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BASES FROM EXPONENTS IN LEBESGUE SPACES OF FUNCTIONS WITH VARIABLE SUMMABILITY EXPONENT

Abstract

In the paper we consider basis properties of the classical system of exponents in ordinary and weight Lebesgue spaces of functions with variable suitability exponent.

Theory of the Fourier series in classic system of exponents

$$e_n(t) \equiv e^{int}, n = 0; \pm 1; ..., \tag{1}$$

has been well studied in different spaces. Fundamental monographs of such known mathematicians as A.Zigmund [1,2], N.K.Bari [3], R.Edwards [4, 5] and others have been devoted to these problems.

Basis properties of the system (1) in the Lebesgue spaces of summable functions $L_p \equiv L_p(-\pi, \pi)$ have been completely studied (for $p \in (1, +\infty)$ the basicity; in L_1 and $C \equiv L_\infty$ completeness and minimality).

As present, in connection with problems of the theory of nonlinear differential equations, the theory of the Lebesgue spaces of functions with variable summability exponent denoted as $L_{p(\cdot)}$ rapidly develops. Naturally, there arises a question on study of these or other properties of the system of exponents (1) in these spaces. The represented paper is devoted to the basicity problem of the system (1) in $L_{p(\cdot)}$. Earlier this problem was considered in [6]. We'll give independent proof and consider the weight case.

1) First of all we determine the spaces $L_{p(\cdot)}$ and $L_{p(\cdot),\rho(\cdot)}$. Let $p : [-\pi, \pi] \to [1, +\infty)$ be a measurable function. By $L_{p(\cdot)}$ we denote a class of measurable on the segment $[-\pi, \pi]$ functions $f(x)$ for which

$$I_p(f) \overset{def}{\equiv} \int_{-\pi}^{\pi} |f(x)|^{p(x)} dx < +\infty.$$

With ordinary operations of addition of functions and multiplication by the number, $L_{p(\cdot)}$ is a linear space.

Proceeding from the Minkowskii functional we introduce in $L_{p(x\cdot)}$ the norm $\|\cdot\|_{p(\cdot)}$:

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\}. \tag{2}$$

$L_{p(\cdot)}$ is a Banach space with respect to this norm.

Now, let's determine the weight space $L_{p(\cdot),\rho(\cdot)}$. Let $\rho : [-\pi, \pi] \to [0, +\infty)$ be a measurable function. Under $L_{p(\cdot),\rho(\cdot)}$ we usually understand a Banach space of measurable on $(-\pi, \pi)$ functions $f(x)$ with the norm

$$\|f\|_{p(\cdot),\rho(\cdot)} \stackrel{def}{\equiv} \|\rho f\|_{p(\cdot)}.$$

Denote

$$p^- = \underset{[-\pi,\pi]}{ess\ \ inf}\ p(x); \qquad p^+ = \underset{[-\pi,\pi]}{ess\ sup}\ p(x).$$

Under the class LN we'll understand a class of measurable on $[-\pi, \pi]$ functions $f(x)$ for which the inequality

$$|f(x_1) - f(x_2)| \leq \frac{A}{\ln \frac{1}{|x_1 - x_2|}}, \quad |x_1 - x_2| \leq \frac{1}{2}$$

is fulfilled. A is a constant dependent only on f.

Let $q(t)$ be a function conjugated to $p(t)$ in the sense $\frac{1}{q(t)} + \frac{1}{p(t)} \equiv 1$.

The following statement is true.

Statement A [7]. Let $1 \leq p^- \leq p^+ < +\infty$. Then $C_0^\infty[-\pi, \pi]$ is dense in $L_{p(\cdot)}$. In particular $C[-\pi, \pi]$ is dense in $L_{p(\cdot)}$.

The similar statement holds in the weight case.

Statement B [7]. Let $1 \leq p^- \leq p^+ < +\infty$ and $\rho(\cdot)$ be a weight function and

$$|\rho(x)|^{p(x)} \in L_1.$$

Then $C_0^\infty[-\pi, \pi]$ and at the same time $C[-\pi, \pi]$ is dense in $L_{p(\cdot),\rho(\cdot)}$.

Let's consider the Hilbert transformation

$$\Gamma f \stackrel{def}{\equiv} \int_{-\pi}^{\pi} \frac{f(y)\, dy}{y - x}.$$

In sequel, we'll need the following fact

Statement C [7]. Let $1 < p^- \leq p^+ < +\infty$ and $p(\cdot) \in LN$. The weight function $\rho(\cdot)$ is of the form :

$$\rho(t) \equiv \prod_{k=1}^{l} |t - t_k|^{\alpha_k}, \quad \{t_k\}_{k=1}^{l} \subset [-\pi, \pi]. \tag{3}$$

If the inequalities

$$-\frac{1}{p(t_k)} < \alpha_k < \frac{1}{q(t_k)}, \quad k = \overline{1, l}; \tag{4}$$

are fulfilled, then Γ boundedly acts from $L_{p(\cdot),\rho(\cdot)}$ to $L_{p(\cdot),\rho(\cdot)}$.

In future, for simplicity of notation, by L_{p_t}; L_{p_t,ρ_t}; $\|\cdot\|_{p_t}$, $\|\cdot\|_{p_t,\rho_t}$ we'll denote appropriate notions.

We'll need the following.

Statement D [8]. The system $\{x_n\}_{n\in N} \subset B$ forms a basis in B if and only if the following conditions are satisfied.

1) $\{x_n\}_{n\in N}$ is complete in B;

2) $\{x_n\}_{n\in N}$ is minimal in B;

3)
$$\left\| \sum_{n=1}^{m} x_n^* (x) x_n \right\|_B \leq M \|x\|_B , \quad \forall m \in N, \quad \forall x \in B,$$

where M is an absolute constant.

2. The non-weight case. Now we pass to the statement of the main results. We consider approximate properties of the system (1) in L_{p_t}. Let $1 \leq p^- \leq p^+ < +\infty$. Then by the statement A $C[-\pi, \pi]$ is dense in L_{p_t} and as a result the system (1) is complete in L_{p_t}. Lets consider a family of functionals $\{e_n^*\}_{n\in N}$:

$$e_n^* (f) \overset{def}{\equiv} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} f (t) \, dt, \quad n \in Z. \tag{5}$$

Let $q(\cdot)$ be a function conjugated to $p(\cdot)$. It is known that if $1 < p^- \leq p^+ < +\infty$ then L_{q_t} is a space conjugated to L_{p_t}. From the Hölders generalized inequality it follows that the functionals e_n^*, $n \in Z$ are continuous on L_{p_t} and moreover

$$e_n^* (e_k) = \delta_{nk}, \forall n, k \in Z, \text{ where } e_k \equiv e^{ikt}.$$

Thus, the system $\{e_n\}_{n\in Z}$ is minimal in L_{p_t}. As a result, it holds.

Lemma 1. *Let $1 < p^- \leq p^+ < +\infty$. Then the system (1) in complete and minimal in L_{p_t}.*

Now, lets assume that $1 < p^- \leq p^+ < +\infty$ and $p \in LN$. By lemma 1 the system (1) in complete and minimal in L_{p_t}. Let $\{e_n^*\}_{n\in Z}$ be functionals determined by (5). Take $\forall f \in L_{p_t}$ and consider the following partial sum

$$S_m = \sum_{n=-m}^{m} e_n^* (f) \, e_n.$$

Obviously,

$$S_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{f} (x - t) \, D_m (t) \, dt,$$

where

$$D_m (t) \equiv \frac{\sin \left(m + \frac{1}{2}\right) t}{2 \sin \frac{t}{2}}$$

is a Dirichlet kernel, $\tilde{f} (t)$ is the function $f (t)$ extended periodically on the axis. As it follows from the results of the paper [7] by fulfilling the above-formulated conditions for the function $p(t)$ the Hilbert transformation Γ boundedly acts from L_{p_t} to L_{p_t}.

Obviously, if $|f_x(t)| \le M |g(t)|$ a.e. on $(-\pi, \pi)$, then $\|f_x\|_{p_t} \le M \|g\|_{p_t}$, where M is a constant independent of x. Thus, denoting by $f_x^m(t)$ the function

$$f_x^m(t) = \frac{1}{4\pi} \frac{x-t}{\sin \frac{x-t}{2}} \sin \left[\left(m + \frac{1}{2} \right) (x-t) \right] f(t)$$

we have

$$|f_x^m(t)| \le M |f(t)|,$$

where M is a constant independent of m and x.

Allowing for these relations, from the boundedness of Γ in L_{p_t} we get

$$\left\| \sum_{n=-m}^{m} e_n^*(f) e_n \right\|_{p_t} \le M_1 \|f\|_{p_t},$$

where M_1 is a constant independent of m and f. As a result, by the statement D we obtain the basicity of the system (1) in L_{p_t}, i.e. it holds.

Theorem 1. *Let $1 < p^- \le p^+ < +\infty$ and $p(t) \in LN$. Then the system (1) forms a basis in L_{p_t}.*

3. The weight case. Take the weight $\rho(t)$ of the form (3). We'll need the following almost obvious

Statement E. Let $p(t) \in C[0,1]$ and $p(t) > 0$, $\forall t \in [0,1]$. Then the integral

$$\int_0^1 t^{\alpha p(t)} dt$$

exists, if $\alpha > -\frac{1}{p(0)}$.

From this statement it immediately follows that if $p(t)$: $1 \le p^- \le p^+ < +\infty$ is a function continuous on $[-\pi, \pi]$ and the inequalities

$$\alpha_k > -\frac{1}{p(t_k)}, \quad k = \overline{1,l}; \tag{6}$$

hold, the system (1) belongs to the space L_{p_t,ρ_t}. It follows from the results of the paper [7] that by fulfilling these conditions $C[-\pi, \pi]$ is dense in L_{p_t,ρ_t}. As a result we get the completeness of the system (1) in L_{p_t,ρ_t}. Thus, it holds.

Lemma 2. *Let $p(t)$: $1 \le p^- \le p^+ < +\infty$ be a continuous on $[-\pi, \pi]$, function, $\rho(t)$ be determined by the formula (3) and inequalities (6) hold. Then the system of exponents (1) is complete in the weight space L_{p_t,ρ_t}.*

Now, we pass to the minimality of the system (1) in L_{p_t,ρ_t}. Let $q(t)$ be a function conjugated to $p(t)$. It a easy to notice that $L_{q_t,\rho_t^{-1}}$ is the space conjugated to L_{p_t,ρ_t}. Pay attention to the fact that if the inequalities (4) are fulfilled, the functionals (5) are continuous on L_{p_t,ρ_t}, and moreover $e_n^*(e_k) = \delta_{nk}$, $\forall n, k \in Z$. This proves the following

Lemma 3. *Let $p(t)$: $1 \le p^- \le p^+ < +\infty$ be a function continuous on $[-\pi, \pi]$, the weight $\rho(t)$ be determined by the formula (3) and the inequalities (4) be satisfied. Then the system (1) is minimal in L_{p_t,ρ_t}.*

Further, we elucidate a question on the basicity of the system in the weight space L_{p_t,ρ_t}. Assume that $1 < p^- \le p^+ < +\infty$ and moreover $p \in LN$. If the inequalities (4) are fulfilled, then as it follows from the lemmas 2 and 3, the system (1) is complete and minimal in L_{p_t,ρ_t}. Similar to section 1 we consider the partial sum

$$S_m = \sum_{n=-m}^{m} e_n^*(f)\, e_n$$

where $f \in L_{p_t,\rho_t}$ is an arbitrary function, $\{e_n^*\}_{n\in Z}$ are the functionals determined by (5). As it follows from the results of the paper [7], if $\rho(t)$ satisfies all the conditions enumerated above, then the Hilbert transformation Γ boundedly acts from L_{p_t,ρ_t} to L_{p_t,ρ_t}. Granting this property, similar to section 1, we prove that it holds the inequality

$$\left\| \sum_{n=-m}^{m} e_n^*(f)\, e_n \right\|_{p_t,\rho_t} \le M_2 \|f\|_{p_t,\rho_t}$$

where M_2 is a constant independent of m and f. And by the statement D we get the basicity of the system (1) in L_{p_t,ρ_t} i.e. it holds

Theorem 2. *Let*

$$p(t) \in LN$$

and $1 < p^- \le p^+ < +\infty$. If the weight $\rho(t)$ determined by the formula (3) satisfies the inequalities (4), the system of exponents (1) forms a basis in the weight space L_{p_t,ρ_t}.

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