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ON CONTINUATION OF BOUNDED OPERATORS

Abstract

In the paper, the notion F_T-invariance of Banach space is introduced and using this notion sufficient condition on possibility of minimal continuation of a bounded operator is given.

Hahn-Banach classic theorem on continuation with preservation of norm (minimal continuation) of a bounded functional in a Banach space is known well. In this relation there arises a question if one get similar result for bounded (not everywhere defined) operators. It is founded that it is not always possible. It is known that if a subspace X_0 of the Banach space X is not topologically complemented, there exists a Banach space Y and a bounded operator $T \in \mathcal{L}(X_0; Y)$ that has no minimal continuation (here $\mathcal{L}(X;Y)$ is a Banach space of bounded operators acting from X to Y). We can be acquainted with this fact in detail in [1, p.184]. Hilbert spaces are excepted since any subspace is topologically complemented therein.

There are also results on possibility of minimal continuation of bounded operators in concrete cases. The notion of Banach space of type $\mathfrak{M}$ was introduced by Nakhbin and he gave criterion on possibility of minimal continuation of bounded operators in real spaces [2, p.241]. Notice that consideration of the real case is essential in these results, and the criterion doesn't cover complex analogy of Hahn-Banach classic theorem.

Let $X_0 \subset X$ be some sub-space and

$$X_f^* \equiv \left\{ g \in X^* \colon \|g\|_{X^*} = \|f\|_{X_0^*}, g\,|_{X_0} = f \right\}, \quad \forall f \in X_0^*.$$

Denote by π an operator that associates to each $f \in X_0^*$ its minimal continuation $g \in X_f^*$: $\pi f = g$. The paper [3] is devoted to study of a uniqueness and linearity of minimal continuation operator π in the terms of weak differentiability of the norm and complementary subspace.

In the present paper we prove a sufficient condition on the existence of minimal continuation operator. Cite some necessary denotation and definitions:

B-space –a Banach space;

$\|\cdot\|_X$-the norm in X;

X^*-a space adjoint to X;

$T\,|_M$-contraction of T on M;

T^*-an operator adjoint to T;

D_T-domain of definition of the operator T;

R_T-a set of values of the operator T;

$\overline{M}$-closure of the set M;

$\mathcal{L}(X;Y)$-a Banach space of bounded operators acting from X to Y;

o-a composition sign.

Definition. *Let X be a B-space and $\mathcal{F} \equiv \{F \colon F \subset X^*\}$ be some linear structure from the subsets X^*. X is said to be $\mathcal{F}$-additive, if for $\forall x \in X$ it holds*

$$x(F_1 + F_2) = x(F_1) + x(F_2), \quad \forall F_i \in \mathcal{F}, \quad i = 1, 2,$$

where

$$x(F) = \inf_{f \in F} f(x) + \sup_{f \in F} f(x).$$

Before we pass to the statement of main results, we form some constructions. So, let $T \in \mathcal{L}(D_T, Y)$, $D_T \subset X$, X and Y be some B-spaces. In Y^* we introduce an equivalence ratio $\sim$ in the following form:

$$g_1 \sim g_2 \iff g_1(Tx) = g_2(Tx),$$

where

$$\forall x \in D_T \ \text{ and } \ g_i \in Y^* \quad \text{for} \quad i = 1, 2.$$

Thus, Y^* is divided into the co-sets $Y^* |_\sim$. By G we denote the elements of $Y^* |_\sim$. Let $g \in Y^*$. Clearly, $g(Tx)$ is a bounded linear functional on D_T, i.e. $g \circ T \in D_T^*$. Denote the norm of the functional $g \circ T$ on D_T by $\|g \circ T\|_{D_T^*}$. Obviously

$$\|g_1 \circ T\|_{D_T^*} = \|g_2 \circ T\|_{D_T^*}, \quad \forall g_i \in G, \ i = 1, 2;$$

i.e. the norms of the functional of the same class co-sets coincide. The class G_0 containing $0 \in Y$ is a zero element of $Y^* |_\sim$. Thus, for $\forall x \in D_T$: we can determine $G(x) = g(Tx)$ where $\forall g \in G$. Introduce in $Y^* |_\sim$ a linear structure: under λG we'll understand a class containing an element λg for some $g \in G$. Such a definition is correct, since it follows from $g_1 \sim g_2$ that $\lambda g_1 \sim \lambda g_2$ and vice versa, if $\lambda g \in \lambda G$ then $\lambda f \in \lambda G$ for $\forall f \in G$.

By G_g we denote a class G containing an element g. Define $G_{g_1} + G_{g_2}$ as $G_{g_1+g_2}$ i.e. $G_{g_1} + G_{g_2} \stackrel{def}{=} G_{g_1+g_2}$. It is easy to see that such definition of the sum is also correct and thus $Y^* |_\sim$ becomes a linear space. Accepting $\|G\|_{D_T^*} = \|g \circ T\|_{D_T^*}$ for $\forall g \in G$ and show that $\|G\|_{D_T^*}$ is a norm in $Y^* |_\sim$. We have:

$$\|\lambda G_g\|_{D_T^*} = \|\lambda G_{\lambda g}\|_{D_T^*} = \|(\lambda g) \circ T\|_{D_T^*} =$$

$$= \|\lambda (g \circ T)\|_{D_T^*} = |\lambda| \cdot \|g \circ T\|_{D_T^*} = |\lambda| \cdot \|G_g\|_{D_T^*}$$

$$\|G_{g_1} + G_{g_2}\|_{D_T^*} = \|G_{g_1+g_2}\|_{D_T^*} = \|(g_1 + g_2) \circ T\|_{D_T^*} \le$$

$$\le \|g_1 \circ T\|_{D_T^*} + \|g_2 \circ T\|_{D_T^*} = \|G_{g_1}\|_{D_T^*} + \|G_{g_2}\|_{D_T^*}.$$

$$\|G_g\|_{D_T^*} = 0 \implies \|g \circ T\|_{D_T^*} = 0 \implies g(Tx) = 0, \quad \forall x \in D_T.$$

Hence we have $g \in G_0$, consequently $G_g = G_0$.

So, $Y^* |_\sim$ with the norm $\|\cdot\|_{D_T^*}$ is a normed space.

Obviously, a $G_1 \ne G_2$ will mean $\exists x_0 \in D_T$:

$$g_1(x_0) \ne g_2(x_0), \quad \text{where} \quad \forall g_i \in G_i, \quad i = 1, 2.$$

Moreover, it is easy to see that $G \in D_T^*$ i.e. $Y^* /_\sim \subset D_T^*$. Take some Hamel basis $\mathcal{F} \equiv \{G_\alpha\}_{\alpha \in M} \subset Y^* |_\sim$ in $Y^* |_\sim$ and $\forall G \in Y^* |_\sim$. Assuming

$$X_G^* \equiv \left\{ x^* \in X^* : {}^{x^*}/_{D_T} = G; \ \|x^*\|_{X^*} = \|G\|_{D_T^*} \right\},$$

we consider a family $\left\{X^*_{G_\alpha}\right\}_{\alpha\in M}$. Determine

$$\lambda X^*_{G_\alpha} \overset{det}{=} X^*_{\lambda G_\alpha}, \quad \forall\lambda \in C$$

and

$$X^*_{G_\alpha} + X^*_{G_\beta} \overset{det}{=} X^*_{G_\alpha+G_\beta}.$$

Thus, under $\lambda X^*_{G_\alpha} + \mu X^*_{G_\beta}$ we'll understand $X^*_{\lambda G_\alpha+\mu G_\beta}$:

$$\lambda X^*_{G_\alpha} + \mu X^*_{G_\beta} \overset{det}{=} X^*_{\lambda G_\alpha+\mu G_\beta}.$$

We denote the obtained linear space by $\mathcal{F}_T$. In other words, on the base of Hamel basis $\mathcal{F}_T$ we take $\left\{X^*_{G_\alpha}\right\}_{\alpha\in M}$. It is easy to see that with respect to the introduced linear operations, the family $\left\{X^*_{G_\alpha}\right\}_{\alpha\in M}$ is an independent system. Thus, for $\forall F \in \mathcal{F}_T$ there exists a unique finite sequence of numbers $\{\lambda_k\} \subset C$: $F = X^*_{\sum\limits_k \lambda_k G_{\alpha_k}}$, and vice versa, for $\forall G \in Y^* |_\sim$ there exists a unique element $F \in \mathcal{F}_T$: $F = X^*_G$. Considering that for $G \quad \exists! \{\lambda_k\} \subset C : G = \sum\limits_k \lambda_k G_{\alpha_k}$ we have $X^*_G = X^*_{\sum\limits_k \lambda_k G_{\alpha_k}} \in \mathcal{F}_T$. We get that between $Y^* |_\sim$ and $\mathcal{F}_T$ there exists a linear one-to-one correspondence $\widetilde{T}$: $Y^* |_\sim \leftrightarrow \mathcal{F}_T$ defined by the expression:

$$\widetilde{T}\left(\sum_k \lambda_k G_{\alpha_k}\right) = X^*_{\sum\limits_k \lambda_k G_{\alpha_k}} \in \mathcal{F}_T.$$

Determine the norm $\mathcal{F}_T$ in $\|\cdot\|_{\mathcal{F}_T}$ in the following way:

$$\|F\|_{\mathcal{F}_T} = \left\|\widetilde{T}^{-1}F\right\|_{D^*_T}, \quad \forall F \in \mathcal{F}_T. \tag{1}$$

Now, show that $\|\cdot\|_{\mathcal{F}_T}$ is really a norm in $\mathcal{F}_T$. Notice that $X^*_{G_0} = \{0\}$ i.e. $X^*_{G_0}$ contains only zero functional from X^*. In fact, it follows from $\|G_0\| = 0$ that $\|x^*\|_{X^*} = 0$ for $\forall x^* \in X^*_{G_0}$ and so $x^* = 0$. Consequently,$\widetilde{T}^{-1}0 = G_0$ whence $\widetilde{T}G_0 = 0$. So, we have:

$$\|F\|_{\mathcal{F}_T} = 0 \implies \left\|\widetilde{T}^{-1}F\right\|_{D^*_T} = 0 \implies \widetilde{T}^{-1}F = G_0 \implies F = 0.$$

The relation $\|\lambda F\|_{\mathcal{F}_T} = |\lambda| \cdot \|F\|_{\mathcal{F}_T}$ is obvious. Further,

$$\|F_1 + F_2\|_{\mathcal{F}_T} = \left\|\widetilde{T}^{-1}F_1 + \widetilde{T}^{-1}F_2\right\|_{D^*_T} \leq$$

$$\leq \left\|\widetilde{T}^{-1}F_1\right\|_{D^*_T} + \left\|\widetilde{T}^{-1}F_2\right\|_{D^*_T} = \|F_1\|_{\mathcal{F}_T} + \|F_2\|_{\mathcal{F}_T}.$$

So, $\|\cdot\|_{\mathcal{F}_T}$ is a norm in $\mathcal{F}_T$. It directly follows from (1) that $\widetilde{T}$ is an isometric isomorphism between $Y^* |_\sim$ and $\mathcal{F}_T$: $\widetilde{T} \in \mathcal{L}\left(Y^* |_\sim ; \mathcal{F}_T\right)$ Let's consider $\widetilde{T}^* : \mathcal{F}^*_T \to (Y^* |_\sim)^*$. By the definition of adjoint operator we have:

$$F^*(\widetilde{T}G) = (\widetilde{T}^*F^*)G, \quad \forall G \in Y^* |_\sim, \quad \forall F^* \in \mathcal{F}^*_T. \tag{2}$$

Now we formulate the main lemma.

Lemma. *Let* X; Y *be real* B*-spaces,* $Y = Y^{**}$. $T : D_T \to Y$ *be a bounded operator,* D_T *be a subspace of the space* X*. If* X *is* $\mathcal{F}_T$ *invariant and* $\overline{\Re}_T = Y$ *then we can continue the operator* T *on the whole of the space* X *preserving the norm.*

Proof. Let's take $\forall x \in X$ and consider the functional $x : \mathcal{F}_T \to R$ determined by the expression:

$$x(F) = \frac{1}{2}\left[\inf_{f\in F} f(x) + \sup_{f\in F} f(x)\right], \quad F \in \mathcal{F}_T.$$

Show that this functional is bounded. Really, if $f \in F$, then

$$|f(x)| \le \|f\|_{X^*} \cdot \|x\|_X = \left\|\widetilde{T}^{-1} F\right\|_{D_T^*} \cdot \|x\|_X = \|x\|_X \cdot \|F\|_{\mathcal{F}_T},$$

and as the result,

$$|x(F)| \le \|x\|_X \cdot \|F\|_{\mathcal{F}_T}, \quad \forall F \in \mathcal{F}_T. \tag{3}$$

Consequently, the functional $x(F)$ is bounded. Obviously, if $\lambda \ge 0$ then

$$x(\lambda F) = \lambda x(F).$$

We have:

$$2x(-F) = \inf_{f\in -F} f(x) + \sup_{f\in -F} f(x) = \inf_{f\in F} (-f)(x) + \sup_{f\in F}(-f)(x) =$$

$$= -\sup_{f\in F} f(x) + \left(-\inf_{f\in F} f(x)\right) = -2x(F).$$

It directly follows from this relation that $x(\lambda F) = \lambda x(F)$, $\lambda < 0$ and so $x(F)$ is a homogeneous functional. It follows from the condition of the lemma that it is additive as well. As the result we get that X is imbedded into $\mathcal{F}_T^*$. Now in (2) as F we take $x \in X$;

$$x(\widetilde{T}G) = (\widetilde{T}^* x)G, \quad \forall G \in Y^* |_\sim . \tag{4}$$

If $x \in D_T$, clearly

$$x(\widetilde{T}G) = g(Tx), \quad \forall g \in G. \tag{5}$$

Really, $\widetilde{T}G \in \mathcal{F}_T$ consists of functionals $f \in X^*$ for which $f/D_T = G$ and consequently $f(x) = G(x) = g(Tx)$, $\forall f \in \widetilde{T}G$ and $\forall g \in G$.

On the other hand, it is easy to see that if $\overline{\Re}_T = Y$ then each class of $G \in Y^* |_\sim$ consists of a unique element $g \in Y^*$. Thus, in this case we get the imbedding $Y^* \subset Y^* |_\sim$ and this imbedding is continuous since for $\forall g \in Y^*$ we have:

$$\|g\|_{Y^*/_\sim} = \|Gg\|_{D_T^*} = \|g \circ T\|_{D_T^*} \le \|T\|_{D_T} \cdot \|g\|_{Y^*}.$$

Consequently, $(Y^* |_\sim)^* \subset Y^{**} = Y$ and as the result, $\widetilde{T}^*{:}\mathcal{F}_T^* \longmapsto Y$. Taking into account (5) in (2) we get:

$$g(Tx) = g(\widetilde{T}^* x), \quad \forall g \in Y^*,$$

and so

$$Tx = \widetilde{T}^* x, \quad \forall x \in D_T. \tag{6}$$

Thus, the operator $\widetilde{T}^*$ is a bounded continuation of the operator T from D_T to $\mathcal{F}_T^*$, moreover $X \hookrightarrow \mathcal{F}_T^*$.

Further, the operator T^* acting from $\mathcal{F}_T^*$ to $(Y^* |_\sim)^*$ has the norm equal to a unit, since

$$\left\| \widetilde{T}^* \right\|_{\mathcal{L}\left(\mathcal{F}_T^*;(Y^*|_\sim)^*\right)} = \left\| \widetilde{T} \right\|_{\mathcal{L}(Y^*|_\sim;\mathcal{F}_T)} = 1.$$

On the other hand we can consider $\widetilde{T}^*$ as an operator acting from X to Y. In this case we'll have directly from (6):

$$\left\| \widetilde{T}^* \right\|_{\mathcal{L}(X;Y)} \geq \|T\|_{D_T}.$$

Moreover, from (4) we have:

$$\left| \left(\widetilde{T}^* x \right) G_g \right| \leq \|x\|_{\mathcal{F}_T^*} \cdot \left\| \widetilde{T} G_g \right\|_{\mathcal{F}_T} = \|x\|_{\mathcal{F}_T^*} \cdot \|G_g\|_{Y^*|_\sim} =$$

$$= \|x\|_{\mathcal{F}_T^*} \cdot \|g \circ T\|_{D_T^*} \leq \|x\|_{\mathcal{F}_T^*} \cdot \|T\|_{D_T} \cdot \|g\|_{Y^*}, \quad \forall g \in Y^*.$$

Considering $\widetilde{T}^* x$ as a functional acting from Y^* we get

$$\left\| \widetilde{T}^* x \right\|_{Y^{**}} \leq \|T\|_{D_T} \cdot \|x\|_{\mathcal{F}_T^*}.$$

Identifying Y^{**} with Y we get:

$$\left\| \widetilde{T}^* x \right\|_{Y} \leq \|T\|_{D_T} \cdot \|x\|_{\mathcal{F}_T^*}.$$

On the other hand, it follows from (3) that

$$\|x\|_{\mathcal{F}_T^*} \leq \|x\|_X.$$

As the result

$$\left\| \widetilde{T}^* x \right\|_{Y} \leq \|T\|_{D_T} \cdot \|x\|_X \implies \left\| \widetilde{T}^* \right\|_{\mathcal{L}(X;Y)} \leq \|T\|_{D_T}.$$

From the last two relations we get

$$\left\| \widetilde{T} \right\|_{\mathcal{L}(X;Y)} = \|T\|_{D_T},$$

that completely proves the lemma.

Now, let's consider the general case, i.e. let, generally speaking, $\overline{\Re}_T = Y_0 \neq \neq Y$. It follows from $Y = Y^{**}$ that $Y_0 = Y_0^{**}$. In the previous lemma considering instead of the operator $T{:}D_T \to Y$ the operator $T{:}D_T \to Y_0$ we get the main result of the paper.

Theorem. *Let $X;Y$ be real B-spaces, $Y = Y^{**}$ and the operator $T{:}X \to Y$ be bounded on $D_T \subset X$. It the space is X $\mathcal{F}_T$-invariant, then we can continue T on the whole of X preserving the norm.*

References

[1]. Helemskiy A.Ya. *Lectures on functional analysis,* M., 2004, 552 p. (Russian)

[2]. Kantorvich l.V., Akilov G.P. *Functional analysis,* M., Nauka, 1984, 752 p. (Russian)

[3]. Belobrov P.K. *On an operator of minimal continuation of linear functionals* // Mat. zametki, 1977, vol 21, No4, pp. 539-555. (Russian)

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Received February 20, 2008; Revised May 23, 2008.