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BIFURCATION FROM ZERO OR INFINITY OF SOME FOURTH ORDER NONLINEAR PROBLEMS WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

Abstract

The fourth order nonlinear spectral problem with spectral parameter in the boundary condition is considered. Existence of a global continua of solutions bifurcating from zero or infinity, is proved.

We consider the following fourth order nonlinear problem

$$y^{(4)}(x) - \left(q(x)\, y'(x)\right)' = \lambda y(x) +$$

$$+g\left(x, y(x), y'(x), y''(x), y'''(x), \lambda\right), \qquad 0 < x < l, \tag{1}$$

$$y(0) = y'(0) = y''(l) = 0, \tag{2}$$

$$(a\lambda + b)\, y(l) = (c\lambda + d)\, Ty(l), \tag{3}$$

where λ is a spectral parameter, $q(x)$ is strictly positive absolutely continuous function on the interval [0,l], $Ty \equiv y''' - qy'$, a, b, c, d are real constants, and $\delta = bc - ad > 0$, the function $g(x, y, u, \upsilon, w, \lambda)$ is defined on $[0,l] \times \mathbb{R}^5$, is continuous in all variables, and satisfies the condition:

$$g(x, y, u, \upsilon, w, \lambda) = o\left|(y, u, \upsilon, w)\right| \quad \text{as} \quad \left|(y, u, \upsilon, w)\right| \to 0, \tag{4}$$

or

$$g(x, y, u, \upsilon, w, \lambda) = o\left|(y, u, \upsilon, w)\right| \quad \text{as} \quad \left|(y, u, \upsilon, w)\right| \to \infty, \tag{4*}$$

uniformly for $(x, \lambda) \in [0, l] \times \Lambda$ for any compact interval $\Lambda \subset \mathbb{R}$, where $|(\cdot,\cdot,\cdot,\cdot)|$ is the Euclidean norm of an element $(\cdot,\cdot,\cdot,\cdot)$.

Note, that the fourth order nonlinear Sturm - Liouville problem (when spectral parameter is not included into boundary conditions) are investigated in papers [1, 2] of author (jointly with A.P.Makhmudov).

The present paper is devoted to studying of structure of solution set of nonlinear problem(1) - (3) and is continuation of author's researches carried out in papers [1, 2].

Under condition (4) or (4 *) problem (1) - (3) is linearized and the corresponding linear problem is

$$\begin{cases} y^{(4)}(x) - (q(x)\, y'(x))' = \lambda y(x), \;\; 0 < x < l, \\ y(0) = y'(0) = y''(l) = 0, \\ (a\lambda + b)\, y(l) = (c\lambda + d)\, Ty(l). \end{cases} \tag{5}$$

Alongside with problem (5) we consider the following problem

$$\begin{cases} y^{(4)}(x) - (q(x)\, y'(x))' = \lambda y(x), \;\; 0 < x < l, \\ y(0) = y'(0) = y''(l) = y(l) = 0. \end{cases} \tag{6}$$

Eigenvalues of problem(6) are positive, simple and form infinitely increasing sequence $\mu_1, \mu_2, ..., \mu_n, ...$; the eigenfunction $\upsilon_n(x)$ corresponding to the eigenvalue μ_n, has $n-1$ simple zeros in the interval $(0,l)$ [3].

Let's define number N from the inequality $\mu_{N-1} < -\frac{d}{c} \leq \mu_N$. Problem (5) is in details investigated in paper [4] (see also [5]) where in particular it is proved: there is exist unbounded increasing sequence of eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n, ...$, and $\lambda_n > 0$, for $n \geq 2$; the eigenfunctions of this problem possess the following oscillation properties:

a) if $c = 0$, the eigenfunction $y_n(x)$ corresponding to the eigenvalue λ_n, has equally $n-1$ simple zeros in the interval $(0,l)$;

b) if $c \neq 0$, the eigenfunction $y_n(x)$ corresponding to the eigenvalue λ_n, for $n \leq N-1$ has exactly $n-1$ simple zeros, and for $n > N$ exactly $n-2$ simple zeros in the interval $(0,l)$.

By virtue of lemma 2.2 from [3] and lemma 3.3 from [4], we have:

i) if $a \in (0,l)$ is a zero of $y_n(x)$ or $y_n''(x)$, then $y_n'(x)\,Ty_n(x) < 0$ in a neighborhood of a;

ii) if $b \in (0,l)$ is a zero of $y_n'(x)$ or $Ty_n(x)$, then $y_n(x)\,y_n''(x) < 0$ in a neighborhood of b.

It is known [6,7], that problem(5) is reduced to problem on eigenvalues for the linear operator L in Hilbert space $H = L_2(0,l) \oplus \mathbb{C}$ with the scalar product

$$(\hat{y}, \hat{u})_H = (\{y,m\}, \{u,s\})_H = (y,u)_{L_2} + \delta^{-1} m\bar{s},$$

where $(\cdot,\cdot)_{L_2}$ is scalar product in $L_2(0,l)$,

$$L\hat{y} = L\{y,m\} = \left\{ y^{(4)} - (qy')', dTy(l) - by(l) \right\}$$

with the domain $D(L) = \{\{y,m\} \in H :\ y, y', y'', y''' \in AC[0,l],\ \ (Ty)' \in L_2(0,l),\ y(0) = y'(0) = y''(l) = 0,\ m = ay(l) - cTy(l)\}$, which is dense in H (see [6]).

The operator $G : \mathbb{R} \times H \to H$ we define as follows:

$$G(\lambda, \hat{y}) = G(\lambda, \{y,m\}) = \left\{ g\left(x, y(x), y'(x), y''(x), y'''(x), \lambda\right), 0 \right\}.$$

Then, the problem (1) - (3) is adequate to the following nonlinear problem

$$L\hat{y} = \lambda\hat{y} + G(\lambda, \hat{y}). \tag{7}$$

The operator $L_1 : D(L_1) \subset L_2(0,l) \to L_2(0,l)$ we shall define in the following way:

$$D(L_1) = \left\{ y \in L_2(0,l) : y, y', y'', y''' \in AC[0,1], y^{(4)} - (qy')' \in L_2(0,l), \right.$$

$$\left. y(0) = y'(0) = y''(l) = 0, by(l) = dTy(l) \right\},$$

$$L_1 y = y^{(4)} - (qy')'.$$

The operator L is self-adjoint, discrete and semibounded below in H and, therefore, $L + \lambda J$ ($J : H \to H$ is the identical operator) is convertible for sufficiently great negative values of λ. Without loosing generality, it is possible to consider, that $\lambda = 0$. It follows, that $\lambda = 0$ can not be an eigenvalue of the operator L_1. Thus L_1^{-1} exists and is an integral operator of the kernel we denote as $k(x,t)$. Using method of paper [8] it is possible to prove, that

$$L^{-1}\hat{y} = L^{-1}\{y,m\} = \left\{ \int_0^l K(x,t)\,y(t)\,d\mu, \int_0^l K(l,t)\,y(t)\,d\mu \right\},$$

where

$$K(x,t)=k(x,t)\,,\ (x,t)\in(0,l)^2\,,\ K(l,x)=ak(l,x)-cT_xk(l,x)\,,$$

$$K(l,l)=\lim_{x\to l}\{a(ak(l,x)-cT_xK(l,x))-c(aT_tk(l,x)-cT_tT_xK(l,x))\}\,,$$

$$T_xk(l,x)=K'''_{x^3}(l,x)-qK'_x(l,x)\,,\quad T_tk(l,x)=K'''_{t^3}(l,x)-qK'_t(l,x)$$

$$T_tT_xk(l,x)=K^{(6)}_{t^3x^3}(l,x)\,,$$

μ is positive measure

$$\mu(\Omega)=\begin{cases}\int\limits_{\Omega}dx, & \text{if } \Omega\subset(0,l)\,,\\ \frac{1}{\delta}, & \text{if } \Omega=\{l\}\,.\end{cases}$$

Hence, problem (1) - (3) (or (7)) is equivalent to the following problem

$$\hat{y}=L^{-1}\hat{y}+L^{-1}G(\lambda,\hat{y})\,, \tag{8}$$

Therefore, it is enough to investigate structures of solution set of problem (1) - (3) in the space $C^3[0,l]$.

Through BC_0 we denote set of functions satisfying boundary conditions (2).

Let $\hat{E}=\left\{\hat{u}=\{u,m\}\in C^3[0,l]\oplus\mathbb{C}|\ u\in BC_0,\ m=ay(l)-cTy(l)\right\}$ be banach space with the norm

$$\|\hat{u}\|=\|\{u,m\}\|=|u|_3+|m|\,,$$

where

$$|u|_3=|u|_0+|u'|_0+|u''|_0+|u'''|_0\,,\quad |\cdot|_0=\max|\cdot|\,.$$

Denote: $\mathcal{L}=L^{-1},\ \mathcal{H}(\lambda,\hat{y})=L^{-1}G(\lambda,\hat{y})$.

Thus, problem (1) - (3) (or (7)) can be written in the following equivalent form:

$$\hat{y}=\mathcal{L}\hat{y}+\mathcal{H}(\lambda,\hat{y})\,. \tag{9}$$

Define the sets $\hat{S}_k^{\nu}=\Big\{\hat{u}=\{u,m\}\in\hat{E}:\ u(x)$ has $k-1$ zeros in the interval $(0,l)$; if $u(\xi)u''(\xi)=0,\ \xi\in(0,l)$, then $u'(x)Tu(x)<0$ in neighborhood of ξ; if $u'(\mu)Tu(\mu)=0,\ \mu\in(0,l)$, then $u'(x)u''(x)<0$ in neighborhood of μ; zeros of the functions $u(x)$ and $u'(x)$, $u(x)$ and $Tu(x)$; $u'(x)$ and $u''(x)$ interlaced, $\lim\limits_{x\to 0}\nu u(x)=1\Big\}$, $\nu=+$ or $-$, $\hat{S}_k=\hat{S}_k^+\cup\hat{S}_k^-$.

Lemma 1. *The sets $\hat{S}_k^+$, $\hat{S}_k^-$ and $\hat{S}_k$ are open in $\hat{E}$. If $\hat{y}\in\partial\hat{S}_k\left(\partial\hat{S}_k^+,\partial\hat{S}_k^-\right)$, then $y(x)$ has at least one quadruple zero.*

Proof. The openness of the sets $\hat{S}_k^+$, $\hat{S}_k^-$, $\hat{S}_k$ in $\hat{E}$ is obvious.

To find boundaries of these sets we use Prŭfer-type transformation of the following form:

$$\begin{cases} y(x)=r(x)\sin\psi(x)\cos\theta(x)\\ y'(x)=r(x)\cos\psi(x)\sin\varphi(x)\\ y''(x)=r(x)\cos\psi(x)\cos\varphi(x)\\ Ty(x)=r(x)\sin\psi(x)\sin\theta(x)\end{cases} \tag{10}$$

If $\hat{y}=\{y,m\}\in\hat{S}_k\left(\hat{S}_k^+,\hat{S}_k^-\right)$, then the Jacobian $J[y]=r^3\sin\psi\cos\psi$ of transformation (10) is does not vanish in $x\in(0,l)$.

Without losing a generality, the function ψ can be chosen so that $\psi(x) \in \left(0, \frac{\pi}{2}\right)$ or $\psi(x) \in \left(\frac{\pi}{2}, \pi\right)$ for $x \in (0, l)$. Initial values of the functions $\theta(x)$, $\varphi(x)$ can be defined as follows

$$\theta(0, \lambda) = -\frac{\pi}{2} \operatorname{sgn} \psi(0), \quad \varphi(0) = 0, \quad \varphi(l) = 2k \pm \frac{\pi}{2}.$$

If $\hat{y} \in \partial \hat{S}_k$, there exists a point $x_0 \in (0, l)$ such, that
a) or $\sin \psi(x_0) = 0$;
b) or $\cos \psi(x_0) = 0$;
c) or $r(x_0) = 0$.
Let's prove, that the cases a) and b) are impossible.

I. In case $\sin \psi(x_0) = 0$ by (10) we have $y(x_0) = Ty(x_0) = 0$. Thus either $y'(x_0) = 0$, or $y'(x_0) \neq 0$.

Let's assume, that $y'(x_0) = 0$. Without loosing a generality it is possible to consider, that $y(x_0) > 0$ for $x \in (0, x_0)$. Then $y'(x)$ increases in the point x_0 and, hence $y''(x_0) > 0$.

Let $\hat{y} = \lim_{n \to \infty} \hat{y}_n$, $\hat{y}_n \in \hat{S}_k$, $y_n = \{y_n(x), m_n\}$. For sufficiently great n in small neighborhood of the point x_0 there exists a point $x_0^{(n)}$such, that $y_n'\left(x_0^{(n)}\right) = 0$. It is obvious, that in neighborhood $U\left(x_0^{(n)}\right)$ of the point $x_0^{(n)}$ $y_n(x) > 0$, $y_n''(x) > 0$, it is means $y_n(x) y_n''(x) > 0$. The obtained contradiction shows, that $\sin \psi(x_0) \neq 0$.

Now we assume, that $y'(x_0) \neq 0$. Without loosing a generality, it is possible to consider, that x_0 is the point nearest to zero where function $y(x)$ accepts zero value and $y(x) > 0$ for $x \in (0, x_0)$. Then there exists a point $x_1 \in (0, x_0)$ such, that $y'(x_1) = 0$. By (2) there exists point $x_2 \in (0, x_1)$ such, that $y''(x_2) = 0$. Besides, by (2) (or more exactly by $y''(l) = 0$) there exists point $x_3 \in (x_0, l)$ such, that $y''(x_3) = 0$. It is obvious, that $y'(x_2) > 0$, $y'(x_3) < 0$, $y'''(x_2) < 0$, $y'''(x_3) > 0$. Hence $Ty(x_2) < 0$, $Ty(x_3) > 0$. It means that $Ty(x)$ accepts values of zero in the point x_0 strictly increasing. Since $\hat{y}_n \to \hat{y}$ in $\hat{E}$, for sufficiently great n in neighborhood of x_0 there exists point $x_0^{(n)}$ such that, $y_n\left(x_0^{(n)}\right) = 0$ and in the left hand side neighborhood $U^-\left(x_0^{(n)}\right)$ of the point $x_0^{(n)}$ $y_n'(x) Ty(x) > 0$. The obtained contradiction shows, that $\sin \psi(x_0) \neq 0$.

II. In case $\cos \psi(x_0) = 0$ we have $y'(x_0) = y''(x_0) = 0$. Thus either $Ty(x_0) = 0$, or $Ty(x_0) \neq 0$.

If $Ty(x_0) \neq 0$, repeating the above-mentioned reasonings, we receive the contradiction.

Let's assume, that $Ty(x_0) = 0$.

We consider following Cauchy problem for the second order equation:

$$\left.\begin{array}{l} -\sigma''(x) + q(x)\sigma(x) = 0, \ 0 < x < l, \\ \sigma(l) = 1, \ \sigma'(l) = 0. \end{array}\right\} \tag{11}$$

It is known [9, lemma 2.1. (see also [10, theorem 12.1])], that $\sigma(x) > 0$ for $x \in [0, l]$.

Change of variable

$$t = t(x) = \frac{l}{\omega} \int_0^x \sigma(s)\, ds, \quad \omega = \int_0^l \sigma(s)\, ds. \tag{12}$$

specifies bounded and bounded invertible operator V in the space $E = C^3[0,l] \cap BC_0$ with the norm $|\cdot|_3$. Notably any function $y(x) \in E$ passes, under action of this operator in the function $y(t) \in E$ of the following form

$$y(t) = y(x(t))$$

$$\dot{y}(t) = y'(x(t)) \cdot \frac{\omega}{l\sigma(x(t))}$$

$$\ddot{y}(t) = \frac{\omega^2}{l^2}\sigma^2(x(t))\left\{y''(x(t)) - \sigma^{-1}(x(t))\sigma'(x(t))y'(x(t))\right\}$$

$$\left(\frac{l^3}{\omega^3}\sigma^3(x(t))\ddot{y}(t)\right)^{\cdot} = y'''(x(t)) - q(x(t))y'(x(t)).$$

Without loosing a generality, it is possible to consider, that x_0 is the point nearest to zero, in which $y'(x)$ accepts value of zero and $y'(x) > 0$ for $x \in (0, x_0)$. By (2) there exists points $x_1 \in (0, x_0)$ and $x_2 \in (x_0, l]$ such, that $y''(x_1) = y''(x_2) = 0$, $y''(x) < 0$ for $x \in (x_1, x_2) \setminus \{x_0\}$. Therefore, there exist the points $x_3 \in (x_1, x_0)$ and $x_4 \in (x_0, x_2)$ such, that $y'''(x_3) = y'''(x_4) = 0$. Obviously, that $y'''(x_0) = 0$ and $y'''(x) < 0$ in the left hand side neighborhood $U^-(x_3)$ of the point x_3, $y'''(x) > 0$ for $x \in (x_3, x_0)$, $y'''(x) < 0$ for $x \in (x_0, x_4)$ and $y'''(x) > 0$ in the right hand side neighborhood $U^+(x_4)$ of the point x_4. Hence $Ty(x_3) < 0$, $Ty(x_4) > 0$.

At change of variable (12) we have: $\dot{y}(t_0) = \ddot{y}(t_0) = (p\ddot{y})(t_0) = 0$, where $p(t) = \dfrac{l^3}{\omega^3}\sigma^3(x(t))$, $t_0 = t(x_0)$, $\dot{y}(t) > 0$ for $t \in (0, t_0)$, $(p\ddot{y})(t_1) = (p\ddot{y})(t_2) = 0$, $t_1 = t(x_1)$, $t_2 = t(x_2)$; $(p\ddot{y})^{\cdot}(t_3) = (p\ddot{y})^{\cdot}(t_4) = 0$, $(p\ddot{y})(t) > 0$ for $t \in (t_3, t_0)$, $(p\ddot{y})^{\cdot}(t) < 0$ for $t \in (t_0, t_4)$. Denote $x(t_3) = x_5$, $x(t_4) = x_6$. It is obvious, that $x_5 \in (x_3, x_0)$, $x_6 \in (x_0, x_4)$, $Ty(x_5) = Ty(x_6) = 0$, $Ty(x) > 0$ for $x \in (x_5, x_0)$, $Ty(x) < 0$ for $x \in (x_0, x_6)$. Since zeros of the functions $y(x)$ and $Ty(x)$ interlaced, there exists $x_7 \in (x_5, x_0)$ and $x_8 \in (x_0, x_6)$ such, that $y(x_7) = y(x_8) = 0$, and $y(x) > 0$ for $x \in (x_7, x_8)$ and $y(x) < 0$ in the right hand side neighborhood $U^+(x_8)$ of the point x_8.

Since $\hat{y}_n \to \hat{y}$ in $\hat{E}$, in the neighborhood of the point x_6 there exists the point $x_6^{(n)}$(for sufficiently great n) such that $Ty_n\left(x_6^{(n)}\right) = 0$ and $y_n(x) < 0$, $y_n''(x) < 0$ in neighborhood of the point $x_6^{(n)}$. The obtained contradiction shows, that $\cos\psi(x_0) \neq 0$.

Thus, if $y \in \partial\hat{S}_k\left(\partial\hat{S}_k^-, \partial\hat{S}_k^+\right)$, then $r(x_0) = 0$, i.e. $y(x)$ has at least one quadruple zero.

The lemma is proved.

Let $\widehat{\mathcal{E}} = \mathbb{R} \times \hat{E}$, $\hat{Y}_k^+ = \mathbb{R} \times \hat{S}_k^+$, $\hat{Y}_k^- = \mathbb{R} \times \hat{S}_k^-$ and $\hat{Y}_k = \mathbb{R} \times \hat{S}_k$. Denote by $\hat{\upsilon}_k^+ = \left\{\upsilon_k^+(x), m_k\right\}$ a unique eigenfunction of the problem (5), corresponding to the simple eigenvalue λ_k, such that $\hat{\upsilon}_k^+ \in \hat{S}_k^+$, $\left\|\hat{\upsilon}_k^+\right\| = 1$; and denote by $\hat{\mathcal{T}}$ $(\mathcal{T})$ the solution set of problem(7) ((1) - (3)) in $\widehat{\mathcal{E}}$ (in $\mathcal{E}$).

There holds the following

Theorem 1. *Let condition (4) be fulfilled . Then for every* $k \in \mathbb{N}$, $k \neq N$, $N+1$*and* $\nu = +$ *or* $-$, *there exists continuum* $\hat{D}_k^\nu \subset \hat{\mathcal{T}}$ *such that* $\hat{D}_k^\nu \subset \hat{Y}_k^\nu \cup \{(\lambda_k, \theta)\}$, $\theta = (0,0)$, *which contains* (λ_k, θ) *and is unbounded in* $\widehat{\mathcal{E}}$.

Proof. Note, that if $(\lambda, \hat{y})$ is a solution of problem (7), and $\hat{y} \in \partial\hat{S}_k^\nu$ (+ or $-$), then by the method of the proof of the theorem 2.1 from [2] it is possible to prove, that $\hat{y} \equiv 0$.

Let's assume, that $\lambda = 0$ doesn't be eigenvalue of the operator L. Then problem (7) is equivalent to problem (9). Eigenvalues of a problem (5) are characteristic numbers of the operator L and all of them are simple. Under condition (4) we have: $H(\lambda, \hat{y}) = o(\|\hat{y}\|)$ for $\hat{y} \to 0$ in $\hat{E}$. Therefore all points (λ_k, θ), $k \in \mathbb{N}$, are bifurcation points of problems (9) (or (7)) [11]. By theorem 1.3 from [12] (see also [13]) there exists a continuum $\hat{D}_{\lambda_k} \equiv \hat{D}_k$, $(\lambda_k, \theta) \in \hat{D}_k$, for which Rabinovich's alternative is fulfilled. If $(\lambda, \hat{y}) \in \hat{D}_k$ and in the neighborhood of (λ_k, θ) $\hat{y} = \alpha \upsilon_k + \hat{w}$, by lemma 1.24 from [12] $\hat{w} = o(|\alpha|)$. Since $\hat{S}_k^\nu$ it is open in $\hat{E}$ and $\upsilon_k \in \hat{S}_k$, then $(\lambda, \hat{y}) \in \hat{Y}_k$ and $\left(\hat{D}_k \backslash \{(\lambda_k, \theta)\}\right) \cap \hat{B}_\xi \subset \hat{Y}_k$, for all small $\xi > 0$, where $\hat{B}_\xi$ is an open sphere in $\widehat{\mathcal{E}}$ of radius ξ with the center in a point (λ_k, θ). By the above-stated remark $\left(\hat{D}_k \backslash \{(\lambda_k, \theta)\}\right) \cap \partial \hat{Y}_k = \varnothing$. Hence, $\hat{D}_k \subset \hat{Y}_k \cup \{(\lambda_k, \theta)\}$ and the statement (ii) of Rabinovich's alternative doesn't holds for $k \neq N, N+1$.

It is reamin to develop $\hat{D}_k$ on two subcontinuums, which contain the point (λ_k, θ), are contained in $\hat{Y}_k^+ \cup \{(\lambda_k, \theta)\}$ and $\hat{Y}_k^- \cup \{(\lambda_k, \theta)\}$, respectively, and are unbounded in $\widehat{\mathcal{E}}$. If $(\lambda, \hat{y})$ is in a small neighborhood of the point (λ_k, θ) and $(\lambda, \hat{y}) \in D_k \backslash \{(\lambda_k, \theta)\}$, then $\hat{y} = \alpha \hat{\upsilon}_k + \hat{w}$ and, we have $\alpha \hat{\upsilon}_k \in \hat{Y}_k^\nu$ if $0 \neq \alpha \in \mathbb{R}^\nu$ and, hence, $\left(\hat{D}_k^+ \backslash \{(\lambda_k, \theta)\}\right) \cap \hat{B}_\xi \subset \hat{Y}_k^+$, $\left(\hat{D}_k^- \backslash \{(\lambda_k, \theta)\}\right) \cap \hat{B}_\xi \subset \hat{Y}_k^-$ for all small $\xi > 0$. Since $\hat{D}_k^\nu \backslash \{(\lambda_k, \theta)\}$ can not exceed $\hat{Y}_k^\nu$ in the neighborhood of (λ_k, θ) and $\hat{Y}_k^\nu$ cannot contain a pair of points $(\lambda, \hat{y})$, $(\lambda, -\hat{y})$, then by theorems 1.27 and 1.40 from [12] $\hat{D}_k^\nu$, $\nu = +$ or $-$, is unbounded in $\hat{Y}_k^\nu$.

If zero is eigenvalue of the operator L, the result is trivial for $\lambda_k = 0$. Let $\lambda_k \neq 0$ ($k \neq N, N+1$). Denote $L_\varepsilon = L + J$. If by $\lambda_k(\varepsilon)$, $k \in \mathbb{N}$, we denote the eigenvalues of the problem

$$L_\varepsilon \hat{y} = \lambda \hat{y}, \tag{13}$$

then $\lambda_k(\varepsilon) = \lambda_k + \varepsilon$. For small $\varepsilon > 0$ zero isn't be eigenvalue of problem (13) and, hence, the above-mentioned reasoningsare valid for the nonlinear spectral problem

$$L_\varepsilon \hat{y} = \lambda \hat{y} + G(\lambda, \hat{y}). \tag{14}$$

It is easy to see, that solution set of a problem (14) $\hat{\mathcal{T}}_\varepsilon = \{(\lambda(\varepsilon), \hat{y}(\varepsilon)) : 0 < \varepsilon < \varepsilon_0\}$ is precompact in $\widehat{\mathcal{E}}$. Therefore there exists the subsequence $\left(\lambda_{\varepsilon_{n_k}}, \hat{y}_{\varepsilon_{n_k}}\right)$ of elements sequence $(\lambda_{\varepsilon_n}, \hat{y}_{\varepsilon_n})$ of the set $\hat{\mathcal{T}}_\varepsilon$, which for $\varepsilon_{n_k} \to 0$ converges in $\hat{Y}_k^\nu$ to the solution $(\lambda, \hat{y})$ of problem (7). Hence, the statement of the theorem is correct for a limit problem as well, i.e. for problem (7).

The theorem is proved.

Let $S_k^\nu = \left\{y \in E | \ \hat{y} = \{y, m\} \in \hat{S}_k^\nu\right\}$, $\nu = +$ or $-$, $\mathcal{E} = \mathbb{R} \times E$, $Y_k^+ = \mathbb{R} \times S_k^+$, $Y_k^- = \mathbb{R} \times S_k^+$, $Y_k = \mathbb{R} \times S_k$.

Since between eigen pairs problem (7) and (1)-(3) there exists an isomorphism $(\lambda, \hat{y}) \leftrightarrow (\lambda, y)$, then by substituting $\widehat{\mathcal{E}}$, $\hat{Y}_k^\nu$, $\hat{Y}_k$, $\hat{D}_k^\nu$, $\hat{D}_k$ by $\mathcal{E}$, Y_k^ν,Y_k, D_k^ν, D_k we get validity of the following theorem.

Theorem 2. *Let condition (4) be fulfilled. Then for every $k \in \mathbb{N}$, $k \neq N$, $N+1$, and $\nu = +$ or $-$, there exists continuum $D_k^\nu \subset \mathcal{T}$ such, that $D_k^\nu \subset Y_k^\nu \cup \{(\lambda_k, 0)\}$, containing $(\lambda_k, 0)$ and is unbounded in $\mathcal{E}$.*

Remark 1. The structure of the set D_k^ν, $\nu = +$ or $-$, for $k = N, N+1$ will be investigated in future.

Theorem 3. *Let condition (4*) be fulfilled. Then for every* $k \in \mathbb{N}$ *there exists unbounded component* $\hat{D}_k \subset \hat{\mathcal{T}}$ *, which contains* $(\lambda_k, \infty) \in \mathbb{R} \times \hat{E}$*. Moreover, if an interval* $\Delta \subset \mathbb{R}$ *such that* $\Delta \cap \sigma(L) = \{\lambda_k\}$ *(*$\sigma(L)$ *is spectrum of operator* L*) and* $\hat{M} \subset \widehat{\mathcal{E}}$ *is a neighborhood of* (λ_k, ∞) *whose projection on* $\mathbb{R}$ *lies in* Δ *and whose projection on* $\hat{E}$ *is bounded away from zero, or either*

1^0. $\hat{D}_k \backslash \hat{M}$ is bounded in $\widehat{\mathcal{E}}$ in which case $\left(\hat{D} \backslash \hat{M}\right) \cap \mathcal{R} \neq \varnothing$, where $\mathcal{R} = \{(\lambda, \theta) \mid \lambda \in \mathbb{R}\}$ or

2^0. $\hat{D} \backslash \hat{M}$ is unbounded in $\widehat{\mathcal{E}}$.

If 1^0 occurs and $\hat{D} \backslash \hat{M}$ has a bounded projection on $\mathbb{R}$, then $\hat{D} \backslash \hat{M}$ contains (λ_s, ∞), where $\lambda_s \in \sigma(L)$, $\lambda_s \neq \lambda_k$.

Component $\hat{D}_k$ can be develop on two subcontinua $\hat{D}_k^+$, $\hat{D}_k^-$ and there exists a neighborhood $\hat{Q} \subset \hat{M}$ of (λ_k, ∞) such that $(\lambda, \hat{y}) \in \hat{D}_k^+ \left(\hat{D}_k^-\right) \cap \hat{Q}$ and $(\lambda, \hat{y}) \neq (\lambda_k, \infty)$, implies $(\lambda, \hat{y}) = (\lambda, \alpha \hat{v}_k + \hat{w})$, where $\alpha > 0$ ($\alpha < 0$), and $|\lambda - \lambda_k| = o(1)$, $\|\hat{w}\| = o(\alpha)$ at $\alpha = \infty$. Thus $\hat{Q}$ can be chosen so that $\hat{D}_k^\nu \cap \hat{Q} \subset \hat{Y}_k^\nu \cup (\lambda_k, \infty)$.

Proof. Assume, that $\lambda = 0$ isn't the eigenvalue of the operator L (problem (1) - (3)). Then problem (7) is equivalent to problem (9). Let condition (4*) be fulfilled. Let's prove, that $\mathcal{H}(\lambda, \hat{y}) = o(\|\hat{y}\|)$ at $\hat{y} = \infty$ uniformly on $(x, \lambda) \in [0, l] \times \Lambda$.

We have

$$\mathcal{H}(\lambda, \hat{y}) = L^{-1} G(\lambda, \hat{y}) =$$

$$= \left\{ \int_0^l K(x,t)\, g\left(t, y(t), y'(t), y''(t), y'''(t), \lambda\right) d\mu, o \right\}. \tag{15}$$

Let

$$\varphi(M) = \max_{\substack{\lambda \in \Lambda,\ x \in [0,l] \\ \xi^2 + \mu^2 + \tau^2 + \eta^2 \leq M^2}} \left(|g(x, \xi, \mu, \tau, \eta, \lambda)| + M \right).$$

It is obvious, that $\varphi(M)$ is strictly increasing function and $\lim\limits_{M \to +\infty} \varphi(M) = +\infty$. Let $\hat{S} = \left\{ y \in \hat{E} \,\middle|\, \|\hat{y}\| \geq \varphi\left(\overline{M}\right) \right\}$, where $\overline{M}$ is some fixed number. By (4*), for any small $\varepsilon > 0$ there exists the number $M = M(\varepsilon) > 0$ such, that if $(x, \lambda) \in [0, l] \times \Lambda$, $\xi^2 + \mu^2 + \tau^2 + \eta^2 \geq M^2$, then $|g(x, \xi, \mu, \tau, \eta, \lambda)| \leq \varepsilon \left(\xi^2 + \mu^2 + \tau^2 + \eta^2\right)^{\frac{1}{2}}$. Hence, by (15) for $(\lambda, \hat{y}) \in \Lambda \times \hat{S}$ we have

$$\|\mathcal{H}(\lambda, \hat{y})\| = \sum_{k=0}^{3} \max_{x \in [0,l]} \left| \int_0^l K_{x^s}^{(s)}(x,t)\, g\left(t, y(t), y'(t), y''(t), y'''(t), \lambda\right) d\mu \right| \leq$$

$$\leq c \int_0^l \left| g\left(t, y(t), y'(t), y''(t), y'''(t), \lambda\right) \right| d\mu =$$

$$= c \left\{ \int_{y^2(t) + y'^2(t) + y''^2(t) + y'''^2(t) \leq M^2} \left| g\left(t, y(t), y'(t), y''(t), y'''(t)\, \lambda\right) \right| d\mu + \right.$$

$$\left. + \int_{y^2(t) + y'^2(t) + y''^2(t) + y'''^2(t) \geq M^2} \left| g\left(t, y(t), y'(t), y''(t), y'''(t), \lambda\right) \right| d\mu \right\} \leq$$

$$\leq c\left(l+\delta^{-1}\right)\left(\varphi\left(M\right)+\varepsilon\left\|\hat{y}\right\|\right), \tag{16}$$

where $c=\sup\limits_{(x,t)\in[0,l]^2}\left\{\left|K_{x^s}^{(s)}\left(x,t\right)\right|,\ s=\overline{0,3}\right\}$. It is possible to choose number $\overline{M}$ so great that the inequality $\varphi\left(M\right)/\varphi\left(\overline{M}\right)\leq\varepsilon$ is fulfilled. Then from (16) we obtain

$$\left\|\mathcal{H}\left(\lambda,\hat{y}\right)\right\|\leq 2c\left(l+\delta^{-1}\right)\varepsilon\left\|\hat{y}\right\|,\quad (\lambda,\hat{y})\in\Lambda\times\hat{S}, \tag{17}$$

which means, that $\mathcal{H}\left(\lambda,\hat{y}\right)=o\left(\left\|\hat{y}\right\|\right)$ at $\hat{y}=\infty$ in $\hat{E}$.

Now we prove, that $\left\|\hat{v}\right\|^2\mathcal{H}\left(\lambda,\frac{\hat{v}}{\left\|\hat{v}\right\|^2}\right)\equiv W\left(\lambda,\hat{v}\right)$ is compact. If $W\left(\lambda,\hat{v}\right)$ is compact, we note, that the image of the set $\left\{(\lambda,\hat{v})\in\widehat{\mathcal{E}}\,|\lambda\in\Lambda,\ \sigma\leq\left\|\hat{v}\right\|\leq\rho\right\}$ for display W is precompact in $\hat{E}$ for any $0<\sigma\leq\rho<\infty$. Therefore, it is enough to prove, that $W\left(\Lambda\times\hat{B}_\sigma\right)$ is precompact in $\hat{E}$, where $\hat{B}_\sigma=\left\{\hat{v}\in\hat{E}\,|\left\|\hat{v}\right\|\leq\sigma\right\}$, $\sigma>0$ is some number. From (17) we get

$$\left\|\hat{v}\right\|^2\left\|\mathcal{H}\left(\lambda,\frac{\hat{v}}{\left\|\hat{v}\right\|^2}\right)\right\|\leq 2c\left(l+\delta^{-1}\right)\varepsilon \tag{18}$$

i.e., the set $W\left(\Lambda\times\hat{B}_\delta\right)$ is bounded in $\hat{E}$.

Assume $\hat{w}=W\left(\lambda,\hat{v}\right)$; $\hat{w}=\left\{w\left(x\right),aw\left(l\right)-cTw\left(l\right)\right\}$. It is obvious, that function $w\left(x\right)$ is solution of the differential equation.

$$w^{(4)}\left(x\right)-\left(q\left(x\right)w'\left(x\right)\right)'=\left\|\hat{v}\right\|^2g\left(x,\frac{v}{\left\|\hat{v}\right\|^2},\frac{v'}{\left\|\hat{v}\right\|^2},\frac{v''}{\left\|\hat{v}\right\|^2},\frac{v'''}{\left\|\hat{v}\right\|^2},\lambda\right). \tag{19}$$

By (18) we have $\left\|w\right\|_3\leq 2c\left(l+\delta^{-1}\right)\varepsilon\overline{M}^{-1}$, and by (4*) we have

$$\left\|\hat{v}\right\|^2g\left(x,\frac{v}{\left\|\hat{v}\right\|^2},\frac{v'}{\left\|\hat{v}\right\|^2},\frac{v''}{\left\|\hat{v}\right\|^2},\frac{v'''}{\left\|\hat{v}\right\|^2},\lambda\right)=o\left(\left(\left|v\right|^2+\left|v'\right|^2+\left|v''\right|^2+\left|v'''\right|^2\right)^{\frac{1}{2}}\right)$$

for $\left\|v\right\|_3\to 0$. Then from (19) it follows the inequality

$$\left|w^{(4)}\left(x\right)\right|\leq c_0,\quad x\in[0,l],$$

where

$$c_0=2c_1c\left(l+\delta^{-1}\right)\varepsilon\sigma+c_2,\quad c_1=\sup_{x\in[0,\pi]}\left|q\left(x\right)\right|+\sup_{x\in[0,\pi]}\left|q'\left(x\right)\right|$$

$$c_2=\sigma^2\sup_{\substack{x\in[0,l],\lambda\in\Lambda\\(\xi,\mu,\tau,\eta)\in\left[\delta^{-1},\infty\right)^4}}\left|g\left(x,\xi,\mu,\tau,\eta,\lambda\right)\right|,$$

hence, by Artsel-Askoly theorem, the set $\{w\}$ is precompact in E, and in turn the set $\{\hat{w}\}$ is precompact in $\hat{E}$, i.e., W is the compact operator from $\widehat{\mathcal{E}}$ in $\hat{E}$. Thus, all conditions of theorem 1.6 and consequences 1.8 of paper [14] are fulfilled, whence it follows the validity of the statement of theorem 3 .

If $\lambda=0$ is eigenvalue of the operator L, then we again substitute the operator L by the operator $L_\varepsilon=L+\varepsilon J$, where $\varepsilon>0$ is sufficiently small. It is obvious, that zero is not the eigenvalue of the operator L_ε, and therefore L_ε has bounded inverse L_ε^{-1}, hence problem (14) is equivalent to the problem

$$\hat{y} = \lambda \mathcal{L}_\varepsilon \hat{y} + \mathcal{H}_\varepsilon (\lambda, \hat{y}), \tag{20}$$

where $\mathcal{L}_\varepsilon = L_\varepsilon^{-1}$, $\mathcal{H}_\varepsilon = L_\varepsilon^{-1} G$. Then there exists a continuum $\hat{D}_{k\varepsilon}$, which satisfies to the statement of theorem 3. Let $\overset{\wedge}{\mathcal{Y}}_{k\varepsilon}$be image of the set $\hat{D}_{k\varepsilon}$ under transformation $\hat{y} \to \hat{\upsilon} = \dfrac{\hat{y}}{\|\hat{y}\|^2}$. Thus $\overset{\wedge}{\mathcal{Y}}_{k\varepsilon}$is continua containing $(\lambda_k(\varepsilon), \theta)$, and for which the statements of lemmas of 1.2 and 1.3 of paper [14] are true. Let A be an open bounded. neighborhood of the point $(\lambda_k(\varepsilon), \theta)$, such that $(\lambda_k(\varepsilon), \theta) \notin A$, for $j \neq k$. Then $(\lambda_k(\varepsilon)) \in A$ for small ε and, hence, there exists $(\mu_k(\varepsilon), \upsilon_k(\varepsilon)) \in \partial A \cap \overset{\wedge}{\mathcal{Y}}_{k\varepsilon}$. As here in before, it is easy to show, that the fourth order derivative from $\upsilon_k(\varepsilon)$ is uniformly bounded. The set $\{\|\hat{\upsilon}_k(\varepsilon)\|\}$ is separated from zero and is bounded, and the functions $w_k(\varepsilon) = \|\hat{\upsilon}_k(\varepsilon)\|^{-1} \upsilon_k(\varepsilon)$ satisfy the equation

$$w_k^{(4)}(\varepsilon) - (q w_k(\varepsilon))' = \mu_k(\varepsilon) w_k(\varepsilon) +$$
$$+ \|\hat{\upsilon}_k(\varepsilon)\| g\left(x, y_k(\varepsilon), y_k'(\varepsilon), y_k''(\varepsilon), y_k'''(\varepsilon), \lambda\right). \tag{21}$$

From (21) it follows, that $\{w_k(\varepsilon)\}$ is uniformly bounded in E, therefore, without loosing a generality it is possible to consider, that $\lim\limits_{\varepsilon \to 0} w_k(\varepsilon) = w$ in E, $\|\hat{w}\| = 1$ and $\lim\limits_{\varepsilon \to 0} \mu_k(\varepsilon) = \lambda$. Moreover, from (21) it follows, that $w_k(\varepsilon) \to w$ in $C^4[0, l]$. If $(\mu_k(\varepsilon), w_k(\varepsilon)) \to (\lambda, 0)$ for $\varepsilon \to 0$, then from (21) we get

$$w^{(4)}(x) - (q(x) w(x))' = \lambda w(x), \quad 0 < x < l, \tag{22}$$

and hence, $\lambda = \lambda_j$ and $w = y_j$ or $-y_j$ for some $j \in \mathbb{N}$. In this case we have $(\mu_k(\varepsilon), \hat{w}_k(\varepsilon)) \to (\lambda_j, \theta)$ as $\varepsilon \to 0$ and $(\lambda_j; \theta) \in \partial A$, which contrary to the construction of A. Thus $(\lambda_k(\varepsilon), \hat{w}_k(\varepsilon)) \to (\lambda, \hat{w})$ as $\varepsilon \to 0$, where $w \neq 0$ and $(\lambda, \hat{y}) = \left(\lambda, \|\hat{w}\|^{-2} \hat{w}\right)$ satisfies equation (7).Since it is true for each such set A, it follows from an elementary argument from point set topology, that problem (7) has unbounded component $\hat{D}_k$, which satisfies statements of the given theorem.

Existence of $\hat{D}_k^\nu, \nu = +$ or $-$ and neighborhood $\hat{Q}$ of (λ_k, ∞) is proved similarly, using continua $\overset{\wedge}{\mathcal{Y}}_{k\varepsilon}$and neighborhood $\hat{Q}_\varepsilon$.

We have: if $(\lambda, \hat{y}) \in \hat{Q} \cap \hat{D}_k^\nu$, then $(\lambda, \hat{y}) = (\lambda, \alpha \hat{\upsilon}_k + \hat{w})$, where $\nu\alpha > 0$ and $\|\hat{w}\| = o(\nu\alpha)$ at $\alpha = \infty$. Since $\hat{S}_k^\nu$ is open and $(\nu\alpha)^{-1} \hat{w}$ is sufficiently small for α near $\alpha = \infty$, relative to $\hat{\upsilon}_k \in \hat{S}_k^\nu$ then $\hat{\upsilon}_k + \alpha^{-1} \hat{w}$, and also $\hat{y} = \alpha \hat{\upsilon}_k + \hat{w} \in \hat{S}_k^\nu$ for α near ∞.Theorem 3 is proved.

Isomorphism $(\lambda, \hat{y}) \to (\lambda, y)$ shows, that the statements of theorem 3 are also true for problem (1) - (3). Thus $\hat{T}$, $\hat{D}_k$, $\hat{M}$, $\hat{E}$, $\widehat{\mathcal{E}}$, $\hat{D}_k^\nu$, $\hat{Q}$ are substituted by T, D_k, M, E, $\mathcal{E}$, D_k^ν, Q.

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