Aydin A. GASYMOV

ON SOLVABILITY OF A CLASS OF COMPLICATED CHARACTERISTIC OPERATOR-DIFFERENTIAL EQUATIONS OF FOURTH ORDER

Abstract

In the paper we find sufficient conditions of regular solvability of fourth order operator-differential equations considered on the real axis. These conditions are expressed only by the operator coefficients of the equation. Therewith we estimate the norms of the intermediate derivatives operators by the principal part of the equation. Notice that the principal part of the investigated equation possesses complicated characteristic.

In the separable Hilbert space H we consider the following equation:

$$Pu(t) \equiv P_0 u(t) + P_1 u(t) = f(t), \quad t \in R = (-\infty; +\infty), \tag{1}$$

where

$$P_0 = \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3, \quad P_1 = \sum_{s=1}^{3} A_s \frac{d^{4-s}}{dt^{4-s}},$$

moreover A is a self-adjoint positive-definite operator, A_s, $s = \overline{1,3}$, are linear, generally speaking, unbounded operators.

Here and in sequel, all derivatives are understood in the sense of distributions theory.

Further, we assume $f(t) \in L_2(R; H)$, where

$$L_2(R; H) = \left\{ f(t) : \|f\|^2_{L_2(R;H)} = \int_{-\infty}^{+\infty} \|f(t)\|^2_H \, dt < +\infty \right\}$$

(see [1, 2]), and $u(t) \in W_2^4(R; H)$, that are determined as follows:

$$W_2^4(R; H) = \left\{ u(t) : \frac{d^4 u(t)}{dt^4} \in L_2(R; H), \quad A^4 u(t) \in L_2(R; H) \right\}$$

with the norm

$$\|u\|_{W_2^4(R;H)} = \left(\left\| \frac{d^4 u}{dt^4} \right\|^2_{L_2(R;H)} + \left\| A^4 u \right\|^2_{L_2(R;H)} \right)^{1/2}$$

(see [2]).

Definition 1. *If for* $f(t) \in L_2(R; H)$ *there exists a vector-function* $u(t) \in W_2^4(R; H)$, *satisfying equation* (1) *almost everywhere, we'll call it a regular solution of equation* (1).

Definition 2. *If for any $f(t) \in L_2(R;H)$ there exists a regular solution of equation* (1), *and it holds the inequality*

$$\|u\|_{W_2^4(R;H)} \leq const\, \|f\|_{L_2(R;H)} \ ,$$

we'll call equation (1) *regularly solvable.*

In the present paper we'll get sufficient conditions on regular solvability of equation (1) that are expressed by its operator coefficients. Earlier this problem for equation (1) was studied in the case when $P_0 = \dfrac{d^4}{dt^4} + A^4$ in the paper [3], in the case when $P_0 = \left(\dfrac{d}{dt} - A\right)^2 \left(\dfrac{d}{dt} + A\right)^2$ - in the paper [4]. Notice that principal part of the equation (1) differs from previous papers with its complicated characteristic.

It is known that if $u(t) \in W_2^4(R;H)$ then $A^{4-s}\dfrac{d^s u(t)}{dt^s} \in L_2(R;H)$, $s = \overline{1,3}$, and the following inequalities are valid

$$\left\| A^{4-s}\frac{d^s u}{dt^s} \right\|_{L_2(R;H)} \leq c_s \|u\|_{W_2^4(R;H)} \ , \quad s = \overline{1,3} \tag{2}$$

(theorem on intermediate derivatives [2]).

Now, let's prove a theorem that shows that in the space $W_2^4(R;H)$ we can reduce a norm equivalent to the initial norm $\|u\|_{W_2^4(R;H)}$.

Theorem 1. *The operator P_0 isomorphically maps the space $W_2^4(R;H)$ into $L_2(R;H)$.*

Proof. Taking into account inequality (2) we easily prove that the operator P_0 continuously acts from the space $W_2^4(R;H)$ to $L_2(R;H)$. Further, applying the Fourier transformation, from the equation $P_0 u(t) = f(t)$ we get

$$(-i\xi E - A)(-i\xi E + A)^3 \widehat{u}(\xi) = \widehat{f}(\xi)$$

(E is a unit operator), where $\widehat{u}(\xi)$, $\widehat{f}(\xi)$ are the Fourier transformations of the vector functions $u(t)$, $f(t)$, respectively. It is clear that the operator bundle $(-i\xi E - A)(-i\xi E + A)^3$ is invertible. Therefore we find

$$\widehat{u}(\xi) = (-i\xi E + A)^{-3}(-i\xi E - A)^{-1}\widehat{f}(\xi), \tag{3}$$

i.e.

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (-i\xi E + A)^{-3}(-i\xi E - A)^{-1}\widehat{f}(\xi)\, e^{i\xi t} d\xi.$$

Show that $u(t) \in W_2^4(R;H)$. Using the Parseval equality and taking into account (3), we have:

$$\|u\|^2_{W_2^4(R;H)} = \left\|\frac{d^4 u}{dt^4}\right\|^2_{L_2(R;H)} + \left\|A^4 u\right\|^2_{L_2(R;H)} =$$

$$= \left\|\xi^4 \widehat{u}(\xi)\right\|^2_{L_2(R;H)} + \left\|A^4 \widehat{u}(\xi)\right\|^2_{L_2(R;H)} =$$

$$= \left\| \xi^4 \left(-i\xi E + A\right)^{-3} \left(-i\xi E - A\right)^{-1} \widehat{f}\left(\xi\right) \right\|^2_{L_2(R;H)} +$$

$$+ \left\| A^4 \left(-i\xi E + A\right)^{-3} \left(-i\xi E - A\right)^{-1} \widehat{f}\left(\xi\right) \right\|^2_{L_2(R;H)} \leq$$

$$\leq \sup_{\xi \in R} \left\| \xi^4 \left(-i\xi E + A\right)^{-3} \left(-i\xi E - A\right)^{-1} \right\|^2_{H \to H} \left\| \widehat{f}\left(\xi\right) \right\|^2_{L_2(R;H)} +$$

$$+ \sup_{\xi \in R} \left\| A^4 \left(-i\xi E + A\right)^{-3} \left(-i\xi E - A\right)^{-1} \right\|^2_{H \to H} \left\| \widehat{f}\left(\xi\right) \right\|^2_{L_2(R;H)} . \qquad (4)$$

If we consider

$$\sup_{\xi \in R} \left\| \xi^4 \left(-i\xi E + A\right)^{-3} \left(-i\xi E - A\right)^{-1} \right\|_{H \to H} \leq$$

$$\leq \sup_{\xi \in R} \sup_{\sigma \in \sigma(A)} \left| \xi^4 \left(-i\xi + \sigma\right)^{-3} \left(-i\xi - \sigma\right)^{-1} \right| = \sup_{\xi \in R} \frac{\xi^4}{\left(\xi^2 + \sigma^2\right)^2} \leq 1,$$

$$\sup_{\xi \in R} \left\| A^4 \left(-i\xi E + A\right)^{-3} \left(-i\xi E - A\right)^{-1} \right\|_{H \to H} \leq$$

$$\leq \sup_{\xi \in R} \sup_{\sigma \in \sigma(A)} \left| \sigma^4 \left(-i\xi + \sigma\right)^{-3} \left(-i\xi - \sigma\right)^{-1} \right| = \sup_{\xi \in R} \frac{\sigma^4}{\left(\xi^2 + \sigma^2\right)^2} \leq 1$$

($\sigma\left(A\right)$ is a spectrum of the operator A), from inequality (4) we get:

$$\left\| u \right\|^2_{W_2^4(R;H)} \leq 2 \left\| \widehat{f}\left(\xi\right) \right\|^2_{L_2(R;H)} = 2 \left\| f \right\|^2_{L_2(R;H)} .$$

Consequently $u\left(t\right) \in W_2^4\left(R;H\right)$.

Then we use a Banach theorem on the inverse operator and get that the operator P_0 is an isomorphism between the spaces $W_2^4\left(R;H\right)$ and $L_2\left(R;H\right)$. The theorem is proved.

Before we formulate exact conditions on regular solvability of equation (1), expressed only by its operator coefficients, we must estimate the norms of intermediate derivative operators participating in the perturbed part of the given equation. It follows from theorem 1 that the norms $\left\| P_0 u \right\|_{L_2(R;H)}$ and $\left\| u \right\|_{W_2^4(R;H)}$ are equivalent in the space $W_2^4\left(R;H\right)$. Therefore by the norm $\left\| P_0 u \right\|_{L_2(R;H)}$ the theorem on intermediate derivatives is valid as well.

Theorem 2. *Let $u\left(t\right) \in W_2^4\left(R;H\right)$. Then there hold the following inequalities:*

$$\left\| A^{4-s} \frac{d^s u}{dt^s} \right\|_{L_2(R;H)} \leq a_s \left\| P_0 u \right\|_{L_2(R;H)} , \quad s = \overline{1,3}, \qquad (5)$$

where $a_1 = a_3 = \dfrac{3\sqrt{3}}{16}, \quad a_2 = \dfrac{1}{4}.$

Proof. In order to establish the validity of inequality (5) we make change $P_0 u\left(t\right) = f\left(t\right)$ and apply the Fourier transformation. As a result we get

$$\left\| A^{4-s} \left(-i\xi\right)^s \left(-i\xi E + A\right)^{-3} \left(-i\xi E - A\right)^{-1} \widehat{f}\left(\xi\right) \right\|_{L_2(R;H)} \leq$$

$$\leq \sup_{\xi \in R} \left\| A^{4-s} (-i\xi)^s (-i\xi E + A)^{-3} (-i\xi E - A)^{-1} \right\|_{H \to H} \times$$

$$\times \left\| \widehat{f}(\xi) \right\|_{L_2(R;H)}, \quad s = \overline{1,3}. \tag{6}$$

Further, for $\xi \in R$ we estimate the following norms:

$$\left\| A^{4-s} (-i\xi)^s (-i\xi E + A)^{-3} (-i\xi E - A)^{-1} \right\|_{H \to H} \leq$$

$$\leq \sup_{\sigma \in \sigma(A)} \left| \sigma^{4-s} (-i\xi)^s (-i\xi + \sigma)^{-3} (-i\xi - \sigma)^{-1} \right| =$$

$$= \sup_{\sigma \in \sigma(A)} \left| \sigma^{-s} (-i\xi)^s \left(-i\frac{\xi}{\sigma} + 1 \right)^{-3} \left(-i\frac{\xi}{\sigma} - 1 \right)^{-1} \right| \leq$$

$$\leq \sup_{\mu = \frac{\xi^2}{\sigma^2} \geq 0} \frac{\mu^{\frac{s}{2}}}{(\mu+1)^2} = \frac{1}{16} s^{\frac{s}{2}} (4-s)^{\frac{4-s}{2}} = a_s, \quad s = \overline{1,3}.$$

Taking into account the found expressions in inequalities (6), we have

$$\left\| A^{4-s} (-i\xi)^s (-i\xi E + A)^{-3} (-i\xi E - A)^{-1} \widehat{f}(\xi) \right\|_{L_2(R;H)} \leq$$

$$\leq a_s \left\| \widehat{f}(\xi) \right\|_{L_2(R;H)}, \quad s = \overline{1,3},$$

that are equivalent to inequalities (5). The theorem is proved.

Remark. We can show that the numbers a_s, $s = \overline{1,3}$, in inequalities (5) are exact values of the norms of the operators $A^{4-s}\frac{d^s}{dt^s} : W_2^4(R;H) \to L_2(R;H)$, but about this fact we'll speak in another paper.

Taking into attention a theorem on intermediate derivatives [2] we easily prove the following

Lemma. *The operator P_1 continuously acts from $W_2^4(R;H)$ to $L_2(R;H)$ provided that the operators $A_s A^{-s}$, $s = \overline{1,3}$, are bounded in H.*

Taking into account the results found up to now we get possibility to establish regular solvability conditions of equation (1).

Theorem 3. *Let the operators $A_s A^{-s}$, $s = \overline{1,3}$, be bounded in H and it hold the inequality $\sum_{s=1}^{3} a_s \left\| A_{4-s} A^{-(4-s)} \right\|_{H \to H} < 1$, where the numbers a_s, $s = \overline{1,3}$, are determined in theorem 2. Then equation (1) is regularly solvable.*

Proof. It follows from theorem 1 that the operator P_0 has a bounded inverse P_0^{-1} that acts from $L_2(R;H)$ to $W_2^4(R;H)$. Then, after substitution $P_0 u(t) = v(t)$ equation (1) is rewritten in the form $\left(E + P_1 P_0^{-1}\right) v(t) = f(t)$. Further, we show that by fulfilling the conditions of the theorem the norm

$$\left\| P_1 P_0^{-1} \right\|_{L_2(R;H) \to L_2(R;H)} < 1.$$

Really, by theorem 2 we have:

$$\left\|P_1P_0^{-1}v\right\|_{L_2(R;H)} = \left\|P_1u\right\|_{L_2(R;H)} \leq \sum_{s=1}^{3}\left\|A_s\frac{d^{4-s}u}{dt^{4-s}}\right\|_{L_2(R;H)} \leq$$

$$\leq \sum_{s=1}^{3}\left\|A_sA^{-s}\right\|_{H\to H}\left\|A^s\frac{d^{4-s}u}{dt^{4-s}}\right\|_{L_2(R;H)} \leq$$

$$\leq \sum_{s=1}^{3}\left\|A_sA^{-s}\right\|_{H\to H} a_{4-s}\left\|P_0u\right\|_{L_2(R;H)} =$$

$$= \sum_{s=1}^{3} a_{4-s}\left\|A_sA^{-s}\right\|_{H\to H}\left\|v\right\|_{L_2(R;H)} ,$$

whence it follows that

$$\left\|P_1P_0^{-1}\right\|_{L_2(R;H)\to L_2(R;H)} \leq \sum_{s=1}^{3} a_{4-s}\left\|A_sA^{-s}\right\|_{H\to H} < 1.$$

Thus, the operator $E + P_1P_0^{-1}$ is invertible in the space $L_2\left(R;H\right)$ and we can determine $u\left(t\right)$ by the following formula:

$$u\left(t\right) = P_0^{-1}\left(E + P_1P_0^{-1}\right)^{-1} f\left(t\right),$$

moreover

$$\left\|u\right\|_{W_2^4(R;H)} \leq \left\|P_0^{-1}\right\|_{L_2(R;H)\to W_2^4(R;H)} \times$$

$$\times\left\|\left(E + P_1P_0^{-1}\right)^{-1}\right\|_{L_2(R;H)\to L_2(R;H)}\left\|f\right\|_{L_2(R;H)} \leq const\left\|f\right\|_{L_2(R;H)}.$$

The theorem is proved.

Remark. The case when in the perturbed part of equation (1) there participate variable operator coefficients, i.e. $A_s\left(t\right),\ s = \overline{1,3}$, are linear operators determined for all $t \in R$, is investigated in a similar way.

References

[1]. Hille E., Phillips R. *Functional analysis and semi-groups.* Moscow, IL, 1962, 829 pp. (Russian).

[2]. Lions J.-L., Majenes E. *Inhomogeneous boudary value problems and their applications.* Moscow, Mir, 1971, 371 pp. (Russian).

[3]. Mirzoyev S. S. *Matters of the theory of solvability of boundary value problems for operator-differential equations in the Hilbert space and related spectral problems.* Author's thesis for Doctor's degree. BSU, Baku, 1994, 32 pp. (Russian).

[4]. Gumbataliyev R. Z. *On solvability of boundary value problems for fourth order operator-differential equations.* Author's thesis for PhD. BSU, Baku, 2000, 16 pp. (Russian).

Aydin A. Gasymov
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F. Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 492 28 08

Received September 3, 2007; Revised November 22, 2007