}}$, $i = 0,1$ from the holomorphic property of the vector-functions $\sum_{i=0}^{1} A^{2-i-\frac{1}{2}} P^{-1}\left(\overline{\lambda}\right)^* \left(\lambda^i A^{2-i-\frac{1}{2}}\chi_i + \lambda^i e^\lambda A^{2-i-\frac{1}{2}}\theta_i\right)$ in the complex plane Π it follows $\chi_i = \theta_i = 0$, $i = 0,1$.*

The proof of the lemma follows from Loran expansion $\left(P^{-1}\left(\overline{\lambda}\right)\right)^*$ in the vicinity of eigen values (see [3], [5], [6]).

At first we prove that the pencil $P(\lambda)$, whose coefficients satisfy the conditions 1) and 2) from lemma 1, under some additional conditions has a discrete spectrum.

It holds

Lemma 3. *Let the conditions 1), 2) from lemma 1 be satisfied and the operator* $E+B_4$ *be invertible in* H. *Then the operator pencil* $P(\lambda)$ *has a discrete spectrum with a unique limiting point at infinity. If* $A^{-1}=C\in\sigma_p$, $p>0$, *the resolvent* $P^{-1}(\lambda)$ *is represented in the form of ratio of two entire functions of order* p *and minimal type at order* p.

Proof. Obviously

$$P(\lambda)=\left(\lambda^4E+A^4\right)+\sum_{j=1}^{4}\lambda^{4-j}A_j=A^2\left(\left(\lambda^4C^4+E\right)+\sum_{j=1}^{4}\lambda^{4-j}C^2A_jC^2\right)A^2=$$

$$=A^2\left(\left(\lambda^4C^4+E\right)+\lambda^3C^2\left(A_1A^{-1}\right)C+\lambda^2C^2\left(A_2A^{-2}\right)+\right.$$
$$\left.+\lambda\left(A^{-2}A_3A^{-1}\right)C+\left(A^{-2}A_4A^{-2}\right)\right)A^2=$$
$$A^2\left(\left(\lambda^4C^4+E\right)+\lambda^3C^2B_1C+\lambda^2C^2B_2+\lambda B_3C+B_4\right)A^2\equiv A^2L(\lambda)A^2,$$

where $L(\lambda)=\lambda^4C^4+E+\sum_{j=1}^{4}\lambda^{4-j}T_j$, where $T_1=C^2B_1C\in\sigma_{p/3}$, $T_2=C^2BC\in\sigma_{p/3}$, $T_3=B_3C\in\sigma_p$, $T_4=B_4$.

Since $L(0)=E+T_4$ is invertible, then the pencil

$$L(\lambda)=(E+T_4)\left(\left(\lambda^4(E+T_4)^{-1}C^4+\sum_{j=1}^{3}\lambda^{4-j}\left(E+T^4\right)^{-1}T_j+E\right)\right)$$

by the Keldysh theorem is invertible except denumarable points that have a unique limiting point at infinity. Since $(E+T_4)^{-1}C^4\in\sigma_{p/4}$, $(E+T_4)^{-1}T_j\in\sigma_{p/4-j}$, $j=\overline{1,3}$, then by M.G.Gasymov's lemma from [6] $L^{-1}(\lambda)$ is represented in the form of ratio of two entire functions of order p and of minimal type at order p. This property relates to the operator pencil $P(\lambda)$ as well. The lemma is proved.

Lemma 4. *When fulfilling the conditions of theorem 1, for* $\xi\in R$ *and* $\varphi\in H_4$ *the following inequalities*

$$\operatorname{Re}\left(P(i\xi)\varphi,\varphi\right)\geq\delta\left(\left(\xi^4E+A^4\right)\varphi,\varphi\right),\ \xi\in R,\ \varphi\in H_4 \tag{8}$$

$$\operatorname{Re}\left(P(i\xi)\varphi,\varphi\right)\geq\delta_1\left(\left(\xi^4E+A^4\right)\varphi,\varphi\right),\ \xi\in R,\ \varphi\in H_4 \tag{9}$$

hold.

Proof. Let's prove inequality (8). Inequality (9) is proved similar to inequality (8).

Let $g(t)=\eta(t)\varphi$ where $\eta(t)\not\equiv0$ is an infinitely differentiable scalar function, moreover $\eta^{(\kappa)}(t)=0$ for $t\leq0$ and $t\geq1$, $k\geq0$, and $\varphi\in H_4$. Then by theorem 1 we have

$$\operatorname{Re}\left(P(d/dt)\eta(t)\varphi,\eta(t)\varphi\right)_{L_2((0,1);H)}\geq(1-\alpha)\left\|\eta(t)\varphi\right\|^2_{W_2^2((0,1);H)}$$

After the Fourier transformation we have

$$\operatorname{Re}\left(P(i\xi)\varphi,\varphi\right)\left\|\widehat{\eta}(\xi)\right\|^2\geq(1-\alpha)\left(\left(\xi^2\widehat{\eta}(\xi)\varphi,\xi^2\widehat{\eta}(\xi)\varphi\right)_{L_2(R:H)}+\right.$$

$$+A^2\widehat{\eta}(\xi)\varphi), A^2\widehat{\eta}(\xi)\varphi))_{L_2(R:H)} = (1-\alpha)\left(\left(\xi^4E+A^4\right)\varphi,\varphi\right)_{L_2(R:H)}\|\widehat{\eta}(\xi)\|^2_{L_2(R:H)}$$

Hence the truth of inequality (8) follows.

Lemma 5. *Let the conditions of theorem 1 be fulfilled. Then for $\xi \in R$ the following estimates*

$$\left\|A^2P^{-1}(i\xi)A^2\right\| \le const, \quad \xi\in R \tag{10}$$

$$\left\|A^2P^{-1}(\xi)A^2\right\| \le const, \quad \xi\in R \tag{11}$$

hold.

Proof. Let's prove inequality (10). Inequality (11) is proved in the similar way. Obviously,

$$A^2P^{-1}(\xi)A^2 = A^2\left(\xi^2E+A^4+\sum_{j=1}^{4}(i\xi)^{4-j}A_j\right)A^2 =$$

$$= A^2\left(-i\xi^2E+A^2\right)^{-1}\left(E+\sum_{j=1}^{4}(i\xi)^{4-j}\left(i\xi^2E+A^2\right)^{-1}\times\right.$$

$$\left.\times A_j\left(-i\xi^2E+A^2\right)^{-1}\right)^{-1}\left(i\xi^2E+A^2\right)^{-1}A^2 \tag{12}$$

It follows from spectral expansion of the operator A that

$$\left\|A^2\left(-i\xi^2E+A^2\right)^{-1}\right\| \le \sup_{\mu\in\sigma(A)}\frac{\mu^2}{\left(\mu^2+\xi^2\right)^{\frac{1}{2}}} \le \sup_{\mu\ge\mu_0>0}\frac{\mu^2}{\left(\mu^4+\xi^4\right)^{\frac{1}{2}}} \le 1 \tag{13}$$

On the other hand, it follows the equality

$$\left\|(i\xi)^3\left(i\xi^2E+A^2\right)^{-1}A_1\left(-i\xi^2E+A^2\right)^{-1}\right\| =$$

$$= \left\|(i\xi)^2\left(i\xi^2E+A^2\right)^{-1}\left(A_1A^{-1}\right)A\left(-i\xi^2E+A^2\right)^{-1}\right\|$$

Since $\left\|A_1A^{-1}\right\| = \|B_1\|$, and

$$\left\|(i\xi)^2\left(i\xi^2E+A^2\right)^{-1}\right\| \le \sup_{\mu\ge\mu_0}\frac{\xi^2}{\sqrt{\xi^4+\mu^4}} \le 1,$$

and

$$\left\|(Ai\xi)\left(-i\xi^2E+A^2\right)^{-1}\right\| \le \sup_{\mu\ge\mu_0}\frac{\mu|\xi|}{\sqrt{\xi^4+\mu^4}} \le \frac{1}{\sqrt{2}},$$

then

$$\left\|(i\xi)^3\left(i\xi^2E+A^2\right)^{-1}A_1\left(-i\xi^2E+A^2\right)^{-1}\right\| \le \frac{1}{\sqrt{2}}\|B_1\|. \tag{14}$$

Let's estimate the other terms. Obviously

$$\left\|(i\xi)^2\left(i\xi^2E+A^2\right)^{-1}A_2\left(-i\xi^2E+A^2\right)^{-1}\right\| \le$$

$$\le \left\|(i\xi)^2\left(i\xi^2E+A^2\right)^{-1}\left(A_2A^{-2}\right)A^2\left(-i\xi^2E+A^2\right)^{-1}\right\| \le$$

$$\leq \left\| (i\xi)^2 \left(i\xi^2 E + A^2\right)^{-1} \right\| \cdot \|B_2\| \cdot \left\| A^2 \left(-i\xi^2 E + A^2\right)^{-1} \right\| \leq$$

$$\leq \frac{\xi^2}{\sqrt{\xi^4+\mu^4}} \frac{\mu^2}{\sqrt{\xi^4+\mu^4}} \cdot \|B_2\| \leq \frac{1}{2}\|B_2\| \tag{15}$$

In sequel, we have:

$$\left\| (i\xi) \left(i\xi^2 E + A^2\right)^{-1} A_3 \left(-i\xi^2 E + A^2\right)^{-1} \right\| =$$

$$= \left\| \left(i\xi^2 E + A^2\right)^{-1} A^2 \left(A^{-2} A_3 A^{-1}\right) A i\xi \left(-i\xi^2 E + A^2\right)^{-1} \right\| \leq$$

$$\leq \left\| A^2 \left(i\xi^2 E + A^2\right)^{-1} \right\| \cdot \|B_3\| \cdot \left\| A (i\xi) \left(-i\xi^2 E + A^2\right)^{-1} \right\| \leq \frac{1}{\sqrt{2}} \|B_3\| \tag{16}$$

Finally, we have:

$$\left\| \left(i\xi^2 E + A^2\right)^{-1} A_4 \left(-i\xi^2 E + A^2\right)^{-1} \right\| =$$

$$= \left\| \left(i\xi^2 E + A^2\right)^{-1} A^2 \left(A^{-2} A_4 A^{-2}\right) A^2 \left(-i\xi^2 E + A^2\right)^{-1} \right\| \leq \|B_4\| \tag{17}$$

Considering inequalities (13)-(17) in the equality (12), from inequality (5) we get the proof of the lemma.

For proving the four-fold completeness of the system $K(\Pi)$ we'll use the method of the papers [3,4]. Therefore, we'll reduce the unbounded operator pencil to the bounded pencil [3].

Denote $L(\lambda) = A^{-2} P(\lambda) A^{-2} = C^2 P(\lambda) C^2$.

As is seen from the proof of lemma 2

$$L(\lambda) = E + \lambda^2 C^4 + \sum_{j=1}^{4} \lambda^{4-j} T_j,$$

where $T_1 = C^2 B_1 C$, $T_2 = C^2 B$, $T_3 = B_3 C$, $T_4 = B_4$.

Denote the space of generalized solutions of the problem (1), (2) by $\mathcal{P}_O$. It follows from the uniqueness of solutions and from the Banach theorem on the inverse operator that for $u \in \mathcal{P}_O$ it holds the inequality

$$c_1 \|\widetilde{\varphi}\|_{\widetilde{H}} \leq \|u\|_{W_2^2((0,1);H)} \leq c_2 \|\widetilde{\varphi}\|_{\widetilde{H}}, \quad \widetilde{\varphi} = (\varphi_0, \varphi_1, \psi_0, \psi_1) \tag{18}$$

For proving the completeness of elementary solutions of first we'll prove that the system $K(\Pi)$ is four-fold complete in $\widetilde{H}$.

Theorem 2. *Let the conditions of theorem 1 and one of the conditions:*
a) $A^{-1} \in \sigma_p$, $0 < p \leq 2$; *or b)* $B_j \in \sigma_\infty$, $A^{-1} \in \sigma_p$, $0 < p < \infty$; *be fulfilled.*
Then the system $K(\Pi)$ *is four-fold complete in* $\widetilde{H}$.

Proof. Obviously, the four-fold completeness of the system $K(\Pi)$ is equivalent to four-fold completeness in H^4 of the system of all derivative chains of eigen and adjoint vectors of the pencil $L(\lambda)$, responding to eigenvalues λ_k by the collection of operator-functions (see [3], p.24)

$$\left(C^{\frac{1}{2}}, \lambda C^{\frac{3}{2}}, e^{\lambda} C^{\frac{1}{2}}, \lambda e^{\lambda} C^{\frac{3}{2}}\right).$$

If there is no four-fold completeness of the indicated system, then (see [3], p.8) there will be found such non-zero vector $\widetilde{z} = \left(x_0, x_1, y_0, y_1 \in H^4\right)$, that the vector-function

$$R(\lambda) = \left(L^*\left(\overline{\lambda}\right)\right)^{-1}\left(\sum_{j=0}^{1}\lambda^j C^{\frac{1}{2}(2j+1)}x_j + e^{\lambda}\lambda^j C^{\frac{1}{2}(2j+1)}y_j\right) \equiv$$

$$\equiv \left(L^*\left(\overline{\lambda}\right)\right)^{-1}\chi(\lambda)$$

is entire.

Here, taking into account lemma 5 and lemma 3 we get that on an imaginary axis and on negative semi-axis the vector-function $R(\lambda)$ grows no more rapid than a polynomial, but on a positive semi-axis it grows exponentially. Then by the Fragmen-Lindeloff theorem [7] the vector-function $R(\lambda)$ is a vector-function of exponential type and in the left half-plane it grows no rapid than a polynomial.

Now, let's consider the entire scalar function [3,4]

$$F_0(\lambda) = \left(\left(L^*\left(\overline{\lambda}\right)\right)^{-1}\chi(\lambda), \chi\left(\overline{\lambda}\right)\right) = \left(R(\lambda), \chi\left(\overline{\lambda}\right)\right) = F_1(\lambda) + F_2(\lambda)$$

where

$$F_1(\lambda) = \left(R(\lambda), \sum_{j=0}^{1}\overline{\lambda}^j C^{\frac{1}{2}(2j+1)}x_j\right),$$

and

$$F_2(\lambda) = e^{\lambda}\left(R(\lambda), \sum_{j=0}^{1}\overline{\lambda}^j C^{\frac{1}{2}(2j+1)}y_j\right)$$

are entire functions. Let's prove that on an imaginary axis $F_1(\lambda)$ and $F_2(\lambda)$ behave as $o\left(|\lambda|^{-1}\right)$.

Prove it for $F_1(\lambda)$. For $F_2(\lambda)$ it is proved in the similar way. Represent $L(\lambda)$ in the form

$$L(\lambda) = L_R(\lambda) + L_1(\lambda),$$

where

$$L_R(\lambda) = \operatorname{Re}\left(I + \lambda^4 C^4\right) + \operatorname{Re}\sum_{j=1}^{4}\lambda^{4-j}T_j,$$

$$L_1(\lambda) = \operatorname{Im}\left(I + \lambda^4 C^4\right) + \operatorname{Im}\sum_{j=1}^{4}\lambda^{4-j}T_j.$$

It follows from lemma 4 that

$$L_R(i\xi) \geq \sigma\left(\xi^4 C^4 + E\right),\ \xi \in R \tag{19}$$

Similar to the problem [3] for $\xi \in R$ we denote

$$G(i\xi) = \left(I - i\xi^2 C^2\right)^{-1} L_R(i\xi)\left(I + i\xi^2 C^2\right)^{-1}.$$

Then, obviously, for $\xi \in R$

$$(G(i\xi)\varphi,\varphi) = \left(L_R(i\xi)\left(E+i\xi^2C^2\right)^{-1}\varphi, \left(E+i\xi^2C^2\right)^{-1}\varphi\right) \geq$$

$$\geq \sigma\left(\xi^4C^4+E\right)\left(E+i\xi^2C^2\right)^{-1}\varphi\Big), \left(E+i\xi^2C^2\right)^{-1}\varphi\Big) \geq \sigma(\varphi,\varphi)$$

Thus, $G(i\xi) \geq \sigma$ then $G^{-1}(i\xi) \leq \sigma^{-1}$, $\xi \in R$.

For $\xi \in R$ we have (see [3], p.18)

$$L^*\left(i\bar{\xi}\right)^{-1} = \left(E+i\xi^2C^2\right)G^{-\frac{1}{2}}(i\xi)\left[I-i\left(T(i\xi)\right)\right]^{-1}G^{-\frac{1}{2}}(i\xi)\left(E-i\xi^2C^2\right)^{-1},$$

where

$$T(i\xi) = G^{-\frac{1}{2}}(i\xi)\left(E-i\xi^2C^2\right)^{-1}\left[\operatorname{Im}\sum_{j=1}^{4}(i\xi)^{4-j}T_j\right]\times$$

$$\times\left(E+i\xi^2C^2\right)G^{-\frac{1}{2}}(i\xi).$$

Since $T(i\xi)$ is a self-adjoint operator for any $\xi \in R$, then

$$\left\|(E-iT(i\xi))^{-1}\right\| \leq 1,\ \xi \in R$$

Since $G^{-\frac{1}{2}}(i\xi) \leq \sigma^{-\frac{1}{2}}$, then for $\xi \in R$ we have

$$|F_1(i\xi)| \leq c\sum_{i,j=0}^{1}|\xi|^{i+j}\left|\left(L^*\left(i\bar{\xi}\right)^{-1}C^{\frac{1}{2}(2i+1)}f_i, C^{\frac{1}{2}(2i+1)}x_j\right)\right| \leq$$

$$\leq c\sum_{i,j=0}^{1}|\xi|^{i+j}\left|\left(G^{-\frac{1}{2}}(i\xi)(E-iT(i\xi))^{-1}G^{-\frac{1}{2}}(i\xi)\left(E-i\xi^2C^2\right)\right)^{-1}C^{\frac{2i+1}{2}}f_i\left(E-i\xi^2C^2\right)^{-1}\times\right.$$

$$\left.\times C^{\frac{2i+1}{2}}x_j\right)\Big| = c\sum_{i,j=0}^{1}|\xi|^{i+j}\,o\left(|\xi|^{-2\frac{2j+1}{4}}\right) = o\left(|\xi|^{-1}\right),\ \ |\xi| \longrightarrow \infty,$$

where $f_i = x_i$ or $f_i = y_i$ $(i=0,1)$.

Here we used the the following lemma from the paper [3].

Lemma 6 [3, p.13]. *Let $Q>0$, $Q \in \sigma_\infty$ then in the domain $\Lambda_\varepsilon = \{\lambda : |\arg\lambda| \geq \varepsilon\}$, $-\pi < \arg\lambda \leq \pi$ for $\beta \in (0,1)$ and for any $T \in \sigma_\infty$ the estimations*

$$\left\|(E-\lambda Q)^{-1}Q^\beta\right\| \leq c(\varepsilon,\beta)|\lambda|^{-\beta},$$

$$\lim_{\eta\to\infty}\sup_{|\lambda|\geq\eta,\lambda\in\Lambda_\varepsilon}\left\|\lambda^\beta(E-\lambda Q)^{-1}Q^\beta T\right\| = 0.$$

are fulfilled.

In the similar way we can get $|F_1(\xi)| = o\left(|\lambda|^{-1}\right)$ for $\xi \in R_- = (-\infty:0)$, $|\xi| \to \infty$.

Since $F_1(\lambda)$ is an entire function and grows no more than a polynomial, then by the Fragmen-Lindeloff theorem $F_1(\xi) = o\left(|\lambda|^{-1}\right)$ in the left half-plane. $F_2(\lambda)$ has the same property, i.e. $F_2(\xi) = o\left(|\lambda|^{-1}\right)$ in the left half-plane.

Now, let's denote $\Phi(\lambda) = \overline{F_1(\overline{\lambda})} + F_2(\lambda)$ that in the left plane decreases as $o\left(|\lambda|^{-1}\right)$. It is easy to see that $\operatorname{Re}\Phi(\lambda) = \operatorname{Re}F_0(\lambda)$ for $\lambda = i\xi$. Then $\operatorname{Re}\Phi(i\xi) \geq 0$ for $\xi \in R$, i.e. $\operatorname{Re}(-\Phi(i\xi)) \leq 0$. If $\operatorname{Re}\Phi(i\xi) \not\equiv 0$ differs from zero even if at one point, then by the Caratheodory inequality (see [3], p.20, or [7] p.28) for $\xi \in R_- = (-\infty : 0)$ and $|\xi| > 1$ we get $|\Phi(\xi)| > c\,|\xi|^{-1}$, $c > 0$. This contradicts the estimation $o\left(|\lambda|^{-1}\right)$.

So, $\operatorname{Re}\Phi(\lambda) = \operatorname{Re}F_0(\lambda) = 0$ for $\lambda = i\xi$. Hence by inequality (19) we get $\chi(\lambda) \equiv 0$. So, $x_0 = x_1 = y_0 = y_1 = 0$. The theorem is proved.

Now, we can prove a theorem on completeness of elementary solutions.

Theorem 3. *Let all the conditions of theorem 2 be fulfilled. Then the system of all elementary solutions is complete in the space of generalized solutions of problem (1), (2).*

Proof. Let $u(t) \in \mathcal{P}_O$ (a space of generalized solutions). Let $u(0) = \varphi_0$, $u'(0) = \varphi_1$, $u(1) = \psi_0$, $u'(1) = \psi_1$. Then by theorem 2, for any $\varepsilon > 0$ we can find such a number $c_{k,N}(\varepsilon)$ that

$$\left\|\sum_{K=1}^{N} c_{k,N}(\varepsilon)\,\varphi_k^{(\nu)} - \varphi_\nu\right\|_{H_{2-\nu-\frac{1}{2}}} < \frac{\varepsilon}{2c_2},$$

$$\left\|\sum_{K=1}^{N} c_{k,N}(\varepsilon)\,\psi_k^{(\nu)} - \psi_\nu\right\|_{H_{2-\nu-\frac{1}{2}}} < \frac{\varepsilon}{2c_2}, \nu = 0,1$$

Then by inequality (18) we get

$$\left\|u(t) - \sum_{K=1}^{N} c_{k,N}(\varepsilon)\,u_k(t)\right\|_{W_2^2((0,1);H)} < \varepsilon.$$

The theorem is proved.

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