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AN INVERSE BOUNDARY VALUE PROBLEM FOR A FOURTH ORDER EVOLUTIONARY EQUATION ARISING IN HYDROACOUSTICS OF STRATIFIED LIQUID

Abstract

In the paper we study an inverse boundary value problem for a fourth order evolutionary equation arising in hydroacoustics of stratified liquid. At first the initial problem is reduced to an equivalent problem for which a theorem on the existence and uniqueness of the classic solution is proved. Further, using these facts, we prove the existence and uniqueness of the classic solution of the initial problem.

In the domain $D_T = \{(x,t);\ 0 \le x \le 1,\ 0 \le t \le T\}$ we consider the equation [1,2]

$$u_{tttt}(x,t) - u_{ttxx}(x,t) + u_{tt}(x,t) - u_{xx} = a(t)u(x,t) + f(x,t) \tag{1}$$

under conditions

$$\begin{gathered} u(x,0) = \varphi_0(x),\ u_t(x,0) = \varphi_1(x),\ u_{tt}(x,0) = \varphi_2(x), \\ u_{ttt}(x,0) = \varphi_3(x) \quad (0 \le x \le 1), \end{gathered} \tag{2}$$

$$u(0,t) = u_x(1,t) = 0 \quad (0 \le t \le T), \tag{3}$$

$$u(1,t) = h(t) \quad (0 \le t \le T), \tag{4}$$

where $f(x,t),\ \varphi_i(x)\ (i = \overline{0,3}),\ h(t)$ are the given functions, $u(x,t)$ and $a(t)$ are the desired functions.

Accept the following

Definition. *A pair $\{u(x,t),\ a(t)\}$ of functions $u(x,t)$ and $a(t)$ possessing the following properties:*

1) the function $u(x,t)$ is continuous in D_T together with all its derivatives entering in the equation (1);

2) the function $a(t)$ is continuous on $[0,T]$;

3) all the conditions of (1)-(4) are satisfied in the ordinary sense, is said to be a classic solution of problem (1)-(4).

The following lemma is valid.

Lemma 1. *Let $h(t) \in C^4[0,T],\ h(t) \ne 0$ for $t \in [0,T],\ \varphi_0(1) = h(0),\ \varphi_1(1) = h'(0),\ \varphi_2(1) = h''(0),\ \varphi_3(1) = h'''(0)$. Then the problem on finding classic solution of the problem (1)-(4) is equivalent to the problem on determination of the functions $u(x,t)$ and $a(t)$ possessing the properties 1) and 2) of definition of the classic solution of the problem (1)-(4) from (1)-(3) and*

$$a(t)h(t) + f(1,t) = h^{(4)}(t) + h''(t) - u_{ttxx}(1,t) - u_{xx}(1,t) \quad (0 \le t \le T). \tag{5}$$

To investigate the problem (1)-(3), (5) we consider the following spaces. By $B_{2,T}^{\alpha}$ [3] we denote a totality of all functions of the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\lambda_k x,\quad \lambda_k=\frac{\pi}{2}(2k-1),$$

considered in D_T, where each of the functions $u_k(t)$ is continuous on $[0,T]$ and

$$I(u)\equiv\left\{\sum_{k=1}^{\infty}\left(\lambda_k^{\alpha}\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right\}^{1/2}<+\infty, \tag{6}$$

where $\alpha\geq 0$. In this set we determine the norm as follows:

$$\left\|u(x,t)\right\|_{B_{2,T}^{\alpha}}=I(u).$$

By E_T^{α} we denote a space $B_{2,T}^{\alpha}\times C[0,T]$ of the vector-functions $z(x,t)=\{u(x,t),\ a(t)\}$ with norm:

$$\left\|z\right\|_{E_T^{\alpha}}=\left\|u(x,t)\right\|_{B_{2,T}^{\alpha}}+\left\|a(t)\right\|_{C[0,T]}.$$

It is known that $B_{2,T}^{\alpha}$ and E_T^{α} are the Banach spaces.

We'll seek the first component $u(x,t)$ of the solution $\{u(x,t),\ a(t)\}$ of the problem (1)-(2), (5) in the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\lambda_k x,\quad \left(\lambda_k=\frac{\pi}{2}(2k-1)\right), \tag{7}$$

where

$$u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx\quad (k=1,2,...).$$

Then, applying the formal scheme of the Fourier method, from (1) and (2) we get:

$$u_k^{(4)}(t)+(\lambda_k^2+1)u_k''(t)+\lambda_k^2 u_k(t)=F_k(u,a;t)\quad (k=1,2,...) \tag{8}$$

$$u_k(0)=\varphi_{0k},\ u_k'(0)=\varphi_{1k},\ u_k''(0)=\varphi_{2k},\ u_k'''(0)=\varphi_{3k}\ \ (k=0,1,2,...), \tag{9}$$

where

$$F_k(u,a;t)=a(t)u_k(t)+f_k(t),\ f_k(t)=2\int_0^1 f(x,t)\sin\lambda_k x dx\quad (k=1,2,...),$$

$$\varphi_{ik}(x)=2\int_0^1\varphi_i(x)\sin\lambda_k x dx\quad (i=\overline{0,3};\ k=1,2,...).$$

Now, allowing for (7) from(5) we have:

$$a(t)=h^{-1}(t)\left\{h^{(4)}(t)+h''(t)-f(1,t)+\sum_{k=1}^{\infty}\lambda_k^2\left(u_k''(t)+u_k(t)\right)\right\}. \tag{10}$$

After application of the method of the variation of constant solution of the problem (8), (9) we find [4]:

$$u_k(t) = \frac{1}{\lambda_k^2-1}\left[\left(\lambda_k^2\cos t - \cos\lambda_k t\right)\varphi_{0k} + \right.$$
$$+\left(\lambda_k^2\sin t - \frac{\sin\lambda_k t}{\lambda_k}\right)\varphi_{1k} + (\cos t - \cos\lambda_k t)\varphi_{2k} +$$
$$+\left(\sin t - \frac{\sin\lambda_k t}{\lambda_k}\right)\varphi_{3k} + \frac{1}{\lambda_k}\int_0^t ((\lambda_k\sin(t-\tau) - \tag{11}$$
$$\left. - \sin\lambda_k(t-\tau))\, F_k(u,a;\tau)d\tau\right] \quad (k=1,2,...).$$

Substituting $u_k(t)$ from (11) into the representation (7) we get:

$$u(x,t) = \sum_{k=1}^{\infty}\left\{\frac{1}{\lambda_k^2-1}\left[\left(\lambda_k^2\cos t - \cos\lambda_k t\right)\varphi_{0k} + \right.\right.$$
$$+\left(\lambda_k^2\sin t - \frac{\sin\lambda_k t}{\lambda_k}\right)\varphi_{1k} + (\cos t - \cos\lambda_k t)\varphi_{2k} +$$
$$+\left(\sin t - \frac{\sin\lambda_k t}{\lambda_k}\right)\varphi_{3k} + +\frac{1}{\lambda_k}\int_0^t (\lambda_k\sin(t-\tau) - \tag{12}$$
$$\left.\left. - \sin\lambda_k(t-\tau)F_k(u,a;\tau)d\tau\right]\right\}\sin\lambda_k x.$$

Now, from (11) we have:

$$u_k'(t) = \frac{1}{\lambda_k^2-1}\left[\left(-\lambda_k^2\sin t + \lambda_k\sin\lambda_k t\right)\varphi_{0k} + \left(\lambda_k^2\cos t - \cos\lambda_k t\right)\varphi_{1k} + \right.$$
$$+(-\sin t + \lambda_k\sin\lambda_k t)\varphi_{2k} + (\cos t - \cos\lambda_k t)\,\varphi_{3k} + \tag{13}$$
$$\left. + \int_0^t (\cos(t-\tau) - \cos\lambda_k(t-\tau)F_k(u,a;\tau)d\tau\right],$$

$$u_k''(t) = \frac{1}{\lambda_k^2-1}\left[\lambda_k^2\left(-\cos t + \cos\lambda_k t\right)\varphi_{0k} + \lambda_k\left(-\lambda_k\sin t + \sin\lambda_k t\right)\varphi_{1k} + \right.$$
$$+(-\cos t + \lambda_k^2\cos\lambda_k t)\varphi_{2k} + (-\sin t + \lambda_k\sin\lambda_k t)\,\varphi_{3k} + \tag{14}$$
$$\left. + \int_0^t (-\sin(t-\tau) + \lambda_k\sin\lambda_k(t-\tau)F_k(u,a;\tau)d\tau\right],$$

$$u_k'''(t)=\frac{1}{\lambda_k^2-1}\left[\lambda_k^2\left(\sin t-\lambda_k\sin\lambda_k t\right)\varphi_{0k}+\lambda_k^2\left(-\cos t+\cos\lambda_k t\right)\varphi_{1k}+\right.$$
$$+(\sin t-\lambda_k^3\sin\lambda_k t)\varphi_{2k}+\left(-\cos t+\lambda_k^2\cos\lambda_k t\right)\varphi_{3k}+ \tag{15}$$
$$\left.+\int_0^t(-\cos(t-\tau)+\lambda_k^2\cos\lambda_k(t-\tau)F_k(u,a;\tau)d\tau\right],$$

$$u_k^{(4)}(t)=\frac{1}{\lambda_k^2-1}\left[\lambda_k^2\left(\cos t-\lambda_k^2\cos\lambda_k t\right)\varphi_{0k}+\lambda_k^2\left(\sin t-\lambda_k\sin\lambda_k t\right)\varphi_{1k}+\right.$$
$$+(\cos t-\lambda_k^4\cos\lambda_k t)\varphi_{2k}+\left(\sin t-\lambda_k^3\sin\lambda_k t\right)\varphi_{3k}+ \tag{16}$$
$$\left.+\int_0^t(\sin(t-\tau)-\lambda_k^3\sin\lambda_k(t-\tau)F_k(u,a;t)d\tau\right]+F_k(u,a;t).$$

Further, it is seen from (11) and (14) that

$$\nu_k(t)\equiv u_k''(t)+u_k(t)=\varphi_{0k}\cos\lambda_k t+\varphi_{1k}\frac{\sin\lambda_k t}{\lambda_k}+\varphi_{2k}\cos\lambda_k t+$$
$$+\varphi_{3k}\frac{\cos\lambda_k t}{\lambda_k}+\frac{1}{\lambda_k}\int_0^t F_k(u,a;\tau)\sin\lambda_k(t-\tau)d\tau. \tag{17}$$

After substituting the expressions $\nu_k(t)\equiv u_k''(t)+u_k(t)$ from (17) into (10), for determining the components $a(t)$ of the solution of the problem (1)-(3), (5) we find:

$$a(t)=h^{-1}(t)\left\{h^{(4)}(t)+h''(t)-f(1,t)+\sum_{k=1}^{\infty}\lambda_k^2\nu_k(t)\right\}=h^{-1}(t)\times$$
$$\times\left\{h^{(4)}(t)+h''(t)-f(1,t)+\sum_{k=1}^{\infty}\lambda_k^2\times\right.$$
$$\times\left[\varphi_{0k}\cos\lambda_k t+\varphi_{1k}\frac{\sin\lambda_k t}{\lambda_k}-\varphi_{2k}\cos\lambda_k t+\right. \tag{18}$$
$$\left.\left.+\varphi_{3k}\frac{\cos\lambda_k t}{\lambda_k}+\frac{1}{\lambda_k}\int_0^t F_k(u,a;\tau)\sin\lambda_k(t-\tau)d\tau\right]\right\}.$$

Proceeding from the definition of the solution of the problem (1)-(3), (5), we easily prove the following lemma.

Lemma 2. *It $\{u(x,t),a(t)\}$ is any solution of the problem (1)-(3), (5) the functions*

$$u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx\quad(k=1,2,...)$$

satisfy the system (11) on $[0,T]$.

Now, from (11), (13)-(17), we have:

$$|u_k(t)| \le 4|\varphi_{0k}| + 4|\varphi_{1k}| + \frac{4}{\lambda_k^2}|\varphi_{2k}| + \frac{4}{\lambda_k^2}|\varphi_{3k}| + \frac{4}{\lambda_k^2}\sqrt{T}\left(\int_0^T |F_k(u,a;\tau)|^2 d\tau\right)^{1/2},$$

$$|u_k'(t)| \le 4|\varphi_{0k}| + 4|\varphi_{1k}| + 4\lambda_k^{-1}|\varphi_{2k}| + \frac{4}{\lambda_k^2}|\varphi_{3k}| + \frac{4}{\lambda_k^2}\sqrt{T}\left(\int_0^T |F_k(u,a;\tau)|^2 d\tau\right)^{1/2},$$

$$|u_k''(t)| \le 4|\varphi_{0k}| + 4|\varphi_{1k}| + 4|\varphi_{2k}| + \frac{4}{\lambda_k}|\varphi_{3k}| + \frac{4}{\lambda_k}\sqrt{T}\left(\int_0^T |F_k(u,a;\tau)|^2 d\tau\right)^{1/2},$$

$$|u_k'''(t)| \le 4\lambda_k|\varphi_{0k}| + 4|\varphi_{1k}| + 4\lambda_k|\varphi_{2k}| + 4|\varphi_{3k}| + 4\sqrt{T}\left(\int_0^T |F_k(u,a;\tau)|^2 d\tau\right)^{1/2},$$

$$\left|u_k^{(4)}(t)\right| \le 4\lambda_k^2|\varphi_{0k}| + 4\lambda_k|\varphi_{1k}| + 4\lambda_k^2|\varphi_{2k}| + 4\lambda_k|\varphi_{3k}| + 4\sqrt{T}\lambda_k \times$$
$$\times\left(\int_0^T |F_k(u,a;\tau)|^2 d\tau\right)^{1/2} + |F_k(u,a;t)|.$$

$$|\nu_k(t)| \le |\varphi_{0k}| + \frac{1}{\lambda_k}|\varphi_{1k}| + |\varphi_{2k}| + \frac{1}{\lambda_k}|\varphi_{3k}| + \frac{1}{\lambda_k}\sqrt{T}\left(\int_0^T |F_k(u,a;\tau)|^2 d\tau\right)^{1/2}.$$

hence we have:

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_k(t)\|_{C[0,T]}\right)^2\right)^{1/2} \le 4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\varphi_{0k}|\right)^2\right)^{1/2} +$$
$$+4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\varphi_{1k}|\right)^2\right)^{1/2} + 4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k|\varphi_{2k}|\right)^2\right)^{1/2} + \tag{19}$$
$$+4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k|\varphi_{3k}|\right)^2\right)^{1/2} + 4\sqrt{5}\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k|F_k(u,a;\tau)|\right)^2 d\tau\right)^{1/2},$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_k'(t)\|_{C[0,T]}\right)^2\right)^{1/2} \le 4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\varphi_{0k}|\right)^2\right)^{1/2} +$$
$$+4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\varphi_{1k}|\right)^2\right)^{1/2} + 4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2|\varphi_{2k}|\right)^2\right)^{1/2} + \tag{20}$$
$$+4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k|\varphi_{3k}|\right)^2\right)^{1/2} + 4\sqrt{5}\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k|F_k(u,a;\tau)|\right)^2 d\tau\right)^{1/2},$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k''(t)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq 4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{0k}\right|\right)^2\right)^{1/2}+$$
$$+4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{1k}\right|\right)^2\right)^{1/2}+4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{2k}\right|\right)^2\right)^{1/2}+ \tag{21}$$
$$+4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\varphi_{3k}\right|\right)^2\right)^{1/2}+4\sqrt{5}\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^2\left|F_k(u,a;\tau)\right|\right)^2d\tau\right)^{1/2},$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k'''(t)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq 4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\varphi_{0k}\right|\right)^2\right)^{1/2}+$$
$$+4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k\left|\varphi_{1k}\right|\right)^2\right)^{1/2}+4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\varphi_{2k}\right|\right)^2\right)^{1/2}+ \tag{22}$$
$$+4\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k\left|\varphi_{3k}\right|\right)^2\right)^{1/2}+4\sqrt{5}\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\left|F_k(u,a;\tau)\right|\right)^2d\tau\right)^{1/2},$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k^{(4)}(t)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq 4\sqrt{6}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{0k}\right|\right)^2\right)^{1/2}+$$
$$+4\sqrt{6}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\varphi_{1k}\right|\right)^2\right)^{1/2}+4\sqrt{6}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{2k}\right|\right)^2\right)^{1/2}+$$
$$+4\sqrt{6}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\varphi_{3k}\right|\right)^2\right)^{1/2}+4\sqrt{6}\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^2\left|F_k(u,a;\tau)\right|\right)^2d\tau\right)^{1/2}+ \tag{23}$$
$$+\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\left\|\left|F_k(u,a;\tau)\right|\right\|_{C[0,T]}\right)^2\right)^{1/2},$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|\nu_k(t)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq \sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{0k}\right|\right)^2\right)^{1/2}+$$
$$+\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\varphi_{1k}\right|\right)^2\right)^{1/2}+\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{2k}\right|\right)^2\right)^{1/2}+ \tag{24}$$
$$+\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\varphi_{3k}\right|\right)^2\right)^{1/2}+\sqrt{5}\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^2\left|F_k(u,a;\tau)\right|\right)^2d\tau\right)^{1/2}.$$

Assume that the data of the problem (1)-(3), (5) satisfy the following conditions:

1. $\varphi_i(x)\in C^2[0,1]$, $\varphi_i'''(x)\in L_2(0,1)$ and $\varphi_i(0)=\varphi_i'(1)=\varphi_i''(0)=0$ $(i=0,1,2)$;
2. $\varphi_3(x)\in C^1[0,1]$, $\varphi_3''(x)\in L_2(0,1)$, $\varphi_3(0)=\varphi_3'(1)=0$;
3. $f(x,t)\in C_{x,t}^{1,0}(D_T)$, $f_{xx}(x,t)\in L_2(D_T)$, $f(0,t)=f_x(1,t)=0$ $(0\leq t\leq T)$;
4. $h(t)\in C^4[0,T]$, $h(t)\neq 0$ for $t\in[0,T]$.

Then, from (19)-(24) we have:

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq 4\sqrt{5}\left\|\varphi_0'''(x)\right\|_{L_2(0,1)}+4\sqrt{5}\left\|\varphi_1'''(x)\right\|_{L_2(0,1)}+$$
$$+4\sqrt{5}\left\|\varphi_2'(x)\right\|_{L_2(0,1)}+4\sqrt{5}\left\|\varphi_3^1(x)\right\|_{L_2(0,1)}+$$
$$+4\sqrt{5}\sqrt{T}\left\|a(t)u_x(x,t)+f_x(x,t)\right\|_{L_2(0,1)}, \tag{25}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq 4\sqrt{5}\left\|\varphi_0'''(x)\right\|_{L_2(0,1)}+4\sqrt{5}\left\|\varphi_1'''(x)\right\|_{L_2(0,1)}+$$
$$+4\sqrt{5}\left\|\varphi_2''(x)\right\|_{L_2(0,1)}+4\sqrt{5}\left\|\varphi_3^1(x)\right\|_{L_2(0,1)}+$$
$$+4\sqrt{5}\sqrt{T}\left\|a(t)u_x(x,t)+f_x(x,t)\right\|_{L_2(0,1)}, \tag{26}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k''(t)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq 4\sqrt{5}\left\|\varphi_0'''(x)\right\|_{L_2(0,1)}+4\sqrt{5}\left\|\varphi_1'''(x)\right\|_{L_2(0,1)}+$$
$$+4\sqrt{5}\left\|\varphi_2'''(x)\right\|_{L_2(0,1)}+4\sqrt{5}\left\|\varphi_3''(x)\right\|_{L_2(0,1)}+$$
$$+4\sqrt{5}\sqrt{T}\left\|a(t)u_{xx}(x,t)+f_{xx}(x,t)\right\|_{L_2(D_t)}, \tag{27}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k'''(t)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq 4\sqrt{5}\left\|\varphi_0''(x)\right\|_{L_2(0,1)}+4\sqrt{5}\left\|\varphi_1'(x)\right\|_{L_2(0,1)}+$$
$$+4\sqrt{5}\left\|\varphi_2''(x)\right\|_{L_2(0,1)}+4\sqrt{5}\left\|\varphi_3^1(x)\right\|_{L_2(0,1)}+ \tag{28}$$
$$+4\sqrt{5}\sqrt{T}\left\|a(t)u_x(x,t)+f_x(x,t)\right\|_{L_2(D_t)},$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k^{(4)}(t)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq 4\sqrt{6}\left\|\varphi_0'''(x)\right\|_{L_2(0,1)}+4\sqrt{6}\left\|\varphi_1''(x)\right\|_{L_2(0,1)}+$$
$$+4\sqrt{6}\left\|\varphi_2'''(x)\right\|_{L_2(0,1)}+4\sqrt{6}\left\|\varphi''(x)\right\|_{L_2(0,1)}+$$
$$+4\sqrt{6}\sqrt{T}\left\|a(t)u_{xx}(x,t)+f_{xx}(x,t)\right\|_{L_2(D_t)}+$$
$$+4\sqrt{6}\sqrt{T}\left\|\left\|a(t)u_x(x,t)+f_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(D_t)}, \tag{29}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|\nu_k(t)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq\sqrt{5}\left\|\varphi_0'''(x)\right\|_{L_2(0,1)}+\sqrt{5}\left\|\varphi_1''(x)\right\|_{L_2(0,1)}+$$

$$+\sqrt{5}\left\|\varphi_2'''(x)\right\|_{L_2(0,1)}+\sqrt{5}\left\|\varphi''(x)\right\|_{L_2(0,1)}+ \tag{30}$$

$$+\sqrt{5}\sqrt{T}\left\|a(t)u_{xx}(x,t)+f_{xx}(x,t)\right\|_{L_2(D_t)},$$

Further, from (25) we find:

$$\|u(x,t)\|_{B^3_{2,T}}\leq A_1(T)+B_1(T)\|a(t)\|_{C(0,T)}\|u(x,t)\|_{B^3_{2,T}}, \tag{31}$$

where

$$A_1(T)=4\sqrt{5}\left\|\varphi_0''(x)\right\|_{L_2(0,1)}+4\sqrt{5}\left\|\varphi_1'''(x)\right\|_{L_2(0,1)}+$$

$$+4\sqrt{5}\left\|\varphi_2'(x)\right\|_{L_2(0,1)}+4\sqrt{5}\left\|\varphi_3^1(x)\right\|_{L_2(0,1)}+$$

$$+4\sqrt{10T}\left\|f_x(x,t)\right\|_{L_2(0,1)},\quad B_1(T)=4\sqrt{10T}.$$

Now, from (18) allowing for (30) we have:

$$\|a(t)\|_{C(0,T)}\leq A_2(T)+B_2(T)\|a(t)\|_{C(0,T)}\|u(x,t)\|_{B^3_{2,T}}, \tag{32}$$

where

$$A_2(T)=\left\|h^{-1}(t)\right\|_{c[0,T]}\left\{\left\|h^{(4)}(t)\right\|_{c[0,T]}+\left\|h''(t)\right\|_{c[0,T]}+\left\|f(1,t)\right\|_{c[0,T]}+\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{1/2}\left[\sqrt{5}\left\|\varphi_0'''(x)\right\|_{L_2(0,1)}+\sqrt{5}\left\|\varphi_1'''(x)\right\|_{L_2(0,1)}+\sqrt{5}\left\|\varphi_2'''(x)\right\|_{L_2(0,1)}+\right.$$

$$\left.\left.+\sqrt{5}\left\|\varphi_3'''(x)\right\|_{L_2(0,1)}+\sqrt{10T}\left\|f_{xx}(x,t)\right\|_{L_2[D_T]}\right]\right\},\quad B_2(T)=\left\|h^{-1}(t)\right\|_{c[0,T]}\cdot\sqrt{10T}.$$

From inequalities (31) and (32) we deduce:

$$\|u(x,t)\|_{B^3_{2,T}}+\|a(t)\|_{C(0,T)}\leq A(T)+B(T)\|a(t)\|_{C(0,T)}\|u(x,t)\|_{B^3_{2,T}}, \tag{33}$$

where

$$A(T)=A_1(T)+A_2(T),\qquad B(T)=B_1(T)+B_2(T).$$

We prove the following theorem.

Theorem 1. *Let the conditions 1-4 be fulfilled and*

$$B(T)(A(T)+2)^2<1. \tag{34}$$

Then the problem (1)-(3), (5) has a unique solution in the ball

$$K=K_R\left(\|z\|_{E^3_T}\leq R=A(T)+2\right)$$

from E^3_T.

Proof. In the space E_T^3 consider the equation

$$z = \Phi z, \tag{35}$$

where $z = \{u, a\}$, the components $\Phi_i(u, a)$ $(i = 1, 2)$ of the operator $\Phi(u, a)$ are determined by the right hand sides of equations (12), (18), respectively.

Let's consider an operator $\Phi(u, a)$ in a ball $K = K_R$ from E_T^3. Similar to (33) we get that for any $z = \{u, a\}$, $z_1 = \{u_1, a_1\}$, $z_2 = \{u_2, a_2\} \in K_R$ the following estimations are true:

$$\|\Phi z\|_{E_T^3} \leq A(T) + B(T) \|a(t)\|_{C(0,T)} \|u(x,t)\|_{B_{2,T}^3}, \tag{36}$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^3} \leq 2B(T)R \left(\|a_1(t) - a_2(t)\|_{C(0,T)} + \|u(x,t) - u_2(x,t)\|_{B_{2,T}^3} \right). \tag{37}$$

Then allowing for (35), it follows from estimations (36) and (37) that the operator $K = K_R$ acts in the ball K_R and it is contractive. Therefore in the ball K_R the operator Φ has a unique fixed point $\{u, a\}$ and this point is the solution of equation (34).

The function $u(x,t)$ as an element of the space $B_{2,T}^3$ has continuous derivatives $u_x(x, t)$, $u_{xx}(x, t)$.

It follows from inequalities (26)-(29) that $u_t(x, t)$, $u_{tx}(x, t)$, $u_{txx}(x, t)$, $u_{tt}(x, t)$, $u_{ttx}(x, t)$, $u_{ttxx}(x, t)$, $u_{ttt}(x, t)$, $u_{tttt}(x, t)$ are continuous in D_T. Further, it is easy to verify that equation (1) and conditions (2, (3), (5) are satisfied in the ordinary sense. So, $\{u(x, t),\ a(t)\} \in E_T^3$ is a solution of the problem (1)-(3), (5). The theorem is proved.

Thus, by lemma 1, the following theorem is true.

Theorem 2. *Let all the conditions of theorem 1 be fulfilled, and*

$$\varphi_0(1) = h(0),\ \varphi_1(1) = h'(0),\ \varphi_2(1) = h''(0),\ \varphi_3(1) = h'''(0).$$

Then the problem (1)-(4) has a unique classic solution in the ball

$$K = K_R \left(\|z\|_{E_T^3} \leq R = A(T) + 2 \right)$$

from E_T^3.

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