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ASYMPTOTIC INVESTIGATION OF THE ROOTS OF A CHARACTERISTIC EQUATION OBTAINED IN A PROBLEM OF ELASTICITY THEORY FOR TRANSVERSALLY-ISOTROPIC HOLLOW CONE

Abstract

We investigate the roots of a characteristic equation of a problem of elasticity theory for transversally-isotropic hollow cone of variable thickness with respect to spectral parameter. Statement on the existence of three groups of zeros with asymptotic properties is proved.

In the paper [1] an axially-symmetric problem on elastic equilibrium of transversally-isotropic frustum of hollow cone of variable thickness is considered. The following characteristic equation that was not studied completely is obtained.

In the present paper we conduct complete analysis of the roots of the characteristic equation depending on the thinwallness of the construction.

The characteristic equation is of the form [1]:

$$\begin{gathered}\Delta = \frac{2d_{11}d_{22}C_{11}C_{13}}{\sin\theta_1 \sin\theta_2} - d_{22}^2 D_{\gamma_2}^{(1.1)}(\theta_1,\theta_2)\times \\ \Big[C_{11}^2 D_{\gamma_1}^{(0.0)}(\theta_1,\theta_2) + C_{11}C_{12}ctg\theta_1 D_{\gamma_1}^{(1.0)}(\theta_1,\theta_2) + \\ +C_{11}C_{12}ctg\theta_2 D_{\gamma_1}^{(0.1)}(\theta_1,\theta_2) + C_{12}^2 ctg\theta_1 ctg\theta_2 D_{\gamma_1}^{(1.1)}(\theta_1,\theta_2)\Big] + \\ +d_{11}d_{22}\Big[C_{11}D_{\gamma_1}^{(0.1)}(\theta_1,\theta_2) + C_{12}ctg\theta_1 D_{\gamma_1}^{(1.1)}(\theta_1,\theta_2)\Big]\times \\ \times\Big[C_{13}D_{\gamma_2}^{(1.0)}(\theta_1,\theta_2) + C_{14}ctg\theta_2 D_{\gamma_2}^{(1.1)}(\theta_1,\theta_2)\Big] - \\ -d_{11}^2 D_{\gamma_1}^{(1.1)}(\theta_1,\theta_2)\Big[C_{13}^2 D_{\gamma_2}^{(0.0)}(\theta_1,\theta_2) + C_{13}C_{14}ctg\theta_1\times \\ \times D_{\gamma_2}^{(1.0)}(\theta_1,\theta_2) + C_{13}C_{14}ctg\theta_2 D_{\gamma_2}^{(0.1)}(\theta_1,\theta_2) + C_{14}^2 ctg\theta_1\times \\ \times ctg\theta_2 D_{\gamma_2}^{(1.1)}(\theta_1,\theta_2)\Big] + d_{11}d_{22}\Big[C_{13}D_{\gamma_2}^{(0.1)}(\theta_1,\theta_2) + \\ +C_{14}ctg\theta_1 D_{\gamma_2}^{(1.1)}(\theta_1,\theta_2)\Big]\times \\ \times C_{11}D_{\gamma_1}^{(1.0)}(\theta_1,\theta_2) + C_{12}ctg\theta_2 D_{\gamma_1}^{(1.1)}(\theta_1,\theta_2)\Big] = 0,\end{gathered} \tag{1}$$

where

$$d_1 = A_1 + (z-3/2)b_0, \qquad d_2 = A_2 + (z-3/2)b_0$$

$$C_{11} = \left[b_{12}\left(z-\frac{1}{2}\right) + b_{22} + b_{23}\right]A_1 - \gamma_1(\gamma_1+1)b_{22}b_0$$

$$C_{13} = (b_{23}-b_{22})b_0, \qquad C_{14} = C_{13}$$

$$C_{12} = [b_{12}(z-1/2) + b_{22} + b_{23}]\,A_2 - \gamma_2(\gamma_2+1)b_{22}b_0.$$

$$D_t^{(s,l)}(\varphi,\psi) = P_t^{(s)}(\cos\varphi)Q_t^{(l)}(\cos\psi) -$$

$$-P_t^{(s)}(\cos\psi)Q_t^{(l)}(\cos\psi), \qquad (s,l=0,1)$$

As we see from (1) the characteristic equation is of complicated structure. In order to study its roots effectively, we make some assumptions for the geometrical parameters of a conic body, namely we assume

$$\theta_1 = \theta_0 - \varepsilon, \qquad \theta_2 = \theta_0 + \varepsilon \tag{2}$$

where θ_0 is an opening of a surface of a shell, ε is dimensionless parameter defining its thickness. Below we'll assume that ε is a small parameter,

$$0 < \xi_1 < \theta_0 < \xi_2 < \frac{\pi}{2}.$$

Substituting (2) into (1) we get

$$D(z, \varepsilon, \theta_0) = \Delta(z, \theta_1, \theta_2) = 0$$

For the roots of the functions $D(z, \varepsilon, \theta_0)$ we can formulate the following statements.

Statement. The function $D(z, \varepsilon, \theta_0)$ has three groups of roots:

a) the first group consists of two multiple roots $z_1 = -\frac{1}{2}$ and $z_2 = \frac{1}{2}$.

b) the second group consists of four roots that as $\varepsilon \to 0$ have the order $0\left(\varepsilon^{-\frac{1}{2}}\right)$.

c) the third group of roots consists of denumerable set of roots that as $\varepsilon \to 0$ have the order $0\left(\varepsilon^{-1}\right)$.

Let's give the proof scheme of the first statement. For this assuming $\varepsilon z \to 0$ and expanding the function $D_z^{(s,l)}(\varphi, \phi)$ in the vicinity $\theta = \theta_0$ in a series of ε, we get:

$$D_z^{(0,0)}(\theta_1, \theta_2) = \varepsilon \sin^{-1}\theta_0 \left\{ -2 + \frac{1}{3}\left[4z(z+1) - (1 + 2ctg^2\theta_0)\right]\varepsilon^2 + \frac{2}{5}\times \right.$$
$$\times\left[(-16z^2(1+z^2) + 24 + 32ctg^2\theta_0)z(z+1) - \right.$$
$$\left. \left. -24ctg^4\theta_0 + 28ctg^2\theta_0 + 5)\right]\varepsilon^4 + ..., \right\}$$

$$D_z^{(1,1)}(\theta_1, \theta_2) = \frac{-z(z+1)}{\sin\theta_0}\varepsilon \left\{ 2 + \frac{1}{3}\left[5 + 6ctg^2\theta_0 - 4z(z+1)\right]\varepsilon^2 + \right.$$
$$+\frac{2}{5!}\left[110ctg^4\theta_0 + 146ctg^2\theta_0 + 57 - \right. \tag{3}$$
$$\left. \left. -(56 + 64ctg^2\theta_0)z(z+1) + 16z^2(z+1)^2\right]\varepsilon^4 + ... \right\}$$

$$D_z^{(1,0)}(\theta_1, \theta_2) = \sin^{-1}\theta_0 \left\{ 1 + ctg\theta_0\varepsilon + \frac{1}{2}\left[1 + 2ctg^2\theta_0 - 4z(z+1)\right]\varepsilon^2 + \right.$$
$$+\frac{1}{3!}\left[5 + 6ctg^2\theta_0 - 4z(z+1)\right]ctg\theta_0\varepsilon^3 + \frac{1}{4!}\times$$
$$\times\left[24ctg^4\theta_0 + 28ctg^2\theta_0 + 5 - (24 + 32ctg^2\theta_0)z(z+1) + 16z^2(z+1)^2\right]\varepsilon^4 +$$
$$+\frac{1}{5!}\left[74ctg^4\theta_0 + 101ctg^2\theta_0 + 57 - (56 + 64ctg^2\theta_0)z(z+1) + \right.$$
$$\left. +16z^2(z+1)\right]ctg\theta_0\varepsilon^5 + \frac{1}{6!}\left[394ctg^6\theta_0 + 663ctg^4\theta_0 + 286ctg^2\theta_0 + 16 - \right.$$
$$-(776ctg^4\theta_0 + 1028ctg^2\theta_0 + 231)\times$$

$$\times z(z+1)+288ctg^2\theta_0+240)z^2(z+1)^2-64z^3(z+1)^3\right]\varepsilon^6+...\}$$

$$D_z^{(0,1)}(\theta_1,\theta_2)=-\sin^{-1}\theta_0\left\{1-ctg\theta_0\varepsilon+\frac{1}{2}\left[(1+2ctg^2\theta_0)-4z(z+1)\right]\varepsilon^2-\right.$$

$$-\frac{1}{3!}\left[5+6ctg^2\theta_0-4z(z+1)\right]ctg\theta_0\varepsilon^3+\frac{1}{4!}\left[24ctg^4\theta_0+28ctg^2\theta_0+5-\right.$$

$$\left.-(24+32ctg^2\theta_0)z(z+1)+16z^2(z+1)\right]\varepsilon^4-\frac{1}{5!}\times$$

$$\times\left[74ctg^4\theta_0+101ctg^2\theta_0+57-(56+64ctg^2\theta_0)z(z+1)+\right.$$

$$\left.+16z^2(z+1)^2\right]ctg\theta_0\varepsilon^5+\frac{1}{6!}\left[394ctg^6\theta_0+663ctg^4\theta_0+286ctg^2\theta_0+16-\right.$$

$$-(776ctg^4\theta_0+1028ctg^2\theta_0+231)z(z+1)+$$

$$\left.\left.+(228ctg^2\theta_0+240)z^2(z+1)^2-64z^3(z+1)^3\right]\varepsilon^6+...\right\}$$

Substituting (3) into (1) and performing very complicated calculations we represent the function $D(z,\varepsilon,\theta_0)$ in the form:

$$\begin{aligned}
&D(z,\varepsilon,\theta_0)=2^{-1}\sin^{-2}\theta_0C_{13}(d_{22}-d_{11})^2(1-\nu)^2E_0G_0^{-1}b_{22}^{-1}\times\\
&\times\left[z^2+2(G_0-1)\right]\left(z^2-\frac{1}{4}\right)\varepsilon^2\left\{2(1-\nu_1\nu_2)ctg^2\theta_0+\frac{1}{3}\times\right.\\
&\times\left[2E_0z^4+(8\nu_1-5E_0-4-8(1+\nu)(G_0-\nu_2)E_0ctg^2\theta_0)z^2+\right.\\
&+6\nu_1(\nu_2-1)-8\nu_2+5+\frac{2}{E_0}+\frac{9}{8}E_0+(10(1-\nu_1\nu_2)+\\
&\left.+2(1+\nu)(G_0-\nu_2)E_0)ctg^2\theta_0+18(1-\nu_1\nu_2)ctg^4\theta_0\right]\varepsilon^2+\\
&\left.+\frac{1}{45}\left[-16E_0^2\frac{(1+\nu)(G_0-\nu_2)}{1-\nu_1\nu_2}z^6+...\right]\varepsilon^4+...\right\}=0
\end{aligned}\qquad(4)$$

Hence, it follows that

$$\lim_{z\to-\frac{1}{2}}D_1(z,\varepsilon,\theta_0)\neq0,\qquad\lim_{z\to-\frac{1}{2}}\frac{D_{\gamma_2}^{(1.1)}}{z+1/2}\neq0$$

$$\lim_{z\to-\frac{1}{2}}D_2(z,\varepsilon,\theta_0)=0$$

It follows there from that $z_1=-\dfrac{1}{2}$ is a double zero of the function $D(z,\varepsilon,\theta_0)$.

Since $D(z,\varepsilon,\theta_0)$ is an even function of z, it follows that $z_2=\dfrac{1}{2}$ is also a double zero of the function $D(z,\varepsilon,\theta_0)$.

Prove that all the remaining zeros of the function $D(z,\varepsilon,\theta_0)$ infinitely increase as $\varepsilon\to0$. Proceed from the contrary having assumed that $z_k\to z_k^*\neq\infty$ as $\varepsilon\to0$. Then the limiting relation

$$D(z,\varepsilon,\theta_0)\to\varepsilon^2D_0(z_k^*,\theta_0)$$

is valid as $\varepsilon\to0$. Thus, limiting points of the set of the roots z_k as $\varepsilon\to0$ are determined from the equation

$$D_0(z_k^*,\theta_0)=0$$

In the present case

$$D_0(z_k^*,\theta_0)=16(1-\nu^2)G_0\sin^2\theta_0 ctg^2\theta_0\left(z_k^*-\frac{1}{4}\right)=0.$$

It follows from the last equality that besides z_1, z_2 there are no other bounded roots.

So, it is proved that all the remaining zeros of the function $D(z,\varepsilon,\theta_0)$ tend to infinity as $\varepsilon\to 0$.

We can divide them into three groups depending on their behaviour as $\varepsilon\to 0$. The following limiting relations are possible:

1) $\varepsilon z_k\to 0$ as $\varepsilon\to 0$

2) $\varepsilon z_k\to const$ as $\varepsilon\to 0$

3) $\varepsilon z_k\to\infty$ as $\varepsilon\to 0$

At first we determine such z_k that $\varepsilon z_k\to 0$ as $\varepsilon\to 0$. To this end we use expansion (4).

Let the principal term of the asymptotes z_k have the form

$$z_k=\eta_0\varepsilon^{-\alpha},\qquad \eta_0=0(1)\quad \text{as}\quad \varepsilon\to 0 \tag{5}$$

Substituting (5) into (4) and leaving in it only principal terms for η_0, we get the following limit equation:

$$\begin{gathered}2(1-\nu_1\nu_2)ctg^2\theta_0+\frac{1}{3}\left[2E_0\eta_0^4+0(\varepsilon^{2\alpha})\right]\varepsilon^{2-4\alpha}+\\+0\left[\max\left(\varepsilon^{4-6\alpha},\varepsilon^{2-2\alpha}\right)\right]=0\end{gathered} \tag{6}$$

Let's consider three cases:

a) $0<\alpha<\frac{1}{2}$;

b) $\alpha=\frac{1}{2}$;

c) $\frac{1}{2}<\alpha<1$.

In the case a) passing to limit in (6) as $\varepsilon\to 0$ we'll get $\eta_0=0$, this contradicts the assumption (5). Similarly, in the case c) we get $\eta_0=0$ and we also get contradiction. In the case b) we have:

$$\eta_0^4+3(1-\nu_1\nu_2)E_0^{-1}ctg^2\theta_0=0 \tag{7}$$

We'll look for z_k in the form of the expansion

$$z_k=\alpha_k\varepsilon^{-\frac{1}{2}}+\alpha_k^{(0)}+\alpha_k^{(1)}\varepsilon^{\frac{1}{2}}+...,\qquad k=3,4,5,6 \tag{8}$$

where $\alpha_k=\eta_0,\quad \alpha_k^{(0)}=0$

$$\begin{gathered}\alpha_k^{(1)}=(40\alpha k_0)^{-1}\left[24(1+\nu)(G_0-\nu_2)ctg^2\theta_0+\right.\\ \left.+5(4+5E_0-8\nu_1)E_0^{-1}\right]\end{gathered}$$

In order to construct asymptotics of zeros in the case 2) ($\varepsilon z_k\to const$ for $\varepsilon\to 0$) we look for z_n $(n=k-6,\quad k=7,8,...)$

in the form

$$z_n = \varepsilon^{-1}\delta_n + 0(\varepsilon) \tag{9}$$

Substituting (9) into the equation

$$b_{22}\mu^2(\mu+1)^2 - \left[\left(b_{11}b_{22} - b_{12}^2 - 2b_{12}\right)\lambda(\lambda+1) + \right.$$
$$+2b_{22} + 2(b_{12} - b_{22} - b_{23})(G_0 - 1)]\,\mu(\mu+1) +$$
$$+b_{11}\lambda^2(\lambda+1)^2 + 2\left[b_{11}(G_0-1) + b_{12} - b_{22} - b_{23}\right]\lambda(\lambda+1) +$$
$$+4(b_{12} - b_{22} - b_{23})(G_0 - 1)] = 0,$$

we have:

$$\begin{gathered} \tau^2 - 2q_1\delta_n^2\tau + q_2\delta_n^4 = 0 \qquad \mu_i = \sqrt{\tau_i} \\ \tau_i = \delta_n^2 s_i, \quad s_i = \sqrt{q_1 - (-1)^i\sqrt{q_1^2 - q_2}} \quad (i = 1, 2) \\ 2q_1 = b_{22}^{-1}(b_{11}b_{22} - b_{12}^2 - 2b_{12}), \quad q_2 = b_{11}b_{22}^{-1}. \end{gathered} \tag{10}$$

The parameters q_1 and q_2 accept different values depending on mechanical parametres ν, ν_1, ν_2, G_0.

Thus, we arrive at different notation of solutions by the Legendre function and it in its turn leads to different asymptotic representations.

Let's consider the following possible cases:

1) $q_1 > 0,\ q_1^2 - q_2 > 0,\ \mu_{1,2} = \pm s_i\delta_n,\ \mu_{3,4} = \pm s_2, \delta_n,$

$$s_{1,2} = \sqrt{q_1 \pm \sqrt{q_1^2 - q_2}}, \quad q_1^2 > q_2$$

$$s_{1,2} = \eta \pm i\beta = \sqrt{q_1 \pm \sqrt{q_2 - q_1^2}}, \quad q_1^2 < q_2$$

2) The roots of characteristic equation (10) are multiple

$$\mu_{1,2} = \eta_{3,4} = \pm\delta_n p,\ q_1 > 0,\ q_1^2 - q_2 = 0,\ p = \sqrt{q_1}$$

3) $q_1 < 0 \quad q_1^2 - q_2 \neq 0 \quad \mu_{1,2} = \pm is\delta_n, \quad \mu_{3,4} = \pm is_2\delta_n$

$s_{1,2} = \sqrt{|q_1| + i\sqrt{q_2 - q_1^2}}, \quad q_1^2 < q_2$

4) $q_1 < 0 \quad q_1^2 - q_2 = 0 \quad \mu_{1,2} = \mu_{3,4} = \pm i\delta_n p$

$p = \sqrt{|q_1|}$

In cases 1), 2) after substitution (9) into (1) and its transformation by means of asymptotic expansions $P_z(\cos\theta),\ Q_2(\cos\theta)$.

$$\begin{gathered} P_z^k(\cos\theta) = \frac{\Gamma(z+k+1)}{\Gamma(z+3/2)}\left(\frac{\pi}{2}\sin\theta\right)^{-1/2} \times \\ \times\left\{\cos\left[\left(z+\frac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{k\pi}{2}\right] + O(z^{-1})\right\} \end{gathered} \tag{11}$$

$$\begin{gathered} Q_z^k(\cos\theta) = \frac{\Gamma(z+k+1)}{\Gamma(z+3/2)}\left(\frac{\pi}{2\sin\theta}\right)^{1/2} \times \\ \times\left\{\cos\left[\left(z+\frac{1}{2}\right)\theta + \frac{\pi}{4} + \frac{k\pi}{2}\right] + O(z^{-1})\right\}, \end{gathered}$$

for δ_n we get

$$(s_2 - s_1)\sin(s_1 + s_2)\delta_n \pm (s_1 + s_2)\sin(s_2 - s_1)\delta_n = 0 \tag{12}$$

$$\gamma \sin 2\beta\delta_n \pm \beta sh 2\gamma\delta_n = 0 \tag{13}$$

$$\sin 2p\delta_n \pm 2p\delta_n = 0 \tag{14}$$

The results for cases 3) and 4) are obtained from cases 1), 2) replacing s_1, s_2, p by is_1, is_2, ip.

These equations coincide with equations determining the Saint Venant edge conditions in the theory of transversally-isotropic plates.

Asymptotics of roots is obtained in the paper [2]. Character of roots essentially influence on general picture of stress-strain state of a shell.

As in the isotropic case, we can show that the case $\varepsilon z_k \to \infty$ as $\varepsilon \to 0$ is not possible here.

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