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ON SOLUTIONS OF A NONLINEAR BOUNDARY VALUE PROBLEM FOR DIFFERENCE EQUATIONS

Abstract

We study a boundary value problem (BVP) for second order nonlinear difference equations. A condition is established that ensures existence and uniqueness of solution to the BVP under consideration.

1. Introduction

Let $\mathbb{Z}$ denote the set of all integers. For any $l, m \in \mathbb{Z}$ with $l \le m$, $[l, m]$ will denote the *discrete interval* being the set defined by

$$[l,m] = \{n \in \mathbb{Z} : l \le n \le m\} = \{l, l+1, \dots, m\}.$$

Throughout the paper all intervals will be discrete intervals.

In this paper, we consider the nonlinear boundary value problem (BVP)

$$\Delta^2 y(n-1) + f(n, y(n)) = 0, \quad n \in [a,b], \tag{1}$$

$$y(a-1) = y(b+1) = 0, \tag{2}$$

where $a, b \in \mathbb{Z}$ with $a \le b$; $y(n)$ is a desired solution defined for $n \in [a-1, b+1]$; Δ denotes the forward difference operator defined by

$$\Delta y(n) = y(n+1) - y(n)$$

so that

$$\Delta^2 y(n-1) = y(n-1) - 2y(n) + y(n+1);$$

$f : [a,b] \times \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ denotes the set of all real numbers) is a given function.

The main result of this paper is the following theorem.

Theorem 1. *Suppose $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition*

$$|f(n,\xi) - f(n,\eta)| \le L\,|\xi - \eta|\,, \tag{3}$$

for all $n \in [a,b]$ and $\xi, \eta \in \mathbb{R}$, where $L > 0$ is a constant (Lipschitz constant). Suppose further that

$$L < 4\sin^2 \frac{\pi}{2(b-a+2)}. \tag{4}$$

Then the BVP (1), (2) has a unique solution.

Problem (1), (2) under the Lipschitz conditon (3) was earlier studied in [2, Chapt.9] where it is proved that if

$$L < \frac{8}{(b-a+2)^2}, \tag{5}$$

then the BVP (1), (2) has a unique solution.

Since $b-a\geq 0$, using the inequality

$$\sin x\geq \frac{2\sqrt{2}}{\pi}x \quad \text{for} \quad 0\leq x\leq \frac{\pi}{4},$$

we have

$$4\sin^2\frac{\pi}{2(b-a+2)}\geq \frac{8}{(b-a+2)^2}.$$

Therefore it is seen that our condition (4) is better than condition (5).

Proof of Theorem 1. is presented below in Section 3 and it uses a Hilbert space technique.

2. The difference operators

Let $\mathbb{Z}$ denote the set of all integers and let $y:\mathbb{Z}\to\mathbb{R}$ be a given function (sequence). The forward and backward difference operators Δ and ∇ are defined by

$$\Delta y(n)=y(n+1)-y(n) \quad \text{and} \quad \nabla y(n)=y(n)-y(n-1),$$

respectively. We easily see that

$$\nabla y(n)=\Delta y(n-1),$$

$$\Delta^2 y(n)=\Delta(\Delta y(n))=y(n+2)-2y(n+1)+y(n),$$

$$\nabla^2 y(n)=\nabla(\nabla y(n))=y(n)-2y(n-1)+y(n-2),$$

$$\Delta\nabla y(n)=y(n+1)-2y(n)+y(n-1)=\nabla\Delta y(n)=\Delta^2 y(n-1)=\nabla^2 y(n+1).$$

For any integers $a,b\in\mathbb{Z}$ with $a<b$ we have the summation by parts formulas

$$\begin{aligned}\sum_{n=a}^{b}(\Delta y(n))z(n) &= y(n+1)z(n)\mid_{a-1}^{b}-\sum_{n=a}^{b}y(n)\nabla z(n)\\ &= y(b+1)z(b)-y(a)z(a-1)-\sum_{n=a}^{b}y(n)\nabla z(n), \qquad (6)\end{aligned}$$

$$\begin{aligned}\sum_{n=a}^{b}(\nabla y(n))z(n) &= y(n)z(n+1)\mid_{a-1}^{b}-\sum_{n=a}^{b}y(n)\Delta z(n)\\ &= y(b)z(b+1)-y(a-1)z(a)-\sum_{n=a}^{b}y(n)\Delta z(n), \qquad (7)\end{aligned}$$

$$\sum_{n=a}^{b}(\Delta\nabla y(n))z(n)=(\Delta y(n))z(n)\mid_{a-1}^{b}-\sum_{n=a}^{b}(\nabla y(n))\nabla z(n), \qquad (8)$$

$$\sum_{n=a}^{b}(\Delta\nabla y(n))z(n)=(\Delta y(n))z(n+1)\mid_{a-1}^{b}-\sum_{n=a}^{b}(\Delta y(n))\Delta z(n), \qquad (9)$$

$$\sum_{k=a}^{b}[(\Delta\nabla y(n))z(n)-y(n)\Delta\nabla z(n)]=[(\Delta y(n))z(n)-y(n)\Delta z(n)]_{a-1}^{b}$$

$$= [(\Delta y(b))z(b) - y(b)\Delta z(b)] - [(\Delta y(a-1))z(a-1) - y(a-1)\Delta z(a-1)]. \quad (10)$$

3. Proof of Theorem 1

First we prove the following Lemma.

Lemma 2. *Let λ_1 be the least positive eigenvalue of the problem*

$$\Delta^2 y(n-1) + \lambda y(n) = 0, \quad n \in [a, b], \quad (11)$$

$$y(a-1) = y(b+1) = 0, \quad (12)$$

and L be the Lipschitz constant presented in the condition (3). If

$$L < \lambda_1, \quad (13)$$

then the BVP (1), (2) has a unique solution.

Proof. Denote by H the real Hilbert space of all functions (finite sequences) $y : [a, b] \to \mathbb{R}$ with the inner product (scalar product)

$$\langle y, z \rangle = \sum_{n=a}^{b} y(n)z(n)$$

and the norm

$$\|y\| = \sqrt{\langle y, y \rangle} = \left\{ \sum_{n=a}^{b} y^2(n) \right\}^{\frac{1}{2}}.$$

Obviously, H is a finite dimensional real linear space and its dimension is equal to the number $b - a + 1$ of all points of the discrete interval $[a, b]$. Next, we define the operators $A : H \to H$ and $F : H \to H$ as follows. For any $y \in H$ we put

$$(Ay)(n) = -\Delta^2 y(n-1) = -\Delta\nabla y(n) = -y(n-1) + 2y(n) - y(n+1),$$

$$(Fy)(n) = f(n, y(n)),$$

for $n \in [a, b]$, taking into account that when we calculate $(Ay)(a)$ and $(Ay)(b)$ we use the boundary conditions (2) setting $y(a-1) = 0$ and $y(b+1) = 0$, respectively. The latter means that for all $y \in H$ we extend $y(n)$ given for $n \in [a, b]$ to the values $n = a - 1$ and $n = b + 1$ by setting $y(a-1) = y(b+1) = 0$.

Note that the operator A is linear, while F is nonlinear in general. The eigenvalues of problem (11), (12) coincide with the eigenvalues of the operator A.

Using summation by parts formulas (10) and (9) and remembering that, according to the boundary conditions (12), we put

$$y(a-1) = y(b+1) = 0,$$

for all $y \in H$, we find that

$$\langle Ay, z \rangle = \langle y, Az \rangle, \quad (14)$$

$$\langle Ay, y \rangle = y^2(a) + \sum_{n=a}^{b} [\Delta y(n)]^2, \quad (15)$$

for all $y, z \in H$. Relation (14) shows that the operator A is self-adjoint, while (15) shows that it is positive:

$$\langle Ay, y \rangle > 0 \quad \text{for all} \quad y \in H, \ y \neq 0.$$

Therefore each eigenvalue of the operator A is real and positive, and the eigenvectors corresponding to the distinct eigenvalues are orthogonal. It also follows from Linear Algebra (see [1]) that the operator A has exactly $N = \dim H = b-a+1$ orthonormal eigenvectors (eigenfunctions) φ_k, $1 \leq k \leq N$, with the corresponding eigenvalues λ_k, $1 \leq k \leq N$, being real and positive. Note that existence of eigenvalues and basisness of eigenvectors for the operator A can be proved directly. We prove also that the eigenvalues are distinct. In fact, denote by $\varphi(n, \lambda)$ the solution of equation (11) satisfying the initial conditions

$$\varphi(a-1, \lambda) = 0, \quad \varphi(a, \lambda) = 1. \tag{16}$$

Using (16), we can recursively find $\varphi(n, \lambda)$, for $n = a, a+1, \ldots, b+1$, from

$$\varphi(n+1, \lambda) = (2-\lambda)\varphi(n, \lambda) - \varphi(n-1, \lambda), \quad n \in [a, b],$$

and $\varphi(n, \lambda)$ will be a polynomial in λ of degree $n-a$. It is easy to see that every solution $y(n, \lambda)$, $n \in [a-1, b+1]$, of equation (11) satisfying the initial condition $y(a-1, \lambda) = 0$ is equal to $\varphi(n, \lambda)$ up to a constant factor:

$$y(n, \lambda) = c\varphi(n, \lambda), \quad n \in [a-1, b+1],$$

with $c = y(a, \lambda)$. Indeed, the both sides are solutions of (11) and they coincide for $n = a-1$ and $n = a$. Hence they coincide for all n by the uniqueness of solution. It follows that the eigenvalues of (11), (12) coincide with the roots of the polynomial $\varphi(b+1, \lambda)$ and to each eigenvalue λ_0 there corresponds, up to a constant factor, single eigenfunction which can be taken to be the function $\varphi(n, \lambda_0)$, $n \in [a-1, b+1]$. Since $\varphi(b+1, \lambda)$ is a polynomial of degree $b-a+1$, it has $b-a+1$ roots. Now we show that the roots of $\varphi(b+1, \lambda)$ are simple. Hence we will get that there exists $N = b-a+1$ distinct eigenvalues. Differentiating the equation

$$\varphi(n-1, \lambda) + (\lambda - 2)\varphi(n, \lambda) + \varphi(n+1, \lambda) = 0$$

with respect to λ, we get

$$\dot{\varphi}(n-1, \lambda) + \varphi(n, \lambda) + (\lambda - 2)\dot{\varphi}(n, \lambda) + \dot{\varphi}(n+1, \lambda) = 0,$$

where the dot over the function indicates the derivative with respect to λ. Multiplying the first equation by $\dot{\varphi}(n, \lambda)$ and the second one by $\varphi(n, \lambda)$, and subtracting the left and right members of the resulting equations, we get

$$[\varphi(n-1, \lambda)\dot{\varphi}(n, \lambda) - \dot{\varphi}(n-1, \lambda)\varphi(n, \lambda)]$$

$$-[\varphi(n, \lambda)\dot{\varphi}(n+1, \lambda) - \dot{\varphi}(n, \lambda)\varphi(n+1, \lambda)] = \varphi^2(n, \lambda).$$

Summing the last equation for the values $n = a, a+1, \ldots, b$ and using the initial conditions (16), we get

$$-\varphi(b, \lambda)\dot{\varphi}(b+1, \lambda) + \dot{\varphi}(b, \lambda)\varphi(b+1, \lambda) = \sum_{n=a}^{b} \varphi^2(n, \lambda).$$

Setting here $\lambda = \lambda_0$, where λ_0 is a root of polynomial $\varphi(b+1, \lambda)$, that is, $\varphi(b+1, \lambda_0) = 0$, we obtain

$$-\varphi(b, \lambda_0)\dot{\varphi}(b+1, \lambda_0) = \sum_{n=a}^{b} \varphi^2(n, \lambda_0).$$

The right-hand side of the last equation is different from zero because the eigenvalue λ_0 is real, the polynomial $\varphi(b+1, \lambda)$ has real coefficients, and $\varphi(a, \lambda_0) = 1$ by (16). Consequently $\dot{\varphi}(b+1, \lambda_0) \neq 0$, that is, the root λ_0 of the polynomial $\varphi(b+1, \lambda)$ is simple.

Note also that the eigenvalues of the operator A (that is, of problem (11), (12)) coincide with the eigenvalues of the real symmetric Jacobi matrix

$$J = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}.$$

Thus we have that the operator A has $N = b - a + 1 = \dim H$ distinct positive eigenvalues λ_k, $1 \leq k \leq N$, which we arrange in the form

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_N.$$

The corresponding orthonormal eigenvectors φ_k, $1 \leq k \leq N$, form a basis for the space H. Thus

$$A\varphi_k = \lambda_k \varphi_k, \quad 1 \leq k \leq N,$$

$$\langle \varphi_k, \varphi_l \rangle = 0 \text{ if } k \neq l, \text{ and } = 1 \text{ if } k = l.$$

It follows that for arbitrary $y \in H$ we have (expansion formula and Parseval's equality)

$$y = \sum_{k=1}^{N} c_k \varphi_k, \quad c_k = \langle y, \varphi_k \rangle, \tag{17}$$

$$\|y\|^2 = \langle y, y \rangle = \sum_{k=1}^{N} c_k^2.$$

Since the operator A is positive it is invertible. We have

$$Ay = \sum_{k=1}^{N} c_k \lambda_k \varphi_k,$$

$$A^{-1}y = \sum_{k=1}^{N} \frac{c_k}{\lambda_k} \varphi_k,$$

for all $y \in H$, where c_k are defined in (17). Hence

$$\left\|A^{-1}y\right\|^2 = \sum_{k=1}^{N} \frac{c_k^2}{\lambda_k^2} \leq \frac{1}{\lambda_1^2} \sum_{k=1}^{N} c_k^2 = \frac{1}{\lambda_1^2} \|y\|^2.$$

Thus we have established the following result: The operator A is invertible and

$$\left\|A^{-1}y\right\| \leq \frac{1}{\lambda_1}\left\|y\right\| \quad \text{for all} \quad y \in H. \tag{18}$$

The BVP (1), (2) is equivalent to the vector equation

$$Ay = Fy \quad \text{for} \quad y \in H,$$

with the operators A and F defined above. This equation can be written in the form

$$y = A^{-1}Fy \quad \text{for} \quad y \in H.$$

Let us set $S = A^{-1}F$. Then we get that the BVP (1), (2) is equivalent to the equation

$$y = Sy \quad \text{for} \quad y \in H.$$

The last equation is a fixed point problem.

We will use the following well-known contraction mapping theorem: *Let H be a Banach space and suppose that $S : H \rightarrow H$ is a contraction mapping, i.e., there is an α, $0 < \alpha < 1$, such that $\|Sy - Sz\| \leq \alpha \|y - z\|$ for all $y, z \in H$. Then S has a unique fixed point in H.*

It will be sufficient to show that the operator $S = A^{-1}F$ is a contraction mapping on the space H. We have, using (18),

$$\|Sy - Sz\| = \left\|A^{-1}Fy - A^{-1}Fz\right\| = \left\|A^{-1}(Fy - Fz)\right\| \leq \frac{1}{\lambda_1}\|Fy - Fz\|. \tag{19}$$

Next, making use of the Lipschitz condition (3), we get

$$\begin{aligned} \|Fy - Fz\|^2 &= \sum_{n=a}^{b} |f(n, y(n)) - f(n, z(n))|^2 \\ &\leq L^2 \sum_{n=a}^{b} |y(n) - z(n)|^2 \\ &= L^2 \|y - z\|^2 \end{aligned}$$

so that

$$\|Fy - Fz\| \leq L\|y - z\| \quad \text{for all} \quad y, z \in H.$$

Thus, from (19) we obtain

$$\|Sy - Sz\| \leq \frac{L}{\lambda_1}\|y - z\| \quad \text{for all} \quad y, z \in H.$$

Consequently, we see that under the condition (13), S is a contraction mapping and hence it has a unique fixed point in H by the contraction mapping theorem. Lemma 2. is proved. ■

Now we compute the eigenvalues of problem (11), (12). Since the eigenvalues of (11), (12) are real, we can deal only with real values of λ. Consider the equation

$$\Delta^2 y(n-1) + \lambda y(n) = 0, \quad n \in \mathbb{Z},$$

that is,

$$y(n-1) + (\lambda - 2)y(n) + y(n+1) = 0, \quad n \in \mathbb{Z}, \tag{20}$$

where $\lambda \in \mathbb{R}$. Let us look for solutions of (20) of the form

$$y(n) = q^n, \quad n \in \mathbb{Z}, \tag{21}$$

where q is an undetermined complex number. Substituting (21) into (20) we get the characteristic equation

$$q^2 + (\lambda - 2)q + 1 = 0.$$

Hence

$$q = \frac{2 - \lambda \pm \sqrt{(\lambda - 2)^2 - 4}}{2}. \tag{22}$$

Consider possible cases separately.

(a) If $|\lambda - 2| > 2$, then according to (22) we get two values q_1 and q_2 which are real and distinct. A general solution of equation (20) has the form

$$y(n) = c_1 q_1^n + c_2 q_2^n, \quad n \in \mathbb{Z},$$

where c_1, c_2 are constants. Substituting this expression of $y(n)$ into boundary conditions (12), we find that $c_1 = c_2 = 0$. Therefore if $|\lambda - 2| > 2$, then there are no eigenvalues.

(b) If $|\lambda - 2| = 2$, then $\lambda = 0$ or $\lambda = 4$. In the case $\lambda = 0$ a general solution of equation (20) has the form

$$y(n) = c_1 + c_2 n, \quad n \in \mathbb{Z},$$

and in the case $\lambda = 4$ a general solution of equation (20) has the form

$$y(n) = (c_1 + c_2 n)(-1)^n, \quad n \in \mathbb{Z},$$

where c_1, c_2 are constants. Substituting these expressions of $y(n)$ into boundary conditions (12), we again find that $c_1 = c_2 = 0$. Therefore in the case $|\lambda - 2| = 2$ also there are no eigenvalues.

(c) Finally, consider the case $|\lambda - 2| < 2$. We can set

$$2 - \lambda = 2\cos\theta, \quad \theta \neq \pi m, \ m \in \mathbb{Z}. \tag{23}$$

Then

$$q = \cos\theta \pm i \sin\theta = e^{\pm i\theta}.$$

Hence a general solution of equation (20) is

$$y(n) = c_1 \cos n\theta + c_2 \sin n\theta, \quad n \in \mathbb{Z}.$$

From the boundary conditions (12), we have

$$y(a-1) = c_1 \cos(a-1)\theta + c_2 \sin(a-1)\theta = 0,$$

$$y(b+1) = c_1 \cos(b+1)\theta + c_2 \sin(b+1)\theta = 0.$$

This system has a nontrivial solution (c_1, c_2) if and only if its determinant is equal to zero:

$$\cos(a-1)\theta \sin(b+1)\theta - \cos(b+1)\theta \sin(a-1)\theta = 0,$$

that is,

$$\sin(b-a+2)\theta = 0.$$

Hence

$$(b-a+2)\theta = \pi k, \quad k \in \mathbb{Z},$$

and we get the values of θ in the form

$$\theta_k = \frac{\pi k}{b-a+2}, \quad k \in \mathbb{Z} \text{ and } k \neq m(b-a+2) \text{ for all } m \in \mathbb{Z}.$$

Substituting these values of θ into (23), we get the following values for λ:

$$\lambda_k = 2(1-\cos\theta_k) = 4\sin^2\frac{\pi k}{2(b-a+2)},$$

where $k \in \mathbb{Z}$ and $k \neq m(b-a+2)$ for all $m \in \mathbb{Z}$. For $k = 1, 2, \ldots, b-a+1$ we get $N = b-a+1$ distinct values of λ. Further integers values of k would give values already obtained. For instance, $k = b-a+3$ gives

$$\frac{\pi k}{2(b-a+2)} = -\left(\pi - \frac{\pi(b-a+1)}{2(b-a+2)}\right),$$

hence the λ corresponding to $k = b-a+1$, etc. Consequently, problem (11), (12) has the $N = b-a+1$ distinct eigenvalues

$$\lambda_k = 4\sin^2\frac{\pi k}{2(b-a+2)}, \quad k \in \{1, 2, \ldots, b-a+1\}. \tag{24}$$

Obviously we have

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_N,$$

where $N = b-a+1$. Therefore the least (positive) eigenvalue of problem (11), (12) is

$$\lambda_1 = 4\sin^2\frac{\pi}{2(b-a+2)}.$$

Now the statement of Theorem 1. follows from Lemma 2..

Remark 1. The orthonormal eigenfunctions $\varphi_k(n)$, $1 \leq k \leq b-a+1$, of problem (11), (12), corresponding to the eigenvalues (24) have the form

$$\varphi_k(n) = \alpha_k \sin\frac{\pi k(n-a+1)}{b-a+2}, \quad n \in [a-1, b+1],$$

where α_k are normirating constants.

References

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