MATHEMATICS

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ON SOME SPACES OF GENERALIZED SMOOTHNESS

Abstract

In the paper we consider genetralized smoothness spaces determined in the Fourier-Bessel transforms terms. We find sufficient conditions for these spaces to be normed rings and also the conditions for their continuous imbedding into the space of continuous functions.

Let $R_n\,(n\geq 1)$ be an n-dimensional Euclidean space, $R_n^+=\{x\in R^n, x_n>0\}$. $L_{2,\gamma}(R_n^+)$ be a space of functions that are even with respect to x_n and integrable in the square on R_n^+, with weight $x_n^\gamma\,(\gamma>0)$; $\mu(x)=\mu(x_1,x_2,...,x_n)$ be a weight function [1], i.e. continuous in R_n^+ and satisfying the condition $\mu(x)\cdot(\mu(x))^{-1}\leq C(1+(x-y))^l$ for any $x,y\in R_n^+$, with constants C and l that depend only on the function $\mu(x)$ itself;

$$T_x^y f(x)=c_\gamma\int_0^\pi f\left(x' - y', \sqrt{x_n^2+y_n^2-2x_ny_n\cdot\cos\alpha}\right)\sin^{\gamma-1}\alpha\, d\alpha,$$

$c_\gamma^{-1}=\int\limits_0^\pi \sin\alpha d\alpha$, $x', y'\in R_{n-1}$, be a generalized shear generated by the Laplace-Bessel differential operator;

$$(f*g)(x)=\int_{R_n^+}T_x^y f(x)\cdot g(y)\cdot y_n^\gamma dy \qquad (*)$$

be a convolution in the space $L_{2,\gamma}(R_n^+)$.

By $L_\gamma^\mu(R_n^+)$ we denote a set of functions $f(x)$ measurable on R_n^+, such that $f(x)\cdot\mu(x)\in L_{2,\gamma}(R_n^+)$.

Theorem 1. *Let $\mu(x)$ be a weight function. It there exists such a function $\widetilde{\mu}(x,y)>0$ that*

$$T_x^y\widetilde{\mu}^2(x,y)\leq\mu^2(x),\quad \forall y\in R_n^+; \qquad (1)$$

$$\mu^2(x)\cdot\int\limits_{R_n^+}\left(\widetilde{\mu}^2(x,y)\right)^{-2}(\mu(y))^{-2}y_n^\gamma dy<C^2<\infty, \qquad (2)$$

where C is some real number, then for any $f,g\in L_\gamma^\mu(R_n^+)$, $f*g\in L_\gamma^\mu$ and $\|f*g\|_{L_\gamma^\mu}\leq C\cdot\|f\|_{L_\gamma^\mu}\cdot\|g\|_{L_\gamma^\mu}$.

For the proof of the theorem we'll need the following.

Lemma A. *Let $f^2\in L_{2,\gamma}(R_n^+)$; then*

$$\left(T_x^y f(x)\right)^2\leq T_x^y f^2(x). \qquad (3)$$

Really, applying the Holder inequality, we get

$$\left(T_x^y f(x)\right)^2=\left(c_\gamma\int\limits_0^\pi f\left(x' - y', \sqrt{x_n^2+y_n^2-2x_ny_n\cdot\cos\alpha}\right)\sin^{\gamma-1}\alpha\, d\alpha\right)^2\leq$$

$$\leq c_\gamma \int_0^\pi f\left(x' - y', \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cdot \cos\alpha}\right) \sin^{\gamma-1}\alpha \, d\alpha \cdot c_\gamma \int_0^\pi \sin^{\gamma-1}\alpha \, d\alpha =$$

$$= c_\gamma \int_0^\pi f^2\left(x' - y', \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\alpha}\right) \sin^{\gamma-1}\alpha \, d\alpha = T_x^y f^2(x).$$

Q.E.D.

Now, let's prove the theorem. We have

$$\|f * g\|_{L_\gamma^\mu}^2 = \int_{R_n^+} \mu^2(x) \Big(\int_{R_n^+} T_x^y f(x) \cdot \widetilde{\mu}(x,y) \cdot (\widetilde{\mu}(x,y))^{-1} \times$$

$$\times g(y) \cdot \mu(y) \cdot (\mu(y))^{-1} \cdot y_n^\gamma dy \Big)^2 \cdot x_n^\gamma dx$$

Taking into account conditions (1) and (2) of theorem 1, by lemma A we get

$$\|f * g\|_{L_\gamma^\mu}^2 \leq C^2 \int_{R_n^+}\int_{R_n^+} \left(T_x^y f(x) \cdot \widetilde{\mu}(x,y)\right)^2 (g(y) \cdot \mu(y))^2 x_n^\gamma y_n^\gamma dx dy \leq$$

$$\leq C^2 \int_{R_n^+}\int_{R_n^+} g^2(y) \cdot \mu^2(y) \left(\int_{R_n^+} T_x^y f^2(x) \cdot \widetilde{\mu}^2(x,y) \cdot x_n^\gamma dx \right) y_n^\gamma dy \leq$$

$$C^2 \int_{R_n^+} f^2(x) \cdot \cdot T_x^y \mu^2(x) \cdot x_n^\gamma dx \cdot \|g\|_{L_\gamma^\mu}^2 = C^2 \|f\|_{L_\gamma^\mu}^2 \cdot \|g\|_{L_\gamma^\mu}^2.$$

The theorem is proved.

Thus, the space $L_\gamma^\mu(R_n^+)$ becomes a normed ring (with convolution $(*)$ as multiplication), if the conditions of theorem 1 are fulfilled for the weight function μ.

Let $\mu(x) = (1 + |x|)^l$, $l > (n+\gamma)/2$. It is easy to verify that $\mu(x)$ is a weight function. Assume $\widetilde{\mu}(x,y) = (1 + |x + \widetilde{y}|)^l$, where $x \in R_n^+$, $y \in R_n^+$, $\widetilde{y} = (y_1, y_2, ..., y_{n-1}, y_n)$.

Let's show that for the pairs $\mu(x)$ and $\widetilde{\mu}(x,y)$ the conditions 1) and 2) of theorem 1 are fulfilled. At first we prove that for all the values $x_n > 0$, $y_n > 0$ and $\alpha \in [0,\pi]$

$$A^{df} = \left(\left(\sqrt{x_n^2 + y_n^2 - 2x_n y_n \cdot \cos\alpha} - y_n\right)\right)^2 \leq x_n^2 \qquad (**)$$

Consider the cases:
1) $\sqrt{x_n^2 + y_n^2 - 2x_n y_n \cdot \cos\alpha} \geq y_n \Rightarrow A \leq (x_n + y_n - y_n)^2 = x_n^2$;
2) $\sqrt{x_n^2 + y_n^2 - 2x_n y_n \cdot \cos\alpha} < y_n \Rightarrow A \leq (y_n - |x_n - y_n|)^2$ *whence* :
$a)$ *if* $x_n \leq y_n$, *if* $A \leq (y_n + x_n - y_n)^2 = x_n^2$;
$b)$ *if* $x_n > y_n$, *if* $A \leq (y_n - x_n + y_n)^2 = (2y_n - x_n)^2 < (2x_n - x_n)^2 = x_n^2$.
Thus, the estimation $(**)$ is proved.
Taking into account $(**)$ we get

$T_x^y \widetilde{\mu}^2(x,y) =$

$$= c_\gamma \int\limits_0^\pi \left(1 + \sqrt{x_1^2 + x_2^2 + ... + x_{n-1}^2 + \left(\sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\alpha} - y_n \right)^2} \right)^{2l} \times$$

$$\times \sin^{\gamma-1}\alpha \, d\alpha \le c_\gamma \int\limits_0^\pi \left(1 + \sqrt{x_1^2 + x_2^2 + ... + x_n^2} \right)^{2l} \sin^{\gamma-1}\alpha \, d\alpha =$$

$$= \left(1 + \sqrt{x_1^2 + x_2^2 + ... + x_n^2} \right)^{2l} \cdot c_\gamma \cdot \int\limits_0^\pi \sin^{\gamma-1}\alpha \, d\alpha = \mu^2(x)$$

whence it follows that (1) holds.

Further, for $l > (n+\gamma)/2$ we have

$$\int\limits_{R_n^+} \frac{\mu^2(x)}{\widetilde{\mu}^2(x,y) \cdot \mu^2(y)} y_n^\gamma dy = \int\limits_{R_n^+} \left[\frac{1+|x|}{(1+|x+\widetilde{y}|)(1+|y|)} \right]^{2l} y_n^\gamma dy <$$

$$< \int\limits_{R_n^+} \left[\frac{1+|x+\widetilde{y}|+|y|}{(1+|x+\widetilde{y}|)(1+|y|)} \right]^{2l} y_n^\gamma dy <$$

$$< 2^{2l} \int\limits_{R_n^+} \left(\frac{1}{(1+|x+\widetilde{y}|)^{2l}} + \frac{1}{(1+|y|)^{2l}} \right) y_n^\gamma dy < c^2,$$

whence validity of (2) follows.

Obviously, $L_\gamma^\mu(R_n^+)$ is a complete reflexive Banach space.

Introduce the space $H_\gamma^\mu(R_n^+)$. By definition $H_\gamma^\mu(R_n^+)$ is a set of functions $f \in D_{even}$ (D_{even} is a set of infinitely differentiable in R_n^+ functions, even with respect to variable x_n and having a compact support in R_n^+), such that $F_B\, f(x) \in L_\gamma^\mu(R_n^+)$, where

$$F_B f(x) = \int\limits_{R_n^+} f(x) \cdot e^{-i(x', y')} \cdot j(x_n, y_n) \cdot x_n^\gamma dx$$

is a Fourier-Bessel transform [2].

We determine the norm in the space $H_\gamma^\mu(R_n^+)$ in the following way:

$$\|f\|_{H_\gamma^\mu} = \left(\int\limits_{R_n^+} (F_B\, f(x) \cdot \mu(x))^2 x_n^\gamma dx \right)^{1/2} \qquad (4)$$

Obviously [1],

$$F_B H_\gamma^\mu = L_\gamma^\mu. \qquad (5)$$

It follows from the equality (5) that the Fourier-Bessel operator establishes isomorphism between the spaces H_γ^μ and L_γ^μ and by definition of the norm in H_γ^μ , this isomorphism is isometric. Hence, it follows that H_γ^μ is a reflexive Banach space, since the space L_γ^μ possess these properties.

In the space H_γ^μ determine the multiplication operation:

$$f \cdot g = F_B^{-1}\left(F_B f * F_B g\right). \tag{6}$$

Then the space H_γ^μ becomes a normed ring with multiplication operation (6) when the conditions of theorem 1 are fulfilled.

Theorem 2. *For the space H_γ^μ to be continuously imbedded into the space of functions continuous on R_n^+ it suffices*

$$\int_{R_n^+} (\mu(x))^{-2} \cdot x_n^\gamma dx < A^2 < \infty. \tag{7}$$

Proof. By the Fourier-Bessel inversion formula we have; $\left(f \in D_{even}\left(\overline{R}_n^+\right)\right)$

$$f(x) = \left(\pi^{n-1}\alpha^{n+\gamma-2}\Gamma^2\left(\frac{\gamma+1}{2}\right)\right)^{-1} \int_{R_n^+} F_B^{-1} f(y)\, e^{i(x',y')} j(x_n, y_n)\, y_n^2 dy.$$

Applying the Schwartz inequality, we get

$$\|f\|_{C(R_n^+)} = \max|f(x)| \le \left(\pi^{n-1}\alpha^{n+\gamma-2}\Gamma^2\left(\frac{\gamma+1}{2}\right)\right)^{-1} \times$$

$$\times \int_{R_n^+} |F_B f(y) \cdot \mu(y)| \cdot (\mu(y))^{-1} y_n^\gamma dy \le \left(\pi^{n-1}\alpha^{n+\gamma-2}\Gamma^2\left(\frac{\gamma+1}{2}\right)\right)^{-1} \times$$

$$\times \left(\int_{R_n^+} |F_B f(y) \cdot \mu(y)|^2 y_n^\gamma\right)^{1/2} \left(\int_{R_n^+} (\mu(y))^{-2} y_n^\gamma dy\right)^{1/2},$$

whence

$$\|f\|_{C(R_n^+)} \le A\left(\pi^{n-1}\alpha^{n+\gamma-2}\Gamma^2\left(\frac{\gamma+1}{2}\right)\right)^{-1} \cdot \|f\|_{H_\gamma^\mu},\ f \in D_{even}\left(R_n^+\right). \tag{8}$$

Now, let's show that $H_\gamma^\mu \subset C$. Let f be an arbitrary element from the space H_γ^μ. By $\{f_k\} \subset D_{even}\left(\overline{R}_n^+\right)$ denote a sequence of functions converging to f in $H_\gamma^\mu : \|f_k - f\|_{H_\gamma^\mu} \underset{n\to\infty}{\to} 0\,(k \to \infty)$.

By (8) this sequence is fundamental in the space $C\left(R_n^+\right)$ and its limit in $C\left(R_n^+\right)$ also coincides with f.

References

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Received January 29, 2008; Revised April 22, 2008.