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ON ABSOLUTE AND UNIFORM CONVERGENCE OF BIORTHOGONAL EXPANSION RESPONDING TO SCHRODINGER DISCONTINUOUS OPERATOR

Abstract

In the paper we consider a Schrodinger discontinuous operator with a potential summable on $G = (a, b)$. We prove a theorem on uniform and absolute convergence of biorthogonal expansion of functions from the space of type $W_p^1(G)$, $1 < p \leq 2$.

Uniform and absolute convergence of biorthogonal series responding to differential operators were investigated in many papers, for example [1-4]. Mainly, continuous operators with two-point and multi-point boundary conditions were studied in these papers. Therewith in multi-point case expansions of functions from $W_2^1(G)$ were studied. In the present paper the results of [4] relating to the Schrodinger operator are transferred to the case of Schrodinger discontinuous operator.

On an arbitrary finite interval $G = (a, b)$ we consider the Schrodinger operator

$$Lu = -u'' + q(x)u, \tag{1}$$

with a potential $q(x) \in L_1(G)$.

We'll proceed from generalized interpretation of eigen functions of the operator L (see [5]).

Assume that by means of the points $a = \xi_0 < \xi_1 < ...\xi_m = b$ the interval (a, b) is divided into m intervals $G_l = (\xi_{l-1}, \xi_l)$, $l = \overline{1, m}$.

By $D_l (l = \overline{1, m})$ we denote a class of functions that are absolutely continuous together with their first derivatives on the segment $[\xi_{l-1}, \xi_l] = \overline{G}_l$.

Let $D(a, b)$ be a class of functions possessing the property: if $f \in D(a, b)$, for each $l = \overline{1, m}$ there exist such a function $f_l(x) \in D_l$ that $f(x) = f_l(x)$ for $\xi_{l-1} x < \xi_l$.

Under eigen function of the operator (1), responding to eigen value λ, we'll understand any function $\overset{\circ}{y}(x) \in D(a, b)$ differ from zero, satisfying almost everywhere in (a, b) the equation $L\overset{\circ}{y} = \lambda \overset{\circ}{y}$.

Similarly, under the adjoint function of the operator (1) of order l $(l \geq 1)$, responding to the same eigen function $\overset{\circ}{y}$, we'll understand any function $\overset{l}{y}(x) \in D(a, b)$ satisfying the equation $L\overset{l}{y} = \lambda \overset{l}{y} + \overset{l-1}{y}$ almost everywhere in (a, b).

Let's consider an arbitrary system $\{u_k(x)\}_{k=1}^{\infty}$ consisting of eigen and adjoint functions of the operator (1), responding to the system of eigen values $\{\lambda_k\}_{k=1}^{\infty}$ and require that together with each adjoint function of order $l \geq 1$, this system include its corresponding eigen function and all the adjoint functions of order smaller than l and the length of the chains of root functions be uniformly bounded. This means that $u_k(x) \in D(a, b)$ and it satisfies almost everywhere on G the equation $Lu_k = \lambda_k u_k + \theta_k u_{k-1}$, where θ_k equals either to 0 (in this case $u_k(x) \neq 0$ is an eigen function) or 1 (in this case we require $\lambda_k = \lambda_{k-1}$ and call $u_k(x)$ an adjoint function).

By μ_k we denote the square root of the number λ_k for which $\operatorname{Re} \mu_k \geq 0$.

We'll require the system $\{u_k(x)\}_{k=1}^{\infty}$ to satisfy the V.A. Il'in conditions:

1) The system $\{u_k(x)\}_{k=1}^{\infty}$ is complete and minimal in $L_2(G)$;

2)

$$|\mathrm{Im}\,\mu_k| \le C_0,\ \ k=1,2,..., \tag{2}$$

$$\sum_{\tau \le \mathrm{Re}\,\mu_k \le \tau+1} 1 \le C_1, \quad \forall\tau \ge 0; \tag{3}$$

3) there exists such a constant C_2 that

$$\|u_k\|_2\,\|\upsilon_k\|_2 \le C_2,\ \ k=1,2,..., \tag{4}$$

where $\|\cdot\|_p$ is a norm of $L_p(G)$, $\{\upsilon_k\}_{k=1}^{\infty}$ is a biorthogonally adjoint system to the system $\{u_k(x)\}_{k=1}^{\infty}$ and it consists of root functions of formally adjoint operator

$$L^* = -\frac{d^2}{dx^2} + \overline{q(x)}\ \ (i.e.\ L^*\upsilon_k = \lambda_k\upsilon_k + \theta_{k+1}\upsilon_{k+1}).$$

For an arbitrary function $f(x) \in L_p(G)$, $(p \ge 1)$ we make a partial sum of its biorthogonal expansion by the system $\{u_k(x)\}_{k=1}^{\infty}$:

$$\sigma_v(x,f) = \sum_{\rho_k \le v} (f,\upsilon_k)u_k(x),\ \ v>0,\ \ \rho_k = \mathrm{Re}\,\mu_k.$$

By $\overset{\circ}{W}{}_p^1\left(G;\{\xi_l\}_{l=0}^m\right)$, $(p \ge 1)$ we denote a class of functions $f(x)$ from $W_p^1(G)$ such that

$$Q(f,\{\xi_l\},\upsilon_k) = f(\xi_m)\overline{\upsilon_k'(\xi_m-0)} - f(\xi_0)\overline{\upsilon_k'(\xi_m+0)} +$$
$$+\sum_{i=1}^{m-1} f(\xi_i)\overline{\left[\upsilon_k'(\xi_i-0)-\upsilon_k'(\xi_i+0)\right]} = 0.$$

Assume $R_v(x,f) = f(x) - \sigma_v(x,f)$.

Theorem. *Let the system $\{u_k(x)\}_{k=1}^{\infty}$ of root functions of the operator L satisfy the conditions 1)-3). Then, biorthogonal expansion of the function $f(x) \in \overset{\circ}{W}{}_p^1\left(G;\{\xi_l\}_{l=0}^m\right)$, $1 < p \le 2$ by the system $\{u_k(x)\}_{k=1}^{\infty}$ converges absolutely and uniformly on G_l, $l=\overline{1,m}$ and the relations*

$$f(x) = \sum_{k=1}^{\infty}(f,\upsilon_k)u_k(x), \quad x \in G_l,\ \ l=\overline{1,m} \tag{5}$$

are valid,

$$\sup_{x\in G_l} |R_v(x,f)| \le const\, v^{-1/q}\left(\|f\|_{W_p^1(G)} + \|f\|_\infty\right) \tag{6}$$

$$\sup_{x\in G_l} |R_v(x,f)| \le o(v^{-1/q}),\, v \to +\infty \tag{7}$$

where $p^{-1}+q^{-1}=1$, the symbol "o" depends on $f(x)$, const is independent of $f(x)$.

If at some point ξ_l, $1 \le l \le m-1$ the $u_k(\xi_l-0) = u_k(\xi_l+0) = u_k(\xi_l)$, $k = 1,2,...$, is fulfilled then relations (5)-(7) are true at the point $x=\xi_l$ as well.

The proof of this theorem is based on the following lemma.

Lemma 1. *For eigen and adjoint functions of the operator L on each interval $G_l(\xi_{l-1},\xi_l)$, $l=\overline{1,m}$ the formula*

$$u_k'(x+t) = -\mu_k u_k(x)\sin\mu_k t + u_k'(x)\cos\mu_k t +$$
$$+\int_x^{x+t}\{q(\xi)u_k(\xi) - \theta_k u_{k-1}(\xi)\}\cos\mu_k\left(|x-\xi|-t\right)d\xi, \tag{8}$$

where $x,\ x+t\in\overline{G_l}$*, is valid. Therewith, under the values* $u_k'(x)$ *at the points* ξ_{l-1} *and* ξ_l *we understand limiting values* $u'(\xi_{l-1}+0)$ *and* $u_k'(\xi_l-0)$*.*

Notice that the formula (8) is a result of differentiation of shear formula (7) of the paper [6].

Lemma 2. *Let conditions 2) be fulfilled. Then the system* $\left\{u_k'(x)\mu_k^{-1}\left\|u_k\right\|_q^{-1}\right\}$, $|\mu_k|\geq 1$ *satisfies the Riesz inequality, i.e.*

$$\left\{\sum_{|\mu_k|\geq 1}\left|\left(f,u_k'\right)\right|^q|\mu_k|^{-q}\left\|u_k\right\|_q^{-q}\right\}^{1/q}\leq const\left\|f\right\|_p, \tag{9}$$

where $f(x)\in L_p(G)$ *is an arbitrary function,* $1<p\leq 2,\ q=\dfrac{p}{p-1}$.

Proof. Represent the Fourier ratio of the function $f(x)\in L_p(G)$ by the system

$$\left\{u_k'(x)\mu_k^{-1}\left\|u_k\right\|_q^{-1}\right\},|\mu_k|\geq 1$$

in the form

$$(f,u_k')\mu_k^{-1}\left\|u_k\right\|_q^{-1}=\sum_{l=1}^{m}\int_{\xi_{l-1}}^{\xi_l}f(\xi)\overline{u_k'(\xi)}d\xi\mu_k^{-1}\left\|u_k\right\|_q^{-1}$$

and allowing for formula (8) transform the integrals under the sum in the following form

$$|\mu_k|^{-1}\left\|u_k\right\|_q^{-1}\left|\int_{\xi_{l-1}}^{\xi_l}f(\xi)\overline{u_k'(\xi)}d\xi\right|=\left|\int_{\xi_{l-1}}^{\xi_l}\overline{f(\xi)}u_k'(\xi)d\xi\right||\mu_k|^{-1}\left\|u_k\right\|_q^{-1}=$$

$$=\left|\int_0^{\xi_l-\xi_{l-1}}\overline{f(\xi_{l-1}+t)}u_k'(\xi_{l-1}+t)dt\right||\mu_k|^{-1}\left\|u_k\right\|_q^{-1}\leq\frac{u_k(\xi_{l-1})}{\left\|u_k\right\|_q}\times$$

$$\times\left|\int_0^{\xi_l-\xi_{l-1}}\overline{f(\xi_{l-1}+t)}\sin\mu_k t dt\right|+\frac{\left|u_k'(\xi_{l-1})\right|}{|\mu_k|\left\|u_k\right\|_q}\times$$

$$\times\left|\int_0^{\xi_l-\xi_{l-1}}\overline{f(\xi_{l-1}+t)}\cos\mu_k t dt\right|+\frac{1}{|\mu_k|\left\|u_k\right\|_q}\times$$

$$\times\left|\int_0^{\xi_l-\xi_{l-1}}\overline{f(\xi_{l-1}+t)}\int_{\xi_{l-1}}^{\xi_{l-1}+t}\left\{q(\xi)u_k(\xi)-\theta_k u_{k-1}(\xi)\right\}\cos\mu_k(\left|\xi_{l-1}-\xi\right|-t)d\xi dt\right|.$$

Hence it is seen that for the validity of the inequalities

$$\left(\sum_{|\mu_k|\geq 1}|\mu_k|^{-q}\left\|u_k\right\|_q^q\left|\int_{\xi_{l-1}}^{\xi_l}\overline{f(\xi)}u_k'(\xi)d\xi\right|^q\right)^{1/q}\leq C(l)\left\|f\right\|_p, \tag{10}$$

$1 \le l \le m$, it suffices to be convinced of the validity of

$$\left(\sum_{|\mu_k| \ge 1} \frac{\left| u_k(\xi_{l-1}) \right|^q}{\|u_k\|_q^q} \left| \int_0^{\xi_l - \xi_{l-1}} \overline{f(\xi_{l-1}+t)} \sin \mu_k t dt \right|^q \right)^{1/q} \le C(l) \|f\|_p , \qquad (11)$$

$$\left(\sum_{|\mu_k| \ge 1} \frac{\left| u'_k(\xi_{l-1}) \right|^q}{|\mu_k|^q \|u_k\|_q^q} \left| \int_0^{\xi_l - \xi_{l-1}} \overline{f(\xi_{l-1}+t)} \cos \mu_k t dt \right|^q \right)^{1/q} \le C(l) \|f\|_p , \qquad (12)$$

$$\left(\sum_{|\mu_k| \ge 1} \frac{\|u_k\|^q_{L_\infty(\xi_{l-1},\xi_l)}}{|\mu_k|^q \|u_k\|_q^q} \left| \int_{\xi_{l-1}}^{\xi_l} |q(\xi)| \times \right. \right.$$
$$\left. \times \left| \int_{\xi - \xi_{l-1}}^{\xi_l - \xi_{l-1}} \overline{f(\xi_{l-1}+t)} \cos \mu_k (\left|\xi - \xi_{l-1}\right| - t) dt \right|^q d\xi^{1/q} \right) \le C(l) \|f\|_p , \qquad (13)$$

$$\left(\sum_{|\mu_k| \ge 1} \frac{\|\theta u_{k-1}\|^q_{L_\infty(\xi_{l-1},\xi_l)}}{|\mu_k|^q \|u_k\|_q^q} \left(\int_{\xi_{l-1}}^{\xi_l} \left| \int_{\xi - \xi_{l-1}}^{\xi_l - \xi_{l-1}} \overline{f(\xi_{l-1}+t)} \times \right. \right. \right.$$
$$\left. \left. \left. \times \cos \mu_k (\left|\xi - \xi_{l-1}\right| - t) dt \right| d\xi \right)^q \right)^{1/q} \le C(l) \|f\|_p , \qquad (14)$$

Inequality (11) follows from the inequality [7]

$$\|u_k\|_{L_\infty(\xi_{l-1},\xi_l)} \le c_1(l) \|u_k\|_{L_\infty(\xi_{l-1},\xi_l)} \qquad (15)$$

and theorem 3 of the paper [6], or condition (2) is fulfilled and $\{\sin \mu_k t\}$ is a system of eigen functions of the operator

$$L_0 = -\frac{d^2}{dt^2}, \ 0 \le t \le \xi_l - \xi_{l-1}.$$

Inequality (12) follows from the inequality [7]

$$\left\| u'_k \right\|_{L_\infty(\xi_{l-1},\xi_l)} \le C_2(l) |\mu_k| \|u_k\|_{L_q(\xi_{l-1},\xi_l)} \qquad (16)$$

and theorem 3 of the paper [6], or condition 2 is fulfilled and $\{\cos \mu_k t\}$ is also a system of eigen functions of the operator L_0.

By the estimation (15), condition 2) and the Hölder inequality the left hand side of relation (13) is majorized from above by the convergent series

$$C \|q\|_1 \|f\|_p \sum_{|\mu_k| \ge 1} \frac{1}{|\mu_k|^q}.$$

By the estimation (see [7])

$$\|\theta_k u_{k-1}\|_{L_\infty(\xi_{l-1},\xi_l)} \le C_3(l) |\mu_k| \|u_k\|_{L_q(\xi_{l-1},\xi_l)} \qquad (17)$$

and the Holder inequality the left hand side of relation (14) is majorized from above by the series

$$C(l) \left(\sum_{|\mu_k| \ge 1} \int_{\xi_{l-1}}^{\xi_l} \left| \int_{\xi - \xi_{l-1}}^{\xi_l - \xi_{l-1}} \overline{f(\xi_{l-1}+t)} \cos \mu_k (\xi - \xi_l - t) dt \right|^q d\xi \right)^{1/q} .$$

Having applied here theorem 3 of the paper [6], for the systems $\{\cos\mu_k t\}$ and $\{\sin\mu_k t\}$ we get that the left hand side of (14) doesn't exceed the quantity

$$C(l,q)\left(\int_{\xi_{l-1}}^{\xi_l}\sum_{|\mu_k|\geq 1}\left\{\left|\int_{\xi-\xi_{l-1}}^{\xi_l-\xi_{l-1}}\overline{f(\xi_{l-1}+t)}\cos\mu_k tdt\right|^q+\right.\right.$$

$$\left.\left.+\left|\int_{\xi-\xi_{l-1}}^{\xi_l-\xi_{l-1}}\overline{f(\xi_{l-1}+t)}\sin\mu_k tdt\right|^q\right\}d\xi\right)^{1/q}\leq C(l)\left\|f\right\|_p$$

Consequently

$$\left\{\sum_{|\mu_k|\geq 1}\left|(f,u'_k)\right|^q\left|\mu_k\right|^{-q}\left\|u_k\right\|_q^{-q}\right\}^{1/q}=$$

$$=\left(\sum_{|\mu_k|\geq 1}\left|\sum_{l=1}^m\int_{\xi_{l-1}}^{\xi_l}f(\xi)\overline{u'_k(\xi)}d\xi\right|^q\left|\mu_k\right|^{-q}\left\|u_k\right\|_q^{-q}\right)^{1/q}\leq$$

$$\leq C(q)\left(\sum_{l=1}^m\sum_{|\mu_k|\geq 1}\left|\int_{\xi_{l-1}}^{\xi_l}f(\xi)\overline{u'_k(\xi)}d\xi\right|^q\left|\mu_k\right|^{-q}\left\|u_k\right\|_q^{-1}\right)^{1/q}\leq$$

$$\leq C(q)\left(\sum_{l=1}^m\left[C(l)\left\|f\right\|_p\right]^q\right)^{1/q}\leq const\left\|f\right\|_p.$$

Lemma 2 is proved.

Proof of the theorem. Let $f(x)\in\overset{\circ}{W}{}_p^1\left(G;\{\xi_l\}_{l=0}^m\right),\ 1<p\leq 2$. Calculate the Fourier ratio of this function by the system $\{\upsilon_k(x)\}_{k=1}^\infty$. Since

$$-\upsilon''_k(x)+\overline{q(x)}\upsilon_k(x)=\overline{\lambda}_k\upsilon_k(x)+\theta_{k+1}\upsilon_{k+1}(x)$$

for $x\in G_l$, $l=\overline{1,m}$ then

$$\int_a^b\overline{f(\xi)}\upsilon_k(\xi)d\xi=\sum_{l=1}^m\int_{\xi_{l-1}}^{\xi_l}\overline{f(t)}\upsilon_k(t)dt=-\sum_{l=1}^m\frac{1}{\overline{\mu}_k^2}\int_{\xi_{l-1}}^{\xi_l}\overline{f(\xi)}\upsilon''_k(\xi)d\xi+$$

$$+\sum_{l=1}^m\left[\frac{1}{\overline{\mu}_k^2}\int_{\xi_{l-1}}^{\xi_l}\overline{q(\xi)}\upsilon_k(\xi)\overline{f(\xi)}d\xi-\frac{\theta_{k+1}}{\overline{\mu}_k^2}\int_{\xi_{l-1}}^{\xi_l}\upsilon_{k+1}(\xi)\overline{f}(\xi)d\xi\right].$$

Conducting integration by parts in the integrals containing $\upsilon''_k(\xi)$ we get

$$\int_a^b\overline{f(\xi)}\upsilon_k(\xi)d\xi=-\overline{Q(f,\{\xi_l\},\upsilon_k)}+\frac{1}{\overline{\mu}_k^2}\sum_{l=1}^m\int_{\xi_{l-1}}^{\xi_l}\overline{f}'(t)\upsilon'_k(t)dt+$$

$$+\sum_{l=1}^{m}\frac{1}{\overline{\mu}_k^2}\int_{\xi_{l-1}}^{\xi_l}\overline{q(\xi)}\upsilon_k(\xi)\overline{f(\xi)}d\xi+\sum_{l=1}^{m}\frac{\theta_{k+1}}{\overline{\mu}_k^2}\int_{\xi_{l-1}}^{\xi_l}\upsilon_{k+1}(\xi)\overline{f(\xi)}d\xi=$$

$$=\frac{1}{\mu_k^2}(\upsilon_k',f')+\frac{1}{\overline{\mu}_k^2}\int_a^b\overline{q(\xi)}\upsilon_k(\xi)\overline{f(\xi)}d\xi+\frac{\theta_{k+1}}{\mu_k^2}(\upsilon_{k+1},f)$$

Thus

$$(\upsilon_k,f)=\frac{1}{\overline{\mu}_k^2}(\upsilon_k',f')+\frac{1}{\overline{\mu}_k^2}\int_a^b\overline{q}(\xi)\upsilon_k(\xi)\overline{f}(\xi)d\xi+\frac{\theta_{k+1}}{\overline{\mu}_k^2}(\upsilon_{k+1},f) \tag{18}$$

Now, let's estimate the series $\sum_{k=1}^{\infty}|(f,\upsilon_k)\,u_k(x)|\,,\ x\in G_l,\ l=\overline{1,m}.$

$$\begin{aligned}\sum_{k=1}^{\infty}|(f,\upsilon_k)\,u_k(x)|&=\sum_{0\le\operatorname{Re}\mu_k<1}|(f,\upsilon_k)|\,|u_k(x)|+\sum_{\operatorname{Re}\mu_k\ge1}|(f,\upsilon_k)|\,|u_k(x)|\le\\&\le\sum_{0\le\operatorname{Re}\mu_k<1}|u_k(x)|\,\|\upsilon_k\|_{L_\infty(G_l)}\|f\|_1+\sum_{|\mu_k|\ge1}|(f,\upsilon_k)|\,|u_k(x)|\end{aligned} \tag{19}$$

Applying the estimations (3), (4) and (15) we get

$$\sum_{0\le\operatorname{Re}\mu_k\le1}|u_k(x)|\,\|\upsilon_k\|_{L\infty(G_l)}\|f\|_1\le$$

$$\le C\sum_{0\le\operatorname{Re}\mu_k\le1}\|u_k\|_2\,\|\upsilon_k\|_2\,\|f\|_1\le C\,\|f\|_1\sum_{0\le\operatorname{Re}\mu_k\le1}1\le C\,\|f\|_1$$

For the estimation of the second sum in the right hand side of the inequality (19) we apply the estimation (15), (17), the representation (18) and lemma 2.

$$\sum_{|\mu_k|\ge1}|(f,\upsilon_k)|\,|u_k(x)|\le\sum_{|\mu_k|\ge1}\frac{1}{|\mu_k|^2}\left|(\upsilon_k',f')\right|\,|u_k(x)|+$$

$$+\sum_{|\mu_k|\ge1}\frac{1}{|\mu_k|^2}\|\upsilon_k\|_{L\infty(G_l)}\|f\|_{L\infty(G_l)}\|q\|_1\,\|u_k\|_{L\infty(G_l)}+$$

$$+\sum_{|\mu_k|\ge1}\frac{\theta_{k+1}}{|\mu_k|^2}\left|\left(f,\frac{\upsilon_{k+1}}{\|\upsilon_{k+1}\|_q}\right)\right|\|\upsilon_{k+1}\|_q\,\|u_k\|_{L\infty(G_l)}\le$$

$$\le C\sum_{|\mu_k|\ge1}\frac{1}{|\mu_k|}\left(f',\upsilon_k'\mu_k^{-1}\|\upsilon_k\|_q^{-1}\right)|\|u_k\|_2\,\|\upsilon_k\|_2+$$

$$+C\sum_{|\mu_k|\ge1}\frac{1}{|\mu_k|^2}\|u_k\|_2\,\|\upsilon_k\|_2\,\|f\|_\infty+C\sum_{|\mu_k|\ge1}\frac{\theta_{k+1}}{|\mu_k|}\left(f,\upsilon_{k+1}\|\upsilon_{k+1}\|_q^{-1}\right)\Bigg|\times$$

$$\times\|\mu_k\|^{-1}\|\upsilon_{k+1}\|_q\,\|u_k\|_2\le C\left(\sum_{|\mu_k|\ge1}\frac{1}{|\mu_k|^p}\right)^{1/p}\times$$

$$\times\left(\sum_{|\mu_k|\geq 1}\left|\left(f',\upsilon_k'\mu_k^{-1}\|\upsilon_k\|_q^{-1}\right)\right|^q\right)^{1/q}+C\|f\|_\infty\sum_{|\mu_k|\geq 1}\frac{1}{|\mu_k|^2}+$$

$$+C\left(\sum_{|\mu_k|\geq 1}\frac{1}{|\mu_k|^p}\right)^{1/p}\left(\sum_{|\mu_k|\geq 1}\left|\left(f,\theta_{k+1}\upsilon_{k+1}\|\upsilon_{k+1}\|_q^{-1}\right)\right|^q\right)^{1/q}\leq$$

$$\leq C\|f'\|_p\left(\sum_{|\mu_k|\geq 1}\frac{1}{|\mu_k|^p}\right)^{1/p}+C\|f\|_\infty\sum_{|\mu_k|\geq 1}\frac{1}{|\mu_k|^2}+$$

$$+C\|f\|_p\left(\sum_{|\mu_k|\geq 1}\frac{1}{|\mu_k|^p}\right)^{1/p}\leq C\left(\|f\|_{W_p^1(a,b)}+\|f\|_\infty\right)$$

Consequently, hence from (19) we get

$$\sum_{k=1}^{\infty}|(f,\upsilon_k)|\,|u_k(x)|\leq C\left(\|f\|_{W_p^1(G)}+\|f\|_\infty\right)$$

uniform with respect to $x\in G_l$.

Validity of representation (5) follows from the fact that under the conditions 1)-3) the system $\{u_k(x)\}_{k=1}^{\infty}$ forms an unconventional basis in $L_2(G)$ (see [5]) and from the fact that $f(x)$ belonging to $W_p^1(G)$ belongs to $L_2(G)$. Really, let $g(x)=\sum_{k=1}^{\infty}(f,\upsilon_k)u_k(x)$, $x\in G_l$, $l=\overline{1,m}$. Clearly, $g(x)$ is continuous on each G_l, $l=\overline{1,m}$. Multiplying $g(x)$ scalarly by $\upsilon_k(x)$ we get $(g,\upsilon_k)=(f,\upsilon_k)$, $k=$ $=1,2,....$

Hence it follows that $f(x)=g(x)$ almost everywhere in each G_l. Since the both functions are continuous in G_l, they coincide everywhere in G_l. Consequently, representation (5) is true.

Now, let's prove the estimation (6). Since at each G_l the representation (5) is true, then for $x\in G_l$

$$|R_v(x,f)|=|f(x)-\sigma_v(x,f)|=\left|\sum_{\operatorname{Re}\mu_k>v}(f,\upsilon_k)u_k(x)\right|\leq$$

$$\leq\sum_{|\mu_k|>v}|(f,\upsilon_k)|\,|u_k(x)|\leq\sum_{|\mu_k|>v}|\mu_k|^{-2}\left|(f',\upsilon_k')\right|\,|u_k(x)|+$$

$$+\sum_{|\mu_k|>v}|\mu_k|^{-2}\|\upsilon_k\|_2\|u_k\|_2\|f\|_\infty+C\sum_{|\mu_k|>v}|\mu_k|^{-1}|(f,\theta_{k+1}\upsilon_{k+1})|\times$$

$$\times\|\upsilon_{k+1}\|_q^{-1}|\mu_k|^{-1}\|\upsilon_{k+1}\|_q\|u_k\|_2\leq C\left(\|f\|_{W_p^1(G)}\left\{\sum_{|\mu_k|\geq v}\frac{1}{|\mu_k|^p}\right\}^{1/p}+\|f\|_\infty\right)\leq$$

$$\leq const\,v^{-1/q}\left(\|f\|_{W_p^1(G)}+\|f\|_\infty\right)$$

Estimation (6) is established.

Let's prove estimation (7). For that we use the representation (18)

$$|R_v(x,f)| \leq C\left(\sum_{|\mu_k|>v}|\mu_k|^{-p}\right)^{1/p}\left(\sum_{|\mu_k|>v}\left|\left(f',\upsilon_k'\mu_k^{-1}\|\upsilon_k\|_q^{-1}\right)\right|^q\right)^{1/q}+$$

$$+C\|f\|_\infty\sum_{|\mu_k|\geq v}|\mu_k|^{-2}+C\left(\sum_{|\mu_k|>v}|\mu_k|^{-p}\right)^{1/p}\times$$

$$\times\left(\sum_{|\mu_k|>v}\left|\left(f,\theta_{k+1}\upsilon_{k+1}\|\upsilon_{k+1}\|_q^{-1}\right)\right|^q\right)^{1/q}=0(v^{-1/q}),$$

or the remainder of convergent series is an infinitely small quantity.

The theorem is proved.

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