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ON SOLVABILITY OF A BOUNDARY VALUE PROBLEM FOR A CLASS OF THIRD ORDER OPERATOR-DIFFERENTIAL EQUATIONS WITH DISCONNECTED COEFFICIENTS

Abstract

In the paper the sufficient conditions providing the solvability of boundary-value problem for one class operator-differential equations of third order with disconnected coefficients were obtained, and the principal part of the equations possesses a normal operator. These conditions are expressed only by properties of coefficients of operator-differential equations.

In the separable Hilbert space H we consider the following boundary value problem

$$\frac{d^3u}{dt^3}+\rho(t)A^3u+\sum_{j=0}^{3}A_{3-j}u^{(j)}=f(t),\quad t\in R_+, \tag{1}$$

$$u(0)=0, \tag{2}$$

where $u(t)$ and $f(t)$ are the vector-functions deteremined in $R_+=(0,\infty)$ with values in H and the operator coefficients satisfy the following conditions:

1) A is a normal invertible operator with a spectrum contained in an angular sector

$$S_\varepsilon=\{\lambda\,|\arg\lambda|\leq\varepsilon\},\quad 0\leq\varepsilon\leq\frac{\pi}{6};$$

2) the operators $B_j=A_jA^{-j}$ $(j=0,1,2,3)$ are bounded in H;

3) $\rho(t)$ is a scalar function such that

$$\rho(t)=\begin{cases}\alpha^3, & t\in(0,1),\\ \beta^3, & t\in(1,\infty).\end{cases}$$

As in the book [1] the Hilbert spaces $L_2(R_+;H)$ and $W_2^3(R_+;H)$ are determined as follows:

$$L_2(R_+;H)=\left\{f(t)\,|f(t)\in H\,,\|f\|_{L_2}^2=\int_0^\infty\|f(t)\|^2dt<\infty\right\},$$

$$W_2^3(R_+;H)=\left\{u(t)|\,\frac{d^3u}{dt^3},A^3u\in L_2(R_+;H)\,,\|u\|_{W_2^3}^2=\left\|A^3u\right\|_{L_2}^2+\left\|\frac{d^3u}{dt^3}\right\|_{L_2}^2\right\}.$$

Let's determine the following complete subspace of the space $W_2^3(R_+;H)$:

$$\overset{\circ}{W}{}_2^3(R_+;H)=\left\{u(t)|\,u\in W_2^3(R_+;H)\,,u(0)=0\right\}.$$

Here and in sequel the derivatives are understood in the sense of distributions theory [1].

By fulfilling condition 1) the operator A is represented in the form $A = UC$, where U is a unitary and C is a positive definite operator in H, moreover $x \in D(A)$ $\|Ax\| = \|A^*x\| = \|Cx\|$ and $UCx = CUx$.

Further, by H we denote a Hilbert scale of spaces generated by the operator C, i.e. $H_\gamma = D(C^\gamma)$, $\|x\|_\gamma = \|C^\gamma x\|$, $\gamma \geq 0$, $x \in D(C^\gamma)$.

Definition 1. *If for $f(t) \in L_2(R_+; H)$ there exists a vector-function $u(t) \in W_2^3(R_+; H)$ satisfying equation (1) almost everywhere in R_+, we'll call it a regular solution of equation (1).*

Definition 2. *If for any $f(t) \in L_2(R_+; H)$ there exists a regular solution $u(t) \in W_2^3(R_+; H)$ satisfying boundary condition (2) in the sense $\lim\limits_{t\to+\infty} \|u(t)\|_{3/2} = 0$ and it holds the estimation $\|u\|_{W_2^3} \leq const \|f\|$, we'll call problem (1)-(2) regularly solvable.*

In the paper we'll find sufficient conditions on the coefficients of an operator-differential equation by fulfilling of which problem (1), (2) is regularly solvable. Notice that for $\rho(t) = 1$, $\varepsilon = 0$ (A is a positive-definite operator) this problem was studied in [2], and when $\rho(t)$ is a discontinuous function and $\varepsilon = 0$ it was studied in the paper [3].

Write problem (1)-(2) in the form of the equation

$$Pu \equiv P_0 u + P_1 u_0 = f, \tag{3}$$

where $f \in L_2(R_+; H)$, $u \in \overset{\circ}{W}{}_2^3(R_+; H)$, and

$$P_0 u = \frac{d^3u}{dt^3} + \rho(t) A^3 u, \quad P_1 u = \sum_{j=0}^{3} A_{3-j} \frac{d^j u}{dt^j}.$$

At first we research the non-perturbated equation

$$P_0 u = f, \tag{4}$$

where $f \in L_2(R_+; H)$, $u \in \overset{\circ}{W}{}_2^3(R_+; H)$.

It holds

Theorem 1. *Let A satisfy condition 1). Then the operator P_0 maps isomorphically the space $\overset{\circ}{W}{}_2^3(R_+; H)$ onto $L_2(R_+; H)$.*

Proof. We can easily verify that the vector-functions

$$U_\alpha(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(-i\xi^3 E + \alpha^3 A\right)^{-1} \left(\int_0^{\infty} f(s) e^{i(t-\xi)s} ds \right) d\xi, \quad t \in R$$

and

$$U_\beta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(-i\xi^3 E + \beta^3 A\right)^{-1} \left(\int_0^{\infty} f(s) e^{i(t-\xi)s} ds \right) d\xi, \quad t \in R$$

satisfy the equations

$$\frac{d^3u}{dt^3} + \alpha^3 A^3 u = f(t),$$

and

$$\frac{d^3u}{dt^3}+\beta^3A^3u=f(t),$$

respectively, in R almost everywhere. Show that each of these vector-functions belong to the space $W_2^3(R;H)$, where $R=(-\infty,\infty)$, and $W_2^3(R;H)$ is defined similar to the space $W_2^3(R_+;H)$. It follows from the Plancherel theorem that to this end it sufficies to prove that $A^3\hat{u}_\alpha(\xi)$, $\xi^3\hat{u}_\alpha(\xi)$ and $A^3\hat{u}_\beta(\xi)$, $\xi^3\hat{u}_\beta(\xi)$ belong to $L_2(R;H)$. Here $\hat{u}_\alpha(\xi)$, $\hat{u}_\beta(\xi)$ is a Fourier transformation of the vector-functions $u_\alpha(\xi)$, $u_\beta(\xi)$, respectively.

Obviously, if $\hat{f}(\xi)$ is a Fourier transformation of the vector-function $f(t)$, then

$$\left\|A^3\hat{u}_\alpha(\xi)\right\|_{L_2}=\left\|A^3\left(-i\xi^3E+\alpha^3A^3\right)^{-1}\hat{f}_\alpha(\xi)\right\|_{L_2}\leq$$

$$\leq\sup_{\xi\in R}\left\|A^3\left(-i\xi^3E+\alpha^3A^3\right)^{-1}\right\|_{L_2}\cdot\|f\|_{L_2}. \tag{5}$$

From the theory of normal operators it follows that for $\xi\in R$

$$\left\|A^3\left(-i\xi^3E+\alpha^3A^3\right)^{-1}\right\|\leq\sup_{\lambda\in\sigma(A)}\left|\lambda^3\left(-i\xi^3E+\alpha^3\lambda^3\right)^{-1}\right|\leq$$

$$\leq\sup_{\substack{\lambda>0\\|\varphi|<\frac{\pi}{6}}}\left|\lambda^3\left(\xi^6+\alpha^6\lambda^6-2\xi^3\alpha^3\lambda^3\sin3\varphi\right)^{-1/2}\right|\leq$$

$$\leq\sup_{\lambda>0}\left|\lambda^3\left(\xi^6+\alpha^6\lambda^6-2\left|\xi\right|^3\alpha^3\lambda^3\sin3\varepsilon\right)^{-1/2}\right|\leq$$

$$\leq\sup_{\lambda>0}\left(\lambda^3\left(\alpha^6\lambda^6\cos^23\varepsilon\right)^{-1/2}\right)\leq\frac{1}{\alpha^3\cos3\varepsilon}.$$

Thus, from inequality (5) it follows $A^3\hat{u}_\alpha(\xi)\in L_2(R;H)$. It is similarly proved that $\xi^3\hat{u}_\alpha(\xi)\in L_2(R;H)$. Thus, we showed $u_\alpha\in W_2^3(R;H)$. $u_\beta\in W_2^3(R;H)$ is proved in the same way.

We denote by $\psi_1(t)$ and $\psi_2(t)$ a contraction of the vector-functions $u_a(t)$ and $u_\beta(t)$ on $[0,1]$ and $[1,\infty)$, respectively. Then $\psi_1(t)\in W_2^3([0,1];H)$, $\psi_2(t)\in W_2^3([1,\infty);H)$. From the traces theorem [1, p.29] it follows that there are exist the boundary values $\psi_1(0)$, $\psi_1(1)$, $\psi_1(1)$, $\psi_2(1)$, $\psi_2(1)$, moreover $\psi_1(0)$, $\psi_1(1)$, $\psi_2(1)\in H_{3/2}$, $\psi_1(1)$, $\psi_2(1)\in H_{3/2}$.

By means of the vector-functions $\psi_1(t)$ and $\psi_2(t)$ we make up the vector-function

$$u(t)=\begin{cases}\theta_1(t)=\psi_1(t)+e^{\alpha\omega_1At}\varphi_1+e^{\alpha\omega_2A(t-1)}+e^{\alpha\omega_3A(t-1)}\varphi_3, & t\in(0,1),\\ \theta_2(t)=\psi_2(t)+e^{\omega_1\beta At}\varphi_4, & t\in(0,\infty),\end{cases}$$

where $\omega_1=-1$, $\omega_2=\frac{1}{2}\left(1+i\sqrt{3}\right)$, $\omega_3=\frac{1}{2}\left(1-i\sqrt{3}\right)$, and φ_1, φ_2, φ_3, φ_4 are the untill unknown vectors from $H_{3/2}$.

Obviously, $u(t)$ satisfies equation (1) almost everywhere in R_+. In order it satisfy boundary value problem (2) and belong to the space $W_2^3(R_+;H)$ the conditions

$$\theta_1(0)=0,\ \ \theta_1(1)=\theta_2(1),\ \theta_1'(1)=\theta_2'(1),\ \theta_1''(1)=\theta_2''(1),$$

should be satisfied, i.e.

$$\begin{cases} \psi_1(0) + \varphi_1 + e^{-\alpha\omega_2 A}\varphi_2 + e^{-\alpha\omega_3 A}\varphi_3 = 0 \\ \psi_1(1) + e^{\alpha\omega_1 A}\varphi_1 + \varphi_2 + \varphi_3 = \psi_2(1) + \varphi_4 \\ \psi_1'(1) + \alpha\omega_1 A e^{\alpha\omega_1 A}\varphi_1 + \alpha\omega_2 A\varphi_2 + \alpha\omega_3 A\varphi_3 = \psi_2'(1) + \omega_1\beta A\varphi_4 \\ \psi_1''(1) + \alpha^2\omega_1^2 e^{\alpha\omega_1 A}\varphi_1 + \alpha^2\omega_2^2 A^2\varphi_2 + \alpha^2\omega_3^2 A^2\varphi_3 = \psi_2''(1) + \omega_1^2\beta^2 A^2\varphi_4 \end{cases}$$

Hence we get the equation

$$\Delta_0(A)\,\tilde{\varphi} = \tilde{\xi},$$

where

$$\Delta_0(A) = \begin{pmatrix} E & e^{-\alpha\omega_2 A} & e^{-\alpha\omega_3 A} & 0 \\ e^{\alpha\omega_1 A} & E & E & -E \\ \alpha\omega_1 e^{\alpha\omega_1 A} & \alpha\omega_2 E & \alpha\omega_3 E & -\omega_1\beta E \\ \alpha^2\omega_1^2 e^{\alpha\omega_1 A} & \alpha^2\omega_2^2 E & \alpha^2\omega_3^2 E & -\omega_1^2\beta^2 E \end{pmatrix},$$

$$\tilde{\xi} = \begin{pmatrix} -\psi_1(0) \\ \psi_2(1) - \psi_1(1) \\ A^{-1}\left(\psi_2'(1) - \psi_1'(1)\right) \\ A^{-2}\left(\psi_2''(1) - \psi_1''(1)\right) \end{pmatrix}, \quad \tilde{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}.$$

Obviously, $\Delta_0(A)$ is invertible in the space $H^4 = H \times H \times H \times H$ and each from the components of the vector $\tilde{\xi}$ belongs to the space $H_{3/2}$. Therefore, the vectors φ_1, φ_2, φ_3, φ_4 are uniquely determined and belong to the space $H_{3/2}$. Consequently, $u(t) \in \overset{\circ}{W}{}_2^3(R_+; H)$. Obviously, equation $P_0 u = 0$ has only zero solution from the space $\overset{\circ}{W}{}_2^3(R_+; H)$. On the other hand

$$\|P_0 u\|_{L_2(R_+;H)} \le const\, \|u\|_{W_2^3(R_+;H)}.$$

Therefore, affirmation of the theorem follows from Banach theorem on the inverse operator.

Now let's solve equation (3).

It is valid

Lemma 1. *Let condition 1) be fulfilled. Then for any $u \in \overset{\circ}{W}{}_2^3(R_+; H)$ it holds the inequality*

$$\left\|\rho^{-1/2}\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)} + \left\|\rho^{1/2}A^3u\right\|^2_{L_2(R_+;H)} + \cos 3\varepsilon\, \|u'(0)\|^2_{3/2} -$$

$$-2\sin 3\varepsilon \left\|\rho^{1/2}A^3u\right\|_{L_2(R_+;H)} \left\|\rho^{-1/2}\frac{d^3u}{dt^3}\right\|_{L_2(R_+;H)} \le \left\|\rho^{1/2}A^3u\right\|^2_{L_2(R_+;H)}. \quad (6)$$

Proof. Multiplying the equation $P_0 u = f$ by $\rho^{-1/2}(t)$ we have:

$$\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(R_+;H)} = \left\|\rho^{-1/2}\frac{d^3u}{dt^3} + \rho^{1/2}A^3u\right\|^2_{L_2(R_+;H)} = \left\|\rho^{-1/2}\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)} +$$

$$+ \left\|\rho^{1/2}A^3u\right\|^2_{L_2(R_+;H)} + 2\,\mathrm{Re}\left(\rho^{-1/2}\frac{d^3u}{dt^3}, \rho^{1/2}A^3u\right)_{L_2(R_+;H)} =$$

$$= \left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|^2_{L_2(R_+;H)} + \left\| \rho^{1/2} A^3 u \right\|^2_{L_2(R_+;H)} + 2\,\mathrm{Re} \left(\frac{d^3 u}{dt^3}, A^3 u \right)_{L_2(R_+;H)}. \qquad (7)$$

Since for $u \in \overset{\circ}{W}{}_2^3 (R_+; H)$

$$\left(\frac{d^3 u}{dt^3}, A^3 u \right)_{L_2(R_+;H)} = \int_0^\infty \left(\frac{d^3 u}{dt^3}, A^3 u \right) dt = \left(A^{*3/2} u'(0), A^{3/2} u'(0) \right) -$$

$$- \int_0^\infty \left(A^{*3} u, \frac{d^3 u}{dt^3} \right) dt = \left(A^{*3/2} u'(0), A^{*3/2} u'(0) \right) - \left(A^{*3} u, \frac{d^3 u}{dt^3} \right)_{L_2(R_+;H)},$$

then allowing for

$$2\,\mathrm{Re} \left(\frac{d^3 u}{dt^3}, A^3 u \right)_{L_2(R_+;H)} = \left(\frac{d^3 u}{dt^3}, A^3 u \right)_{L_2(R_+;H)} + \left(A^3 u, \frac{d^3 u}{dt^3} \right)_{L_2(R_+;H)} =$$

$$= \left(A^3 u, \frac{d^3 u}{dt^3} \right)_{L_2(R_+;H)} + \left(A^{*3/2} u(0), A^{3/2} u(0) \right) - \left(A^{*3} u, \frac{d^3 u}{dt^3} \right)_{L_2(R_+;H)} \geq$$

$$\geq \cos 3\varepsilon \left\| C^{3/2} u(0) \right\|^2 - \left| \left(\left(A^3 - A^{*3} u \right), \frac{d^3 u}{dt^3} \right)_{L_2(R_+;H)} \right| \geq$$

$$\geq \cos 3\varepsilon \left\| u(0) \right\|_{3/2} - 2 \sin 3\varepsilon \left\| \rho^{1/2} A^3 u \right\|_{L_2(R_+;H)} \cdot \left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|_{L_2(R_+;H)}.$$

from (7) we get the affirmation of the lemma.

This lemma implies

Corollary 1. *For any $u \in \overset{\circ}{W}{}_2^3 (R_+; H)$ the inequalities*

$$\left\| \rho^{1/2} A^3 u \right\|_{L_2(R_+;H)} \leq \frac{1}{\cos 3\varepsilon} \left\| \rho^{-1/2} P_0 u \right\|_{L_2(R_+;H)}, \qquad (8)$$

$$\left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|_{L_2(R_+;H)} \leq \frac{1}{\cos 3\varepsilon} \left\| \rho^{-1/2} P_0 u \right\|_{L_2(R_+;H)}, \qquad (9)$$

$$\left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|^2_{L_2(R_+;H)} + \left\| \rho^{1/2} A^3 u \right\|^2_{L_2(R_+;H)} \leq$$

$$\leq (1 - \sin 3\varepsilon)^{-1} \left\| \rho^{-1/2} P_0 u \right\|^2_{L_2(R_+;H)}, \qquad (10)$$

follow.

Proof. We apply the Cauchy inequality and rrom inequality (6) we get:

$$\left\| \rho^{-1/2} P_0 u \right\|^2_{L_2(R_+;H)} \geq \left\| \rho^{1/2} A^3 u \right\|^2_{L_2(R_+;H)} + \left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|^2_{L_2(R_+;H)} +$$

$$+ \cos 3\varepsilon \left\| u'(0) \right\|^2_{3/2} - \left(\sin 3\varepsilon \left\| \rho^{1/2} A^3 u \right\|^2_{L_2(R_+;H)} + \left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|^2_{L_2(R_+;H)} \right) =$$

$$= \cos^2 3\varepsilon \left\| \rho^{1/2} A^3 u \right\|^2_{L_2(R_+;H)},$$

i.e.

$$\left\| \rho^{1/2} A^3 u \right\|_{L_2(R_+;H)} \leq \frac{1}{\cos 3\varepsilon} \left\| \rho^{-1/2} P_0 u \right\|_{L_2(R_+;H)}.$$

Validity of inequality (8) is proved. Inequality (9) is proved in a similar way. Further,

$$\left\| \rho^{-1/2} P_0 u \right\|^2_{L_2(R_+;H)} \geq \left\| \rho^{1/2} A^3 u \right\|^2_{L_2(R_+;H)} + \left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|^2_{L_2(R_+;H)} -$$

$$- \sin 3\varepsilon \left(\left\| \rho^{1/2} A^3 u \right\|^2_{L_2(R_+;H)} + \left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|^2_{L_2(R_+;H)} \right) =$$

$$= (1 - \sin 3\varepsilon) \left(\left\| \rho^{1/2} A^3 u \right\|^2_{L_2(R_+;H)} + \left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|^2_{L_2(R_+;H)} \right)$$

follows from inequality (6).

Hence, it follows validity of inequality (10). Now, let's estimate the norms of intermediate derivatives.

Lemma 2. *Let condition (1) be fulfilled. Then for any* $u \in \overset{\circ}{W}{}_2^3 (R_+; H)$ *the inequalities*

$$\left\| A^{3-j} u^{(j)} \right\|_{L_2(R_+;H)} \leq C_j (\alpha; \beta; \varepsilon) \| P_0 u \|_{L_2(R_+;H)}, \quad j = \overline{0,3}, \tag{11}$$

where

$$C_0 (\alpha; \beta; \varepsilon) = \frac{1}{\cos 3\varepsilon} \cdot \frac{1}{\min \left(\alpha^3; \beta^3 \right)},$$

$$C_1 (\alpha; \beta; \varepsilon) = \frac{2^{2/3}}{3^{1/2}} (1 - \sin 3\varepsilon)^{-1/2} \frac{\max \left(\alpha^{1/2}, \beta^{1/2} \right)}{\min \left(\alpha^{5/2}; \beta^{5/2} \right)},$$

$$C_2 (\alpha; \beta; \varepsilon) = \frac{2}{3^{1/2}} (1 - \sin 3\varepsilon)^{1/2} \frac{\max (\alpha, \beta)}{\min \left(\alpha^2; \beta^2 \right)},$$

$$C_3 (\alpha; \beta; \varepsilon) = \frac{1}{\cos 3\varepsilon} \cdot \frac{\max \left(\alpha^{3/2}, \beta^{3/2} \right)}{\min \left(\alpha^{3/2}; \beta^{3/2} \right)},$$

hold.

Proof. Inequality (11) for $j = 0$ and $j = 3$ directly follows from inequalities (8) and (9), respectively. Really, for, example, for $j = 0$ we have:

$$\left\| A^3 u \right\|_{L_2(R_+;H)} = \left\| \rho^{-1/2} \rho^{1/2} A^3 u \right\|_{L_2(R_+;H)} \leq \max_t \rho^{-1/2} (t) \left\| \rho^{1/2} A^3 u \right\|_{L_2(R_+;H)} \leq$$

$$\leq \max_t \rho^{-1/2} (t) \cdot \frac{1}{\cos 3\varepsilon} \left\| \rho^{-1/2} P_0 u \right\|_{L_2(R_+;H)} \leq$$

$$\leq \frac{1}{\cos 3\varepsilon}\max_t \rho^{-1}\left\|P_0 u\right\|_{L_2(R_+;H)} = \frac{1}{\cos 3\varepsilon}\cdot\frac{1}{\min\left(\alpha^3;\beta^3\right)}\left\|P_0 u\right\|_{L_2(R_+;H)}.$$

Prove inequality (11) for $j=1$. Obviously, for $u \in \overset{\circ}{W}{}_2^3\left(R_+;H\right)$

$$\left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)} = \left\|C^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)} = \int_0^\infty \left(C^2\frac{du}{dt}, C^2\frac{du}{dt}\right)dt =$$

$$= \int_0^\infty \left(C^3 u, C^2\frac{d^2u}{dt^2}\right)dt \leq \left\|C^2 u\right\|_{L_2(R_+;H)}\cdot\left\|C\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)} \leq$$

$$\leq \max_t \rho^{-1/2}(t)\left\|\rho^{1/2}A^2 u\right\|_{L_2(R_+;H)}\cdot\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}. \qquad (12)$$

On the other hand we have [4]

$$\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)} = \left\|C\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)} \leq 2\left\|C^2\frac{du}{dt}\right\|_{L_2(R_+;H)}\cdot\left\|\frac{d^3u}{dt^3}\right\|_{L_2(R_+;H)} =$$

$$= 2\left\|A^2\frac{du}{dt}\right\|_{L_2(R_+;H)}\cdot\left\|\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)} \leq$$

$$\leq 2\left\|A^2\frac{du}{dt}\right\|_{L_2(R_+;H)}\cdot\max_t \rho^{1/2}(t)\left\|\rho^{-1/2}\frac{d^3u}{dt^3}\right\|_{L_2(R_+;H)}. \qquad (13)$$

Allowing for inequality (13) in (12) we get:

$$\left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)} \leq \max_t \rho^{-1/2}(t)\left\|\rho^{1/2}A^2 u\right\|_{L_2(R_+;H)} 2^{1/2}\max_t \rho^{1/4}\times$$

$$\times\left\|\rho^{-1/2}\frac{d^3u}{dt^3}\right\|_{L_2(R_+;H)}\cdot\left\|A^2\frac{du}{dt}\right\|^{1/2}_{L_2(R_+;H)}$$

or

$$\left\|A^2\frac{du}{dt}\right\|^{3/2}_{L_2(R_+;H)} \leq 2^{1/2}\frac{\max\left(\alpha^{3/4},\beta^{3/4}\right)}{\min\left(\alpha^{3/2},\beta^{3/2}\right)}\left\|\rho^{1/2}A^3 u\right\|_{L_2(R_+;H)}\left\|\rho^{-1/2}\frac{d^3u}{dt^3}\right\|^{1/2}_{L_2(R_+;H)}.$$

Consequently,

$$\left\|A^2\frac{du}{dt}\right\|_{L_2(R_+;H)} \leq 2^{1/3}\frac{\max\left(\alpha^{1/2},\beta^{1/2}\right)}{\min\left(\alpha,\beta\right)}\left\|\rho^{1/2}A^3 u\right\|^{3/2}_{L_2(R_+;H)}\times$$

$$\times\left\|\rho^{-1/2}\frac{d^3u}{dt^3}\right\|^{1/3}_{L_2(R_+;H)}.$$

Hence, for any $\delta > 0$ we have:

$$\left\| A^2 \frac{du}{dt} \right\|^2_{L_2(R_+;H)} \leq 2^{2/3} \frac{\max(\alpha,\beta)}{\min\left(\alpha^2,\beta^2\right)} \left(\delta \left\| \rho^{1/2} A^3 u \right\|^2_{L_2(R_+;H)} \right)^{2/3} \times$$

$$\times \left(\frac{1}{\delta^2} \left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|^2_{L_2(R_+;H)} \right)^{1/3}.$$

Applying the Young inequality we get:

$$\left\| A^2 \frac{du}{dt} \right\|^2_{L_2(R_+;H)} \leq$$

$$\leq 2^{2/3} \frac{\max(\alpha,\beta)}{\min\left(\alpha^2,\beta^2\right)} \left(\frac{2}{3} \delta \left\| \rho^{1/2} A^3 u \right\|^2_{L_2(R_+;H)} + \frac{1}{3\delta^2} \left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|^2_{L_2(R_+;H)} \right).$$

Choosing $\delta = 2^{-1/3}$ $\left(\frac{2}{3}\delta = \frac{1}{3\delta^2} \right)$ we have:

$$\left\| A^2 \frac{du}{dt} \right\|^2_{L_2(R_+;H)} \leq$$

$$\leq \frac{2^{4/3}}{3} \frac{\max(\alpha,\beta)}{\min\left(\alpha^2,\beta^2\right)} \left(\left\| \rho^{1/2} A^3 u \right\|^2_{L_2(R_+;H)} + \left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|^2_{L_2(R_+;H)} \right).$$

Further, applying inequality (10) we get:

$$\left\| A^2 \frac{du}{dt} \right\|_{L_2(R_+;H)} \leq \frac{2^{2/3}}{3^{1/2}} \frac{\max\left(\alpha^{1/2},\beta^{1/2}\right)}{\min(\alpha,\beta)} (1-\sin 3\varepsilon) \left\| p^{-1/2} P_0 u \right\|_{L_2(R_+;H)} \leq$$

$$\leq \frac{2^{2/3}}{3^{1/2}} \frac{\max\left(\alpha^{1/2},\beta^{1/2}\right)}{\min(\alpha,\beta)} (1-\sin 3\varepsilon) \max_t p^{-1/2}(t) \left\| P_0 u \right\|_{L_2(R_+;H)} \leq$$

$$\leq \frac{2^{2/3}}{3^{1/2}} \frac{\max\left(\alpha^{1/2},\beta^{1/2}\right)}{\min\left(\alpha^{5/2},\beta^{5/2}\right)} \left\| P_0 u \right\|_{L_2(R_+;H)}.$$

Now, let's prove inequality (11) for $j = 2$. Aloowing for inequality (12) in (13) we have:

$$\left\| A \frac{d^2 u}{dt^2} \right\|^2_{L_2(R_+;H)} \leq 2 \max_t p^{1/2}(t) \left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|^2_{L_2(R_+;H)} \times$$

$$\times \max_t p^{-1/4}(t) \left\| \rho^{1/2} A^3 u \right\|^{1/2}_{L_2(R_+;H)} \times \left\| A \frac{d^2 u}{dt^2} \right\|^{1/2}_{L_2(R_+;H)} =$$

$$= 2 \frac{\max\left(\alpha^{3/2},\beta^{3/2}\right)}{\min\left(\alpha^{3/4},\beta^{3/4}\right)} \left\| \rho^{-1/2} \frac{d^3 u}{dt^3} \right\|_{L_2(R_+;H)} \left\| \rho^{1/2} A^3 u \right\|^{1/2}_{L_2(R_+;H)} \left\| A \frac{d^2 u}{dt^2} \right\|^{1/2}_{L_2(R_+;H)}.$$

Consequently,

$$\left\|A\frac{d^2u}{dt^2}\right\|^{3/2}\leq 2\frac{\max\left(\alpha^{3/2},\beta^{3/2}\right)}{\min\left(\alpha^{3/4},\beta^{3/4}\right)}\left\|\rho^{1/2}A^3u\right\|^{1/2}_{L_2(R_+;H)}\left\|\rho^{-1/2}\frac{d^3u}{dt^3}\right\|_{L_2(R_+;H)},$$

i.e.

$$\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}\leq 2^{2/3}\frac{\max\left(\alpha,\beta\right)}{\min\left(\alpha^{1/2},\beta^{1/2}\right)}\left\|\rho^{1/2}A^3u\right\|^{1/3}_{L_2(R_+;H)}\left\|\rho^{-1/2}\frac{d^3u}{dt^3}\right\|^{2/3}_{L_2(R_+;H)}.$$

Hence for any $\delta>0$ we have:

$$\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)}\leq 2^{4/3}\frac{\max\left(\alpha^2,\beta^2\right)}{\min\left(\alpha,\beta\right)}\left(\delta^2\left\|\rho^{1/2}A^3u\right\|^2_{L_2(R_+;H)}\right)^{1/3}\times$$

$$\times\left(\frac{1}{\delta}\left\|\rho^{-1/2}\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)}\right)^{2/3}.$$

Here, applying Young inequality we have:

$$\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)}\leq$$

$$\leq 2^{4/3}\frac{\max\left(\alpha^2,\beta^2\right)}{\min\left(\alpha,\beta\right)}\left(\frac{1}{3}\delta^2\left\|\rho^{1/2}A^3u\right\|^{21/3}_{L_2(R_+;H)}+\frac{2}{3\delta}\left\|\rho^{-1/2}\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)}\right).$$

Here, assuming $\delta=2^{1/3}$ $\left(\frac{1}{3}\delta^2=\frac{2}{3\delta}\right)$ we get:

$$\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)}\leq\frac{4}{3}\frac{\max\left(\alpha^2,\beta^2\right)}{\min\left(\alpha,\beta\right)}\left(\left\|\rho^{1/2}A^3u\right\|^2_{L_2(R_+;H)}+\left\|\rho^{-1/2}\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)}\right).$$

Applying inequality (10) we have:

$$\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}\leq\frac{2}{3^{1/2}}\frac{\max\left(\alpha,\beta\right)}{\min\left(\alpha^{1/2},\beta^{1/2}\right)}\left(1-\sin 3\varepsilon\right)^{-1/2}\left\|p^{-1/2}P_0u\right\|_{L_2(R_+;H)}\leq$$

$$\leq\frac{2}{3^{1/2}}\frac{\max\left(\alpha,\beta\right)}{\min\left(\alpha^{1/2},\beta^{1/2}\right)}\left(1-\sin 3\varepsilon\right)^{-1/2}\max_t p^{-1/2}\left(t\right)\left\|P_0u\right\|_{L_2(R_+;H)}=$$

$$=\frac{2}{3^{1/2}}\left(1-\sin 3\varepsilon\right)^{-1/2}\frac{\max\left(\alpha,\beta\right)}{\min\left(\alpha^2,\beta^2\right)}\left\|P_0u\right\|_{L_2(R_+;H)}.$$

The lemma is proved.

Now, we'll prove the main theorem on regular solvability of problem (1)-(2).

Theorem 2. *Let conditions (1)-(2) be fulfilled and the inequality*

$$K\left(\alpha;\beta;\varepsilon\right)=\sum_{j=0}^{3}C_j\left(\alpha;\beta;\varepsilon\right)\left\|B_{3-j}\right\|<1$$

hold.

Then problem (1)-(2) is regularly solvable. Here, the numbers $C_j(\alpha;\beta;\varepsilon)$ $\left(j=\overline{0,3}\right)$ *are defined in lemma 2.*

Proof. After substitution of $P_0u=\upsilon$ we can write equation (3) in the form $\upsilon+P_1P_0^{-1}\upsilon=f$, where $\upsilon\in L_2(R_+;H)$, $f\in L_2(R_+;H)$.

On the other hand for any $\upsilon\in L_2(R_+;H)$ we have:

$$\left\|P_1P_0^{-1}\upsilon\right\|_{L_2(R_+;H)}=\|P_1u\|_{L_2(R_+;H)}\leq\sum_{j=0}^{3}\left\|A_{3-j}u^{(j)}\right\|_{L_2(R_+;H)}\leq$$

$$\leq\sum_{j=0}^{3}\|B_{3-j}\|\left\|A^{3-j}u^{(j)}\right\|_{L_2(R_+;H)}.$$

By lemma 2

$$\left\|A^{3-j}u^{(j)}\right\|_{L_2(R_+;H)}\leq C_j(\alpha;\beta;\varepsilon)\|P_0u\|_{L_2(R_+;H)}.$$

Then

$$\left\|P_1P_0^{-1}\upsilon\right\|_{L_2(R_+;H)}\leq\sum_{j=0}^{3}C_j(\alpha;\beta;\varepsilon)\|B_{3-j}\|\,\|\upsilon\|_{L_2(R_+;H)}=K(\alpha;\beta;\varepsilon)\|\upsilon\|_{L_2(R_+;H)}.$$

Since $K(\alpha;\beta;\varepsilon)<1$, the operator $P_1P_0^{-1}+E$ is invertible in $L_2(R_+;H)$. Therefore $u=P^{-1}\left(E+P_1P_0^{-1}\right)^{-1}f$ and $\|u\|_{W_2^3(R_+;H)}\leq const\,\|f\|_{L_2(R_+;H)}$.

The theorem is proved.

References

[1]. Lions J.L., Majenes E. *Inhomogeneous boundary value problems and their applications.* Moscow, Mir, 1971, 371 pp. (Russian)

[2]. Mirzoyev S.S. *Solvability problems of boundary value problems in a Hilbert space and related spectral problems.*Thesis for Doctor's Degree. Baku, 1993, 208 pp. (Russian)

[3]. Aliyev A.R. *On correct solvability of boundary value problems for operator differential equation with disconnected coefficients.* Thesis for PhD, BSU, Baku, 1998, 13 pp. (Russian)

[4]. Mirzoyev S.S. *On correct solvability of boundary value problem for odd order operator-differential equations on a semi-axis.* "Spectral analysis of operators", Baku, Az.State Univer., 1982, pp.53-59. (Russian)

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