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GRACK NUCLEATION IN THICK-WALLED CYLINDER

Abstract

Mathematical model of crack-formation in an isotropic thick-walled cylinder under the conditions of plane deformation is constructed. The solution of a problem on equilibrium of thick-walled cylinder with germ crack is reduced to the solution of a singular integral equation with Cauchy type kernel. Condition of crack appearance is formulated allowing for criterion of limit opening of prefracture zone focus.

Let the cross section of the considered cylinder fill in the plane $z = x + iy$ the domain S bounded from the outside by a circle of radius R_1 and from the inside-by a circle of radius R. Consider the stress strain state in the annular domain S under the action of load normal and tangent to the contour. It is assumed that plane deformation state holds.

It is accepted that there is a stress concentrator (a zone of weakened interparticle bonds of the material) in the material of the thick-walled cylinder. When a cylinder is loaded (interlayer of overstressed material) plastic flow zone is formed therein. After several number of cycles possibility of plastic deformation in the zone of weakened interparticle bonds of the material is exhausted and opening of faces of plastic flow region sharply increases. If at the maximal concentration point the opening of the faces of prefracture zone (a zone of weakened interpartide bonds of the material) achieves limiting value δ_c for the given material of the cylinder, then fatigue crack originates at this point [1].

As the cylinder operates, there will arise prefracture zones in the metallic cylinder that will be modelled as zones of weakened interparticle bonds of the material. Interaction of prefracture zone surface is modelled [1] by introducing plastic sliding (degenerated zones of plastic deformations) between its faces. It is assumed that under the action of normal and tangent load on external and internal surfaces on the cylinder, interaction between the surfaces in prefracture zones is characterized by the constant normal σ_T and tangent τ_T stresses of cohesion. Such an assumption allows to model plastic flow in the prefracture zones of the material. Physical nature of such bonds and sizes of prefracture region wherein interaction of faces of interparticle bonds zones are realized, depends on the form of the material[2,3].

In the considered case arise of a crack in the cross section of the cylinder is a process of passage of prefracture zone to the zone of destroyed bonds between the surfaces of the cylinder material. The sizes of the prefracture zone are not known beforehand and to be determined in solving the considered problem.

We model the cylinder by an isotropic homofeneous body. A solid body (cylinder) deformed beyond the elasticity limit is represented as a body elastically deformed everywhere except some surfaces . Remind that elastic deformation represents itself as change in distances between elementary particles not changing their location order but plastic deformation changes relative location of atoms with respect to the other one (elementary plastic deformations).

Since the indicated zones (interlayer of overstressed material) are small in comparison with remaining elastic part of the cylinder, one can remove them mentally and replace them by cross-cuts whose surfaces interact between themselves by a low corresponding to the action of the removed material. Thus, a problem on deformation of a cylinder beyond the elasticity limit is reduced to the problem on deformation of some elastic body possessing conventional cross-cuts. The forces acting on the surfaces of these zones are called forces of adhesion of weakened bonds, and the zones wherein they appear, are called prefractute zones or weakened bonds domains.

It is accepted that a zone of weakened interparticle bonds of the material is directed in the direction of action of maximal stretching stresses obtained from the solution of elastic problem. In the centre of prefracture zone we locate the origin of local system of coordinates x_1Oy_1, whose axis x_1 coincides with plastic deformations line and forms an angle α_1 with axis x.

Given external forces acting on the boundary of annular domain S

$$\sigma_r - i\tau_{r\theta} = f_1(\theta) - if_2(\theta) \;\; for \;\; |z| = R$$

$$\sigma_r - i\tau_{r\theta} = f_3(\theta) - if_4(\theta) \;\; for \;\; |z| = R_1 \tag{1}$$

and principal vector and principal moments of these forces equal zero.

Boundary conditions on the surface of the prefracture zone will be

$$\sigma_{y_1} - i\tau_{x_1y_1} = \sigma_T - i\tau_T \;\; for \; y_1 = 0, \;\; |x_1| \leq \ell_1, \tag{2}$$

where l_1 is a half-length of a prefracture zone, to be determined. We use N.I. Muskhelishvili method [4] for solving boundary value problem (1)-(2). Using Kolosov-Muskheleshvili formula we write conditions (1)-(2) in the following form:

$$for \; r = R \;\; \Phi(z) + \overline{\Phi(z)} - e^{2i\theta}\left[\bar{z}\Phi'(z) + \Psi(z)\right] = f_1(\theta) - if_2(\theta)$$

$$for \; r = R_1 \;\; \Phi(z) + \overline{\Phi(z)} - e^{2i\theta}\left[\bar{z}\Phi'(z) + \Psi(z)\right] = f_3(\theta) - if_4(\theta) \tag{3}$$

$$for \; y_1 = 0 \;\; |t| \leq \ell_1 \;\; \Phi(t) + \overline{\Phi(t)} + \bar{t}\Phi'(t) + \Psi(t) = \sigma_T + i\tau_T. \tag{4}$$

We look for the complex potentials $\Phi(z)$ and $\Psi(z)$ describing strese-strain state in an annular domain S in the form:

$$\Phi(z) = \Phi_0(z) + \Phi_1(z) + \Phi_2(z);$$

$$\Psi(z) = \Psi_0(z) + \Psi_1(z) + \Psi_2(z), \tag{5}$$

where

$$\Phi_0(z) = \sum_{k=-\infty}^{\infty} a_k z^k; \;\; \Psi_0(z) = \sum_{k=-\infty}^{\infty} b_k z^k \tag{6}$$

$$\Phi_1(z)=\frac{1}{2\pi}\int\limits_{-\ell_1}^{\ell_1}\frac{g_1(t)\,dt}{t-z_1};$$

$$\Psi_1(z)=\frac{1}{2\pi}e^{-2i\alpha_1}\int\limits_{-\ell_1}^{\ell_1}\left[\frac{\overline{g_1(t)}}{t-z_1}-\frac{\overline{T}_1e^{-2i\alpha_1}}{(t-z_1)^2}g_1(t)\right]dt; \tag{7}$$

Here $g_1(x)$ is a desired function characterizing opening of prefracture zone faces while passing throung the prefracture zone line.

The unknown desired function $g_1(x)$ and complex potentials $\Phi_2(z)$ and $\Psi_2(z)$ should be satisfied from boundary conditions on the prefracture zone faces and on the contour $r=R_1$. After some transformations and calculations of appropriate integrals we find

$$\Phi_2(z)=\frac{1}{2\pi}\int\limits_{-\ell_1}^{\ell_1}\left\{\left(\frac{1}{z\overline{T}_1-1}+\frac{1}{2}\right)\overline{T}_1e^{i\alpha_1}g_1(t)+\right.$$

$$\left.+\left[\frac{T_1}{2}-\frac{z^2\overline{T}_1-2z+T_1}{\left(z\overline{T}_1-1\right)^2}\right]e^{-i\alpha_1}\overline{g_1(t)}\right\}dt;$$

$$\Psi_2(z)\,\frac{1}{2\pi}\int\limits_{-\ell_1}^{\ell_1}\left[\frac{e^{i\alpha_1}\overline{T}_1^3}{\left(z\overline{T}_1-1\right)^2}g_1(t)+\right.$$

$$\left.+\left(z^2\overline{T}_1^2+4-3z\overline{T}_1+zT_1\overline{T}_1^2-3T_1\overline{T}_1\right)\frac{\overline{T}_1e^{-i\alpha_1}}{\left(z\overline{T}_1-1\right)^3}\overline{g_1(t)}\right]dt \tag{8}$$

$$T_1=te^{i\alpha_1}+z_1^0;\qquad z_1=e^{-i\alpha_1}\left(z-z_1^0\right).$$

For finding complex potentials $\Phi_0(z)$ and $\Psi_0(z)$ we can write boundary conditions (3) of the problem in the following form

$$\begin{gathered}\Phi_0(\tau)+\overline{\Phi_0(\tau)}-e^{2i\theta}\left[\overline{\tau}\Phi_0'(\tau)+\Psi_0(\tau)\right]=f_1(\theta)-if_2(\theta)-\\ -\left(f_1^*(\theta)-if_1^*(\theta)\right)\quad for\ \ \tau=R\,e^{i\theta}\\ \Phi_0(\tau_1)+\overline{\Phi_0(\tau_1)}-e^{2i\theta}\left[\overline{\tau}_1\Phi_0'(\tau_1)+\Psi_0(\tau_1)\right]=\\ =f_3(\theta)-if_4(\theta)\quad for\ \ \tau_1=R_1\,e^{i\theta}\end{gathered} \tag{9}$$

Here $f_1^*(\theta)-if_1^*(\theta)=\Phi_*(\tau)+\overline{\Phi_*(\tau)}-e^{2i\theta}\left[\overline{\tau}_1\Phi_*'(\tau)+\Psi_*(\tau)\right],$

$$\Phi_*(z)=\Phi_1(z)+\Phi_2(z);\ \Psi_*(z)=\Psi_1(z)+\Psi_2(z).$$

The solution of boundary value problem (9) is attained by the method of power series [4]. To this end it is necessary to expand the right hand sides of boundary conditions in Fourier series. These expansions are of the form:

$$f_1^*(\theta)-if_1^*(\theta)=\sum_{k=-\infty}^{\infty}A_k^*e^{ik\theta},$$

$$f_1(\theta) - if_2(\theta) = \sum_{k=-\infty}^{\infty} A_k e^{ik\theta}, \ for\ r = R \tag{10}$$

$$f_3(\theta) - if_4(\theta) = \sum_{k=-\infty}^{\infty} B_k e^{ik\theta}, \ for\ r = R_1$$

where

$$A_k^* = \frac{1}{2\pi}\int_0^{2\pi} \left(f_1^*(\theta) - if_2^*(\theta)\right) e^{-ik\theta} d\theta \quad (k = 0, \pm 1, \pm 2, ...)$$

$$A_k = \frac{1}{2\pi}\int_0^{2\pi} \left(f_1(\theta) - if_2(\theta)\right) e^{-ik\theta} d\theta$$

$$B_k = \frac{1}{2\pi}\int_0^{2\pi} \left(f_3(\theta) - if_4(\theta)\right) e^{-ik\theta} d\theta$$

We don't cite here expansion coefficients in obvious form because of their bulky form.

Satisfying boundary conditions (9) by complex potentials (6), after some transformations we get a system of linear algebraic equations with respect to the desired coefficients a_k and b_k:

$$\begin{gathered} a_0 + \overline{a}_0 - b_{-2}R^{-2} = A_0 - A_0^* \\ a_0 + \overline{a}_0 - b_{-2}R_1^{-2} = B_0 \\ (1-k)\, a_k R^k + \overline{a}_{-k} R^{-k} - b_{k-2} R^{k-2} = A_k - A_k^* \\ (1-k)\, a_k R_1^k + \overline{a}_{-k} R_1^{-k} - b_{k-2} R_1^{k-2} = B_k \end{gathered} \tag{11}$$

The solution of infinite linear system (11) is easy and has the following form:

$$a_0 = \frac{B_0 R_1^2 - (A_0 - A_0^*)R^2}{2\left(R_1^2 - R^2\right)};\ a_{-1} = \frac{(\overline{A}_1 - A_1^*)R^2}{1 + k_0};\ k_0 = 3 - 4\nu;$$

$$b_{-1} = -\frac{k_0\left(A_1 - A_1^*\right)R}{1 + k_0};\ a_1 = \frac{\overline{M}_{-1}}{R_1^4 - R^4} - \frac{2\left(A_1 - A_1^*\right)R}{(1 + k_0)\left(R_1^2 + R^2\right)} \tag{12}$$

$$a_k = \frac{(1 + k_0)\left(R_1^2 - R^2\right) M_k - \left(R_1^{-2k+2} - R^{-2k+2}\right)\overline{M}_{-k}}{(1 - k^2)\left(R_1^2 - R^2\right)^2 - \left(R_1^{2k+2} - R^{2k+2}\right)\left(R_1^{-2k+2} - R^{-2k+2}\right)};$$

$$(k = \pm 2, \pm 3, ...)$$

$$M_k = B_k R_1^{-k+2} - \left(A_k - A_k^*\right) R_1^{-k+2};$$

$$b_{-2}R^{-2} = 2a_0 - A_0 + A_0^*;$$

$$b_{k-2}R_1^{k-2} = (1-k)\, a_k R_1^k + \overline{a}_{-k} R_1^{-k} - B_k.$$

Satisfying boundary conditions by the functions (5)-(8) on the prefracture zone faces we get complex singular integral equation with respect to the unknown function $g_1(x_1)$:

$$\int_{-\ell_1}^{\ell_1} \left[R(t,x_1)\, g_1(t) + S(t,x_1)\, \overline{g_1(t)} \right] dt = \pi f(x_1), \quad |x_1| \le \ell_1. \tag{13}$$

Here $f(x_1) = \sigma_T - i\tau_T + f_0(x_1)$,

$$f_0(x_1) - \left[\Phi_0(x_1) + \overline{\Phi_0(x_1)} + x_1 \overline{\Phi_0'(x_1)} + \overline{\Psi_0(x_1)} \right]$$

$$R(t,x_1) = \frac{e^{ia_1}}{2} \left(\frac{1}{T_1 - X_1} + \frac{e^{-2ia_1}}{\overline{T}_1 - \overline{X}_1} \right) -$$

$$-\frac{e^{ia_1}}{2} \left(\frac{X_1 \overline{T}_1^2}{1 - X_1 \overline{T}_1} + \frac{\overline{X}_1^2 T_1 - 2\overline{X}_1 + \overline{T}_1}{\left(1 - T_1 \overline{X}_1\right)^2} + \right.$$

$$\left. + e^{-2ia_1} \frac{2X_1 \left(T_1 \overline{T}_1 - 1\right) + \overline{T}_1^2 \left(\overline{X}_1 + \overline{T}_1\right) \left(\overline{X}_1 T_1 - 3\right) + 4T_1}{\left(1 - T_1 \overline{X}_1\right)^3} \right);$$

$$S(t,x_1) = \frac{e^{-ia_1}}{2} \left[\frac{1}{\overline{T}_1 - \overline{X}_1} - \frac{T_1 - X_1}{\left(\overline{T}_1 - \overline{X}_1\right)^2} e^{-2ia_1} \right] -$$

$$-\frac{e^{-ia_1}}{2} \left[\frac{T_1^2 \overline{X}_1}{1 - T_1 \overline{X}_1} + \frac{X_1^2 \overline{T}_1 - 2X_1 + T_1}{\left(1 - X_1 \overline{T}_1\right)^2} + \frac{T_1^2 \left(X_1 - T_1\right) e^{-2ia_1}}{\left(1 - T_1 \overline{X}_1\right)^3} \right];$$

$$X_1 = x_1 e^{-ia_1} + z_1^0.$$

For internal layer of prefracture the additional condition

$$\int_{-\ell_1}^{\ell_1} g_1(t)\, dt = 0 \tag{14}$$

providing uniqueness of displacement in tracing the prefracture zone boundary should be added to the singular integral condition.

Using change of variables $\xi = t/\ell_1$ and $\eta = t/\ell_1$ we pass to dimensionless variables in the integral equation (13).

We represent the solution of integral equation (13) in the form [5,6]:

$$g_1(\eta) = \frac{\varphi_0(\eta)}{\sqrt{1 - \eta^2}}. \tag{15}$$

Applying the algebraization proceduce [5,6] to singular integral equation (13) and additional condition (14) we get a system of M algebraic equations for determining m unknowns $g_1(t_m) = \upsilon(t_m) - iu_1(t_m)$:

$$\frac{1}{M} \sum_{m=1}^{M} \ell_1 \left[\varphi_0(t_m)\, R(\ell_1 t_m, \ell_1 \eta_r) + \overline{\varphi_0(t_m)} S(\ell_1 t_m, \ell_1 \eta_r) \right] = f(\eta_r);$$

$$(r = 1, 2, ..., \ M - 1) \tag{16}$$

$$\sum_{m=1}^{M} \varphi_0 (t_m) = 0$$

Here $t_m = \cos \dfrac{2m-1}{2M}\pi \quad (m = 1, 2, ..., M)$

$$\eta_r = \cos \frac{\eta r}{M} \ (r = 1, 2, ..., \ M - 1).$$

If in the system (16) to pass to complexly adjoint values, we get some more M algebraic equations.

Since the stresses in the cylinder are restricted everywhere, the solution of singular integral equation should be sought in a class of everywhere bounded functions. Cousequently, it is necessary to add to the system of equation (16) stress finiteness conditions in the vicinity of the ends of prefracture zone.

Writing these conditions

$$\sum_{m=1}^{M} (-1)^m \varphi_0 (t_m) \, ctg \frac{2m-1}{4M}\pi = 0; \tag{17}$$

$$\sum_{m=1}^{M} (-1)^{M+m} \varphi_0 (t_m) \, tg \frac{2m-1}{4M}\pi = 0.$$

we get a finite algebraic system (16),(17) for determining M unknowns

$$\varphi_0 (t_m) \, (m = 1, 2, ..., M)$$

and dimension of prefracture zone.

Because of the unknown dimension of prefracture zone length ℓ_1 algebraic system (16),(17) became non-linear. For solving the obtained algebraic systems (16),(17) we used successive approximations method. For determining limit equilibrium of prefracture zone in the cylinder we use criterian of critical opening of prefracture zone faces.

We accept that break of interparticle bonds of the material on the prefracture zone faces ($for \ x_1 = x_0$) occurs in fulfilling the condition

$$\sqrt{\left[u_1^+ (x_0, 0) - u_1^- (x_0, 0)\right]^2 + \left[\upsilon_1^+ (x_0, 0) - \upsilon_1^- (x_0, 0)\right]^2} = \delta_c, \tag{18}$$

where the parameter δ_c is a characteristics of resistance of material of the cylinder to crackformation, is determined experimentally.

Using the solution of a problem on equilibrium of prefracture zone in the cylinder, we calculate displacements on the prefracture zone faces

$$\upsilon_1^* (x_0, 0) - iu_1^* (x_0, 0) = -\frac{1+k_0}{2\mu} \frac{\pi \ell_1}{M} \sum_{m=1}^{M_1} g_1 (t_m) \tag{19}$$

where $\upsilon_1^*(x_0,0) = \upsilon_1^+(x_0,0) - \upsilon_1^-(x_0,0)$; $u_1^*(x_0,0) = u_1^+(x_0,0) - u_1^-(x_0,0)$;

M_1 is the number of nodal points belonging to the segment $(-\ell_1, x_0)$; μ is a shear modulus of the material.

Obviously break of interparticle bonds of the material will occur in the middle part of the prefracture zone i.e. $M_1 = \frac{1}{2}M$.

From relation (19) separating real and imaginary parts and calculating displacament vector modulus V_1 on the prefracture zone faces for $x = x_0$ we get

$$V_1 = \frac{1+k_0}{2\mu}\frac{\pi\ell_1}{M}\sqrt{C_1^2 + C_2^2},$$

where

$$C_1 = \sum_{m=1}^{M_1} \upsilon_1(t_m); \quad C_2 = \sum_{m=1}^{M_2} u_1(t_m).$$

Thus,

$$\frac{1+k_0}{2\mu}\frac{\pi\ell_1}{M}\sqrt{C_1^2 + C_2^2} = \delta_c \tag{20}$$

is the condition determining limit value of external load under which crack arises.

Joint solution of equations (12),(16),(17) and (20) allows to determine critical value of external load and dimension of prefracture zone ℓ_1^c for limit equilibrium state for the given characteristics of thick-walled cylinder.

Changing the values of parameters α_1 and z_1^0 we can investigate different cases of prefracture zone location in the material of the cylinder.

In the case when prefracture zone arrives at the surface of the cylinder by one end, necessity in additional equality (14) falls off.

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