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SETTLEMENT OF POROUS WATER-SATURATED LAYER OF FRACTAL STRUCTURE

Abstract

The equation of consolidation of two-phase layer at fractal law of filtration has been deduced. Solution has been obtained by numerical method, and calculation has been carried out. Diagrams of thrusts distribution on layer depth for several sequential time values have been constructed.

Theory of consolidation of water-saturated soils attracts growing attention of scientists in connection with increasing volume of construction ground and underground works in compound soils. From the standpoint of mechanics of multiphase systems, it is represented as a problem of unsteady filtration in deformable medium. Subject to variable porosity and permeability of soil, its multiphasicity and complex rheology of phase components, this problem at times can not be solved analytically and in this case to analyze the studied process we use corresponding numerical methods.

Last investigations allow to judge on fractal character of ground massif structure. Estimation of influence of ground structure fractality on processes occurring in this soil, is connected with solution of corresponding problems. In the present time there are not enough such solutions and they have been obtained only for elementary cases of cylindrical and spherical symmetry.

Construction of consolidation theory of ground massif of fractal structure is urgent required. However, it is natural, that the obtained system of equations will be mathematically more complicated. Therefore, the step-by-step way of accounting of fractal structure of the studied object is logical.

In the present paper the two-dimensional equation of consolidation of two-phase entirely water-saturated soil, when it is taken into account the fractality of geometry of porous space only in Darcy's law of filtration, has been obtained. For its derivation we follow the designations accepted in [1].

The filtration law in soil with fractal geometry is accepted according to [2] in the form:

$$\begin{aligned} u_x - \frac{n}{m} v_x &= -k_x \left(\frac{\partial}{\partial x} \left(D^{\alpha}_{a_+} H \right)(x) - i_0 \right); \\ v_z - \frac{n}{m} v_z &= -k_z \left(\frac{\partial}{\partial z} \left(D^{\beta}_{0_+} H \right)(z) - i_0 \right); \end{aligned} \tag{1}$$

where

$$\begin{aligned} \left(D^{\alpha}_{a_+} H \right)(x) &= \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int\limits_a^x \frac{H(\xi)\, d\xi}{(x-\xi)^{\alpha}}, \\ \left(D^{\beta}_{0_+} H \right)(z) &= \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial z} \int\limits_0^z \frac{H(\xi)\, d\xi}{(x-\xi)^{\beta}}, \end{aligned} \tag{2}$$

are the left-side fractional derivatives of orders α; $0<\alpha<1$ and β; $0<\beta<1$, respectively.

Here u_x, u_y are projections of mean fictitious velocity of water motion in pores; v_x, v_y are projections of mean velocity of soil skeleton, n is soil porosity (pore volume in unit of soil volume, m is solid volume in unit of soil, and $m+n=1$; k_x and k_y are filtration coefficients in a direction of axes x and y; i_0 is initial thrust gradient; H is thrust pressure function connected with pressure p and height z of the considered point above the plane of comparison by dependence

$$H=\frac{p}{\gamma}+z, \tag{3}$$

where γ is water specific weight.

Moreover, in (2) the latter α and β are fractality parameters of filtration component;

$$\Gamma\left(1-\alpha\right)$$

is Euler's function.

The continuity equations of liquid and solid components of soil are accepted in classical form (that is our simplification at the first investigation stage):

$$\begin{aligned}\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}+\frac{\partial n}{\partial t}&=0;\\ \frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}+\frac{\partial m}{\partial t}&=0;\end{aligned} \tag{4}$$

From the system of the equations (1) - (4) following the sequence of the actions given in [1] we come to the following consolidation equation for soil with fractal structure.

$$\begin{aligned}\frac{\partial e}{\partial t}=\left(1+e\right)\Bigg[\frac{\partial}{\partial x}k_x\left(\frac{\partial}{\partial x}\left(D^{\alpha}_{a_+}H\right)(x)-i_0\right)+\\ +\frac{\partial}{\partial z}k_z\left(\frac{\partial}{\partial z}\left(D^{\beta}_{0_+}H\right)(z)-i_0\right)\Bigg],\end{aligned} \tag{5}$$

where

$$e=\frac{n}{m}=\frac{n}{1-n}$$

is porosity factor determining the ratio of pore volume to soil (particles) skeleton volume.

Let's simplify this equation, assuming the absence of initial thrust gradient $i_0=0$, constancy and similarity of filtration coefficient in various directions, $k_x=k_z=k$. Then we get

$$\frac{\partial e}{\partial t}=\left(1+e\right)k\left(\frac{\partial^2}{\partial x^2}\left(D^{\alpha}_{a_+}H\right)(x)+\frac{\partial^2}{\partial z^2}\left(D^{\beta}_{0_+}H\right)(z)\right). \tag{6}$$

The obtained equation connects change of soil porosity with process of unsteady filtration of water.

To solve applied problems it is necessary to pass to stressed state of the consolidated medium. In this connection there is a necessity for attraction of additional physical representations on consolidation process. Such essential physical assumption is the N.M.Gersevanov hydrocapacity principle, consisting of the fact that change of soil porosity factor (compacting) is determined only by change of the principal stresses sum $\sigma = \sigma_x + \sigma_y + \sigma_z$, i.e $e = f(\sigma)$.Then equation (6) can be written as

$$\frac{\partial \theta}{\partial t} = \frac{(1+e)\,k}{\dfrac{de}{d\theta}} \left(\frac{\partial^2}{\partial x^2} \left(D^{\alpha}_{a_+} H \right)(x) + \frac{\partial^2}{\partial z^2} \left(D^{\beta}_{0_+} H \right)(z) \right). \tag{7}$$

This equation we'll called the basic equation of two-dimensional consolidation at fractal law of filtration, for it allows to take into account any calculation model of deffected mode of soil, i.e. any dependence $de/d\theta$. In particular for linear dependence between stresses and strains, for example, linear connection of porosity factor with the main stresses sum as the straightened site of compression curve, when it holds

$$e = -\frac{a\sigma}{1+\xi} + b, \tag{8}$$

where a (compacting factor) and b are empirically determined constants, ξ is factor of lateral pressure (thrust), determining the ratio of lateral compressing stresses to longitudinal ones; the consolidation equation will have the form:

$$\frac{\partial \theta}{\partial t} = -\frac{(1+e)(1+\xi)\,k}{a} \left(\frac{\partial^2}{\partial x^2} \left(D^{\alpha}_{a_+} H \right)(x) + \frac{\partial^2}{\partial z^2} \left(D^{\beta}_{0_+} H \right)(z) \right). \tag{9}$$

For coordination of the main stresses sum σ with the thrust function H we accept V.A.Florin's [1] assumption according to which the stress state of the soil medium, on the whole, for any time coincides with stress state of the soil medium, in the assumption of its instant consolidation, i.e.

$$\sigma + 2p = \sigma^* + 2p^*, \tag{10}$$

where σ^* and p^* designate the sum of additional stresses of soil skeleton and pressure in water, which would arise, if water filling the pores, would not prevent the pores volume change. At the instant application of constant in future, loadings or boundary thrusts, the quantities σ^* and p^* don't vary in time, and then from (10) and (3) it follows

$$\frac{\partial \sigma}{\partial t} = -2\frac{\partial p}{\partial t} = -2\gamma \frac{\partial H}{\partial t}.$$

Taking this into account in (9) we finally obtain the following equation of two-dimensional consolidation of completely saturated soil massif at fractal law of filtration:

$$\frac{\partial H}{\partial t}=C_v\left(\frac{\partial^2}{\partial x^2}\left(D^{\alpha}_{a_+}H\right)(x)+\frac{\partial^2}{\partial z^2}\left(D^{\beta}_{0_+}H\right)(z)\right), \tag{11}$$

where

$$C_v=\frac{(1+e)(1+\xi)k}{2\gamma a} \tag{12}$$

is called the consolidation factor.

Estimation of influence of fractal law of filtration on process of consolidation we'll make on example of solution of one-dimensional problem of completely saturated soil consolidation.

Let's direct axis z vertically upwards, having arranged the origin coordinate of on base of layer with thickness h.

Let the soil layer be compressed by the uniformly distributed load q instantly applied at the time $t=0$. Boundary surfaces $z=0;h$ of the layer we assume completely water-permeable.

The consolidation equation (11) for one-dimensional case will be:

$$\frac{\partial H}{\partial t}=\frac{C_v}{\Gamma(1-\beta)}\frac{\partial^3}{\partial z^3}\int\limits_0^z\frac{H(\xi,t)}{(z-\xi)^{\beta}}d\xi. \tag{13}$$

Under the term "instantly applied load" we'll understand such velocity of statical application of load, at which it is possible to assume, that in the soil there is no any water outflow from soil pores. As a result, ignoring water compressibility and even more so of soil mineral particles, its porosity or factor of porosity during the initial moment of the load application $t=0$ doesn't change.

Accepting dependence of porosity factor on stresses in soil skeleton in the form of compression dependences $e=e_0-a\sigma_z$, and taking into account, that for $t=0$ $e-e_0=0$, we obtain at the initial moment $\sigma_z=\sigma_{z_0}=0$. As in the considered case the equilibrium equation for any time has the form $q=\sigma_z+p$, we'll get $p_0=q$. Whence the additional surplus thrust in water for the initial time will be

$$H_0=p_0/\gamma=q/\gamma.$$

Thus, the solution of the considered problem is reduced to search of solution of equation (13) under the following initial and boundary conditions:

$$t=0 \;\; and \;\; 0\le z\le h \;\; H_0=p_0/\gamma=q/\gamma, \tag{14}$$

$$t\neq 0 \;\; and \;\; z=0 \;\; or \;\; z=h \;\; H=0. \tag{15}$$

Let's introduce the following dimensionless quantities:

$$\tilde{H} = \frac{H}{h}; \quad \tilde{z} = \frac{z}{h}; \ \tilde{\xi} = \frac{\xi}{h}; \ \tilde{t} = \frac{C_v}{h^{2+\beta}} t. \tag{16}$$

In dimensionless quantities the of consolidation equation (11), initial and boundary conditions (14) and (15) will be written as (where for simplification of notation the sign $\sim$ above the letters is omitted):

$$\frac{\partial H}{\partial t} = \frac{1}{\Gamma(1-\beta)} \frac{\partial^3}{\partial z^3} \int_0^z \frac{H(\xi, t)}{(z-\xi)^\beta} d\xi, \tag{17}$$

$$t = 0 \quad \text{and} \quad 0 \le z \le 1 \ \ H = H_0 \tag{18}$$

$$t \ne 0 \ \ \textit{and} \ \ z = 0 \ \ \text{or} \ \ z = 1 \quad H = 0. \tag{19}$$

To solve the problem we shall use a numerical method of finite differences. For convenience of further calculations we'll designate:

$$f(z, t) = \int_0^z \frac{H(\xi, t)}{(z-\xi)^\beta} d\xi. \tag{20}$$

Then instead of (18) we get:

$$\frac{\partial H(z, t)}{\partial t} = \frac{1}{\Gamma(1-\beta)} \frac{\partial^3}{\partial z^3} f(z, t) \tag{21}$$

Let's apply to (21) the finite difference method. For that we cover the whole of consolidation domain by coordinate grid with the steps Δz and Δt, having nodes with coordinates $t_k = k\Delta t$, $z_i = i\Delta z$.

Value of the pressure function H in a node with number (i, k) we'll denote by $H_{i,k}$. Having replaced the partial derivatives included in equation (21) by corresponding differences, we'll shall obtain the following numerical analogue:

$$H_{i,k+1} = H_{i,k} + \frac{\Delta t}{\Delta z^3} \frac{1}{\Gamma(1-\beta)} (f_{i+1,k} - 3f_{i,k} + 3f_{i-1,k} - f_{i-2,k}) \tag{22}$$

where

$$f(z, t) \approx f_{i,k} = \sum_{j=0}^{i-1} \frac{H_{j,k}}{(i-j)^\beta} \Delta z.$$

Numerical analogue of initial and boundary conditions will be:

$$k = 0 \quad \textit{and} \quad 0 \le i \le n \ \ H_{i,k} = H_0 \tag{23}$$

$$k \ne 0 \quad \textit{and} \quad i = 0 \quad \textit{or} \ \ i = n \quad H_{i,k} = 0. \tag{24}$$

On the basis of available formula (22), initial condition (23) and boundary condition (24), the program of calculations for numerical values of pressures $H_{i,k}$ in nodes of numerical grid is made and realized.

Calculation has been carried out for the following values of initial parameters: $H_0 = 1$, $\beta = 0.5$ (fractality of filtration law), $\beta = 0$ (the classical law of filtration) $\Delta t = 0.01$, $\Delta z = 0.1$, $k_* = 20$, $n = 10$. By results of calculations the thrust functions profiles by soil layer thickness are constructed for several values of time.

H

Thrust functions profiles by depth for different values of time:

$$t_k = k\Delta t = 0,01t.$$

$$\beta = 0.5\text{;}\ \beta = 0$$

From the table and profiles follows, that the account of fractality of filtration law gives overestimate of calculated values of thrust functions, and this overestimate eventually becomes more and more appreciable.

Separate interest represents influence of filtration law fractality on common settlement of the layer. For this settlement $S(t)$ according to the known formula [1] we have:

$$S(t) = \frac{a}{1+e}\int_0^h (q - \gamma H)\, dz.$$

Or by means of $H_{i,k}$values of thrust function in nodes of the numerical grid:

$$S_k = \frac{a}{1+e}\left(qh - \gamma \sum_{i=1}^{9} H_{i,k}\Delta z\right).$$

As $H_0 = \dfrac{q}{h\gamma} = 1$, for dimensionless settlement $\tilde{S}_k = S_k/h$, allowing for dimensionless coordinate z and dimensionless quantity $\tilde{a} = aq/(1+e)$, omitting, as before, the sign $\sim$ above the dimensionless quantities we 'll obtain:

$$S_k = a\left(1 - \sum_{i=1}^{9} H_{i,k}\Delta z\right).$$

The settlements calculated by this formula for values of times Δt, $5\Delta t$, $10\Delta t$, $15\Delta t$, $20\Delta t$, for the fractality parameter $\beta = 0.5$ are tabulated.

In the same place for comparison there are resulted the values of settlements obtained at the classical law of filtration:

Table

In the last column of the table the differences of these results expressed in percentage have been resulted. They testify to significance of fractality of filtration law influence on the calculated values of soil layer settlement.

References

[1]. Ivanov L.P. *Soils and the bases of hydraulic engineering constructions.* Mekhanika gruntov, Visshaya Shkola, Moscow, 1991, 447 p. (Russian)

[2]. Dinariev O.Yu. *Fluids and gases flow in porous medium with fractal geometry.* Mekhanika zhidkosti i gaza, 1992, No 5, pp.101-109. (Russian)

[3]. Nigmatulin R.R. *Fractional integral and its physical interpretation.* Teoreticheskaya i matematicheskaya fizika, 1992, v.90, No 3, pp.354-367. (Russian)

[4]. Feder E. *Fractals.* Izdatel. "Mir", M., 1991. (Russian)

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