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NECESSARY OPTIMALITY CONDITIONS OF FIRST AND SECOND ORDER FOR SYSTEMS WITH BOUNDARY CONDITIONS

Abstract

We consider an optimal control problem wherein the state of a system is determined from controlled system of ordinary differential equations with two-point boundary conditions. Admissible controls are chosen from a class of bounded and measurable functions. Validity of the Pontryagin's maximum principle is proved for the investigated class of problems. Increment formula of the second order functional is calculated. On the base of the needle-shaped variations control we get necessary optimality conditions for singular controls in the sense of the Pontryagin's maximum principle.

Problem statement. The investigation object of the present paper is optimal control problems in systems of first order nonlinear ordinary differential equations with the boundary conditions:

$$\dot{x} = f(x,u,t), \quad x(t) \in R^n, \quad t \in T = [t_0, t_1], \tag{1.1}$$

$$Ax(t_0) + Bx(t_1) = C. \tag{1.2}$$

Here $f(x,u,t)$ is the given $n-$dimensional vector-function continuous in totality of variables together with respect to x up to the second order inclusively; $A, B \in R^{n\times n}$, $C \in R^{n\times 1}$ are constant matrices,$u(t)$ is $r-$demensional measurable and bounded vector of controlling effects on the segment T.

It is assumed that almost everywhere on this segment the controlling effects satisfy the boundedness of the type of the inclusion:

$$u(t) \in U, \quad t \in T, \tag{1.3}$$

where U is a compact from the space R^r.

The goal of the optimal control problem is the minimization of the functional

$$J(u) = \varphi(x(t_0), x(t_1)) + \int_T F(x,u,t)\,dt \tag{1.4}$$

determined in the solutions of boundary value problem (1.1), (1.2) for admissible control satisfying the condition (1.3). Here we assume that the scalar functions $\varphi(x,y)$ and $F(x,u,t)$ are continuous by their argumets and have continuous and bounded partial derivatives with respect to x and y up to the second order inclusively.

Let under some conditions the boundary value problem (1.1), (1.2) for each admissible process $u(t) \in U$, $t \in T$ have a unique solution $x(t,u)$. Admissible control $\{u(t), x(t,u)\}$, being a solution of the problem $(1.1) - (1.4)$ i.e. delivering minimum to the functional (1.4) under restrictions $(1.1) - (1.3)$ will be said to be optimal process, and $u(t)$-an optimal control.

2. Increment formula of the functional. We can carry out investigation of the optimal control problem (1.1) − (1.4) wih using different variants of increment formula of the aim functional in two admissible processes $\{u, x\}$ and $\{\widetilde{u} = u + \Delta u, \widetilde{x} = x + \Delta x\}$. L. T. Rozonoer's [1] classic method of increments allows to obtain in this paper necessary optimality condition of the Pontragin's maximum principle type [2] . For obtaining necessary optimality conditions of the second order for the Cauchy problem there are methods to obtain second order increment formulae suggested in [3, 4] . In this section we'll obtain increment formulae for the second order functional for the problem (1.1)−(1.4) based on [4] . Notice that in deriving necessary optimality conditions, locality of the increment formula is essential, since the remainder terms are estimated by the quantity characterizing the smallness of the degree of the needle-shaped variation of control.

Necessary optimality conditions for an optimal control problem described by systems of ordinary differential equations have also been obtained in the paper [5 − 8] .

Since non-local conditions (1.2) are present in the considered problem (1.1) − (1.4) , more subtle approach for obtaining increment formula is required to prove this fact.

Let $\{u, x = x(t, u)\}$ and $\{\widetilde{u} = u+\Delta u, \widetilde{x} = x+\Delta x = x(t, \widetilde{u})\}$ be two addmissible processes. We can define the boundary value problem in increments for the problem (1.1), (1.2) :

$$\Delta \dot{x} = \Delta f(x, u, t), \quad t \in T \tag{2.1}$$

$$A\Delta x(t_0) + B\Delta x(t_1) = 0 \tag{2.2}$$

where by

$$\Delta f(x, u, t) = f(\widetilde{x}, \widetilde{u}, t) - f(x, u, t)$$

we denote a complete increment of the function $f(x, u, t)$. For particular increments we'll use the denotation

$$\Delta_{\widetilde{u}} f(x, u, t) = f(x, \widetilde{u}, t) - f(x, u, t)$$

we can represent the increment of the functional in the form:

$$\Delta J(u) = J(\widetilde{u}) - J(u) = \Delta\varphi(x(t_0), x(t_1)) + \int_T \Delta F(x, u, t)\, dt. \tag{2.3}$$

We make some sufficiently standard operations usually used in deriving necessary optimality conditions of first and second orders:

In formula (2.3)

- we add zero summands

$$\int_T \left\langle \psi(t),\ \Delta\dot{x} - \Delta f(x, u, t) \right\rangle dt$$

and

$$\langle \lambda,\ A\Delta x(t_0) + B\Delta x(t_1) \rangle$$

where $\psi(t) \in R^n,\ t \in T;\ \lambda \in R^n$ are some untill undetermined vector-functions and constant vector: by $\langle \cdot, \cdot \rangle$ we denote a scalar product in R^n;

- let's introduce the Pontryagin's function

$$H(\psi, x, u, t) = \langle \psi(t), f(x, u, t) \rangle - F(x, u, t);$$

- expand the increment by the second order Taylor formula

$$\Delta\varphi(x(t_0), x(t_1)) = \left\langle \frac{\partial\varphi}{\partial x(t_0)},\ \Delta x(t_0) \right\rangle + \left\langle \frac{\partial\varphi}{\partial x(t_1)},\ \Delta x(t_1) \right\rangle +$$
$$+\frac{1}{2}\left\langle \frac{\partial^2\varphi}{\partial x(t_1)^2}\Delta x(t_0) + \frac{\partial^2\varphi}{\partial x(t_1)\partial x(t_0)}\Delta x(t_1), \Delta x(t_0) \right\rangle + \tag{2.4}$$
$$+\frac{1}{2}\left\langle \frac{\partial^2\varphi}{\partial x(t_1)^2}\Delta x(t_1) + \frac{\partial^2\varphi}{\partial x(t_0)\partial x(t_1)}\Delta x(t_0), \Delta x(t_1) \right\rangle +$$
$$+o_\varphi\left(\|\Delta x(t_0)\|^2,\ \|\Delta x(t_1)\|^2\right)$$

- expand the partial increment $\Delta_{\widetilde{x}\widetilde{u}}H(\psi, x, u, t)$ by the second order Taylor formula with respect x:

$$\Delta_{\widetilde{x}\widetilde{u}}H(\psi, x, u, t) = \Delta_{\widetilde{u}}H(\psi, x, u, t) + \Delta_{\widetilde{x}}H(\psi, x, \widetilde{u}, t) =$$
$$= \Delta_{\widetilde{u}}H(\psi, x, u, t) + \left\langle \frac{\partial H(\psi, x, \widetilde{u}, t)}{\partial x}, \Delta x(t) \right\rangle + \tag{2.5}$$
$$+\frac{1}{2}\left\langle \frac{\partial^2 H(\psi, x, \widetilde{u}, t)}{\partial x^2}\Delta x(t), \Delta x(t) \right\rangle + o_H\left(\|\Delta x(t)\|^2\right),$$

where it holds

$$\begin{aligned} \frac{\partial H(\psi, x, \widetilde{u}, t)}{\partial x} &= \frac{\partial H(\psi, x, u, t)}{\partial x} + \Delta_{\widetilde{u}}\frac{\partial H(\psi, x, u, t)}{\partial x}, \\ \frac{\partial^2 H(\psi, x, \widetilde{u}, t)}{\partial x^2} &= \frac{\partial^2 H(\psi, x, u, t)}{\partial x^2} + \Delta_{\widetilde{u}}\frac{\partial^2 H(\psi, x, u, t)}{\partial x^2}. \end{aligned} \tag{2.6}$$

- we expand the complete increment $\Delta f(x, u, t)$ by the first order Taylor formula with respect to x :

$$\begin{aligned} \Delta f(x, u, t) &= \Delta_{\widetilde{x}} f(x, \widetilde{u}, t) + \Delta_{\widetilde{u}} f(x, u, t) \\ \Delta_{\widetilde{x}} f(x, \widetilde{u}, t) &= \frac{\partial f(x, \widetilde{u}, t)}{\partial x}\Delta x(t) + o_f(\|\Delta x(t)\|), \\ \frac{\partial f(x, \widetilde{u}, t)}{\partial x} &= \frac{\partial f(x, u, t)}{\partial x} + \Delta_{\widetilde{u}}\frac{\partial f(x, u, t)}{\partial x}. \end{aligned} \tag{2.7}$$

Now, using the introduced denotation and taking into account (2.4) – (2.7) in (2.3) for the increment of the functional we get the formulae

$$\Delta J(u) = J(\widetilde{u}) - J(u) = -\int_T \Delta_{\widetilde{u}}H(\psi, x, u, t)\,dt -$$
$$-\int_T \left\langle \frac{\partial H(\psi, x, u, t)}{\partial x} + \dot{\psi}(t), \Delta x(t) \right\rangle dt + \left\langle \left[\frac{\partial\varphi}{\partial x(t_0)} - \psi(t_0) + A'\lambda\right], \Delta x(t_0) \right\rangle +$$

$$+\left\langle\left[\frac{\partial\varphi}{\partial x(t_1)}-\psi(t_1)+B'\lambda\right],\Delta x(t_1)\right\rangle-$$

$$-\int\limits_T\left\langle\Delta_{\widetilde{u}}\frac{\partial H(\psi,x,u,t)}{\partial x}+\frac{1}{2}\Delta x'(t)\frac{\partial^2 H(\psi,x,u,t)}{\partial x^2},\Delta x(t)\right\rangle dt+$$

$$+\frac{1}{2}\left\langle\Delta x(t_0)'\frac{\partial^2\varphi}{\partial x(t_0)^2}+\Delta x(t_1)'\frac{\partial^2\varphi}{\partial x(t_0)\partial x(t_1)},\Delta x(t_0)\right\rangle+$$

$$+\frac{1}{2}\left\langle\Delta x(t_0)'\frac{\partial^2\varphi}{\partial x(t_1)\partial x(t_0)}+\Delta x(t_1)'\frac{\partial^2\varphi}{\partial x(t_1)^2},\Delta x(t_1)\right\rangle+\eta_{\widetilde{u}} \tag{2.8}$$

where

$$\eta_{\widetilde{u}}=-\int\limits_T o_H\left(\|\Delta x(t)\|^2\right)dt+o_\varphi\left(\|\Delta x(t_0)\|^2,\|\Delta x(t_1)\|^2\right)-$$
$$-\frac{1}{2}\int\limits_T\left\langle\Delta_{\widetilde{u}}\frac{\partial^2 H(\psi,x,u,t)}{\partial x^2}+\Delta x(t),\Delta x(t)\right\rangle dt \tag{2.9}$$

Require the vector function $\psi=\psi(t)\in R^n$ and constant vector $\lambda\in R^n$ be the solutions of the following conjugation problem (stationary state condition of Lagrange function by the state):

$$\dot{\psi}(t)=-\frac{\partial H(\psi,x,u,t)}{\partial x},\quad t\in T \tag{2.10}$$

$$\psi(t_0)=\frac{\partial\varphi}{\partial x(t_0)}+A'\lambda, \tag{2.11}$$

$$\psi(t_1)=-\frac{\partial\varphi}{\partial x(t_1)}+B'\lambda, \tag{2.12}$$

Then increment formula (2.8) accepts the form:

$$\Delta J(u)=-\int\limits_T\Delta_{\widetilde{u}}H(\psi,x,u,t)\,dt-$$

$$-\int\limits_T\left\langle\Delta_{\widetilde{u}}\frac{\partial H(\psi,x,u,t)}{\partial x}+\frac{1}{2}\Delta x'(t)\frac{\partial^2 H(\psi,x,u,t)}{\partial x^2},\Delta x(t)\right\rangle dt+$$

$$+\frac{1}{2}\left\langle\Delta x(t_0)'\frac{\partial^2\varphi}{\partial x(t_0)^2}+\Delta x(t_1)'\frac{\partial^2\varphi}{\partial x(t_0)\partial x(t_1)},\Delta x(t_0)\right\rangle+$$

$$+\frac{1}{2}\left\langle\Delta x(t_0)'\frac{\partial^2\varphi}{\partial x(t_1)\partial x(t_0)}+\Delta x(t_1)'\frac{\partial^2\varphi}{\partial x(t_1)^2},\Delta x(t_1)\right\rangle+\eta_{\widetilde{u}} \tag{2.13}$$

On the other hand, it follows from the equalities (2.1), (2.2) and (2.7) that $\Delta x(t)$ is a solution of the following linearized system

$$\Delta\dot{x}(t)\frac{\partial f(x,u,t)}{\partial x}\Delta x(t)+\Delta_{\widetilde{u}}\frac{\partial f(x,u,t)}{\partial x}+\eta_1(t) \tag{2.14}$$

$$A\Delta x(t_0) + B\Delta x(t_1) = 0, \tag{2.15}$$

where by definition

$$\eta_1(t) = \Delta_{\widetilde{u}} \frac{\partial f(x,u,t)}{\partial x} \Delta x(t) + o_f(\|\Delta x(t)\|). \tag{2.16}$$

Let the matrix function $\Phi(t), t \in T$ be a solution of the following matrix deferential equation

$$\dot{\Phi}(t) \frac{\partial f(x,u,t)}{\partial x} \Phi(t)$$

with initial condition

$$\Phi(t_0) = E$$

where E is a unit matrix of dimension $n \times n$. Then we can represent any solution of problem (2.14), (2.15) in the form:

$$\Delta x(t) = \Phi(t)\Delta x(t_0) + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(\tau)\Delta_{\widetilde{u}} f(x,u,t)\,d\tau + \eta_2(t) \tag{2.17}$$

where

$$\eta_2(t) = \int_{t_0}^{t} \Phi(t)\Phi^{-1}(\tau)\eta_1(t)\,d\tau.$$

Require that the function (2.17) satisfy the condition (2.15). Then we obtain

$$[A + B\Phi(t_1)]\Delta x(t_0) = -B\Phi(t_1)\int_{t_0}^{t} \Phi^{-1}(\tau)\Delta_{\widetilde{u}} f(x,u,t)\,d\tau + \eta_3(t)$$

where $\eta_3 = B\eta_2(t_1)$. Hence

$$\Delta x(t_0) = -[A + B\Phi(t_1)]^{-1} B\Phi(t_1)\int_{t_0}^{t} \Phi^{-1}(\tau)\Delta_{\widetilde{u}} f(x,u,t)\,d\tau + \eta_4 \tag{2.18}$$

where

$$\eta_4 = [A + B\Phi(t_1)]^{-1} B\eta_2(t_1) \tag{2.19}$$

Taking into account (2.19) in (2.17) we have

$$\Delta x(t) = -\Phi(t)[A + B\Phi(t_1)]^{-1} B\Phi(t_1)\int_{T} \Phi^{-1}(\tau)\Delta_{\widetilde{u}} f(x,u,t)\,d\tau +$$

$$+\int_{t_0}^{t} \Phi(t)\Phi^{-1}(\tau)\Delta_{\widetilde{u}} f(x,u,t)\,d\tau + \eta_5(t) \tag{2.20}$$

where

$$\eta_5(t) = \Phi(t)\eta_4 + \eta_2(t). \tag{2.21}$$

Finally, from the last one we have

$$\Delta x(t_1)=\Phi(t_1)\left[A+B\Phi(t_1)\right]^{-1}A\int\limits_T\Phi^{-1}(\tau)\,\Delta_{\widetilde{u}}f(x,u,t)\,d\tau+\eta_5(t_1) \tag{2.22}$$

Now, we rewrite some summands in (2.13) in the following form:

$$\left\langle \Delta x(t_0)'\frac{\partial^2\varphi}{\partial x(t_0)^2},\Delta x(t_0)\right\rangle=$$
$$=\int\limits_T\int\limits_T\Big\langle \Delta_u' f(x,u,\tau)\,\Phi^{-1'}(\tau)\,\Phi(t_1)'\,B'\left(A+B\Phi(t_1)\right)^{-1'}\times$$
$$\times\frac{\partial^2\varphi}{\partial x(t_0)^2}\left(A+B\Phi(t_1)\right)^{-1}B\Phi(t_1)\,\Phi^{-1}(s),\Delta_u f(x,y,s)\Big\rangle\,d\tau ds+$$
$$+\left\langle \eta_4'\frac{\partial^2\varphi}{\partial x(t_0)^2},\Delta x(t_0)\right\rangle+\left\langle \Delta x(t_0)'\frac{\partial^2\varphi}{\partial x(t_0)^2},\eta_4\right\rangle, \tag{2.23}$$

$$\left\langle \Delta x(t_1)'\frac{\partial^2\varphi}{\partial x(t_1)^2},\Delta x(t_1)\right\rangle=$$
$$=\int\limits_T\int\limits_T\Big\langle \Delta_{\widetilde{u}}' f(x,u,\tau)\,\Phi^{-1'}(\tau)\,\Phi(t_1)'\,A'\left(A+B\Phi(t_1)\right)^{-1'}\times$$
$$\times\Phi(t_1)'\frac{\partial^2\varphi}{\partial x(t_1)^2}\Phi(t_1)\left(A+B\Phi(t_1)\right)^{-1}A\Phi^{-1}(s),\Delta_u f(x,u,s)\Big\rangle\,d\tau ds+$$
$$+\left\langle \eta_5(t_1)'\frac{\partial^2\varphi}{\partial x(t_1)^2},\Delta x(t_1)\right\rangle+\left\langle \Delta x(t_1)'\frac{\partial^2\varphi}{\partial x(t_1)^2},\eta_5(t_1)\right\rangle \tag{2.24}$$

$$\left\langle \Delta x(t_0)'\frac{\partial^2\varphi}{\partial x(t_0)\,\partial x(t_1)},\Delta x(t_1)\right\rangle=$$
$$=\int\limits_T\int\limits_T\Big\langle \Delta_u' f(x,u,\tau)\,\Phi^{-1'}(\tau)\,\Phi(t_1)'\,B'\left(A+B\Phi(t_1)\right)^{-1'}\times$$
$$\times\frac{\partial^2\varphi}{\partial x(t_0)\,\partial x(t_1)}\Phi(t_1)\left(A+B\Phi(t_1)\right)^{-1}A\Phi^{-1}(s),\Delta_u f(x,u,s)\Big\rangle\,d\tau ds+$$
$$+\left\langle \eta_4'\frac{\partial^2\varphi}{\partial x(t_0)\,\partial x(t_1)},\Delta x(t_1)\right\rangle+\left\langle \Delta x(t_0)'\frac{\partial^2\varphi}{\partial x(t_0)\,\partial x(t_1)^2},\eta_5(t_1)\right\rangle \tag{2.25}$$

$$\left\langle \Delta x(t_1)'\frac{\partial^2\varphi}{\partial x(t_1)\,\partial x(t_0)},\Delta x(t_0)\right\rangle=$$
$$=\int\limits_T\int\limits_T\Big\langle \Delta_u' f(x,u,\tau)\,\Phi^{-1'}(\tau)\,\Phi(t_1)'\,A'\left(A+B\Phi(t_1)\right)^{-1'}\times$$
$$\times\frac{\partial^2\varphi}{\partial x(t_1)\,\partial x(t_0)}\Phi(t_1)\left(A+B\Phi(t_1)\right)^{-1}B\Phi^{-1}(s),\Delta_u f(x,u,s)\Big\rangle\,d\tau ds+$$

$$+\left\langle \eta_5(t_1)' \frac{\partial^2\varphi}{\partial x(t_0)\,\partial x(t_1)}, \Delta x(t_0)\right\rangle + \left\langle \Delta x(t_1)' \frac{\partial^2\varphi}{\partial x(t_0)\,\partial x(t_1)}, \eta_4\right\rangle \tag{2.26}$$

$$\int\limits_T \left\langle \Delta x(t)' \frac{\partial^2 H(\psi,x,u,\tau)}{\partial x^2}, \Delta x(t)\right\rangle dt =$$

$$= \int\limits_T \int\limits_T \Big\langle \Delta_u f(x,u,\tau) \Big[\Phi^{-1'}(\tau)\,\Phi^{-1}(t)'\,B'\,(A+B\Phi(t_1))^{-1'} \times$$

$$\times \int\limits_T \Phi(t)' \frac{\partial H(\psi,x,u,t)}{\partial x}\Phi(t)\Big\rangle d\tau\,(A+B\Phi(t_1))^{-1} B\Phi(t_1)\times$$

$$\times\Phi^{-1}(s) + \Phi^{-1'}(\tau) \int\limits_{\max(\tau,s)}^{t_1} \Phi'(t)\frac{\partial^2 H(\psi,x,u,t)}{\partial x^2}\Phi(t)\,dt\Phi^{-1}(s) -$$

$$-\Phi^{-1'}(\tau)\int\limits_{\tau}^{t_1}\Phi'(\xi)\frac{\partial^2 H(\psi,x,u,\xi)}{\partial x^2}\Phi(\xi)\,d\xi\,(A+B\Phi(t_1))^{-1}B\Phi(t_1)\,\Phi^{-1}(s) -$$

$$-\Phi^{-1}(\tau)\,\Phi(t_1)'\,B'\,(A+B\Phi(t_1))^{-1'}\int\limits_{\tau}^{t_1}\Phi'(\xi)\times$$

$$\times\frac{\partial^2 H(\psi,x,u,\xi)}{\partial x^2}\Phi(\xi)\,d\xi\Phi^{-1}(s)\Big]\,ds, \Delta_u f(x,u,\tau) \tag{2.27}$$

Using $(2.23)-(2.27)$, following [4] we introduce the function

$$R(\tau,s) = \Phi^{-1'}(\tau)\left[\Phi(t_1)(A+B\Phi(t_1))^{-1}A\right]'\times$$

$$\times\frac{\partial^2\varphi}{\partial x(t_1)^2}\Phi(t_1)(A+B\Phi(t_1))^{-1}A\Phi^{-1}(s)+$$

$$+\Phi^{-1'}(\tau)\left[(A+B\Phi(t_1))^{-1}B\Phi(t_1)\right]'\times$$

$$\times\frac{\partial^2\varphi}{\partial x(t_0)^2}(A+B\Phi(t_1))^{-1}B\Phi(t_1)\,\Phi^{-1}(s)+$$

$$+\Phi^{-1'}(\tau)\left[(A+B\Phi(t_1))^{-1}B\Phi(t_1)\right]'\times$$

$$\times\frac{\partial^2\varphi}{\partial x(t_0)\,\partial x(t_1)}\Phi(t_1)(A+B\Phi(t_1))^{-1}A\Phi^{-1}(s)+$$

$$+\Phi^{-1'}(\tau)\left[(A+B\Phi(t_1))^{-1}A\Phi(t_1)\right]'\times$$

$$\times\frac{\partial^2\varphi}{\partial x(t_0)\,\partial x(t_1)}\Phi(t_1)(A+B\Phi(t_1))^{-1}B\Phi^{-1}(s)+$$

$$+\Phi^{-1'}(\tau)\left[(A+B\Phi(t_1))^{-1}B\Phi(t_1)\right]'\int\limits_T\Phi'(t_1)\frac{\partial^2 H(\psi,x,u,t)}{\partial x^2}\times$$

$$\times\Phi(t)\,dt\,(A+B\Phi(t_1))^{-1}B\Phi(t_1)\,\Phi^{-1}(s)+$$

$$+\Phi^{-1\prime}(\tau)\int\limits_{\max(\tau,s)}^{t_1}\Phi'(t)\,\frac{\partial^2 H(\psi,x,u,\tau)}{\partial x^2}\Phi(t)\,dt\Phi^{-1}(s)-$$

$$-\Phi^{-1\prime}(\tau)\int\limits_{\tau}^{t_1}\Phi'(\xi)\,\frac{\partial^2 H(\psi,x,u,\xi)}{\partial x^2}\Phi(\xi)\,d\xi\,(A+B\Phi(t_1))^{-1}\times$$

$$\times B\Phi(t_1)\,\Phi^{-1}(s)-\Phi^{-1}(\tau)\left[(A+B\Phi(t_1))^{-1}B\Phi(t_1)\right]'\times$$

$$\times\int\limits_{\tau}^{t_1}\Phi'(\xi)\,\frac{\partial^2 H(\psi,x,u,\xi)}{\partial x^2}\Phi(\xi)\,d\xi\Phi^{-1}(s) \tag{2.28}$$

We use the Dirichlet formula [9, p. 204] and get

$$\int\limits_T\left\langle\Delta'_{\widetilde{u}}\frac{\partial H(\psi,x,u,\tau)}{\partial x},\Delta x(t)\right\rangle dt=\int\limits_T\int\limits_T\left\langle\Delta'_{\widetilde{u}}\frac{\partial H(\psi,x,u,t)}{\partial x}\times\right.$$

$$\left.\times\Phi(t)\,(A+B\Phi(t_1))^{-1}B\Phi(t_1)\,\Phi^{-1}(\tau)\,,\Delta_{\widetilde{u}}f(x,u,\tau)\right\rangle d\tau ds+$$

$$+\int\limits_T\int\limits_{\tau}^{t_1}\Delta'_{\widetilde{u}}\frac{\partial H(\psi,x,u,\tau)}{\partial x}\Phi(\tau)\,d\tau\Phi^{-1}(t)\,,\Delta_{\widetilde{u}}f(x,u,t)\,dt+\eta_6\ , \tag{2.29}$$

where

$$\eta_6=\int\limits_T\left\langle\Delta'_{\widetilde{u}}\frac{\partial H(\psi,x,u,\tau)}{\partial x},\eta_2(t)\right\rangle dt.$$

Using denotation (2.28) and transformation (2.29) for the increment of the functional we get the terminal formula

$$\Delta J(u)=J(\widetilde{u})-J(u)=$$

$$=-\frac{1}{2}\int\limits_T\int\limits_T\Delta_{\widetilde{u}}f'(x,u,\tau)\,R(\tau,s)\,\Delta_u f(x,u,s)\,d\tau ds-\int\limits_T\int\limits_T\left\langle\Delta'_{\widetilde{u}}\frac{\partial H(\psi,x,u,t)}{\partial x}\times\right.$$

$$\left.\times\Phi(t)\,(A+B\Phi(t_1))^{-1}B\Phi(t_1)\,\Phi^{-1}(\tau)\,,\Delta_{\widetilde{u}}f(x,u,\tau)\right\rangle d\tau-$$

$$-\int\limits_T\left\langle\int\limits_{\tau}^{t_1}\Delta'_{\widetilde{u}}\frac{\partial H(\varphi,x,u,\tau)}{\partial x}\Phi(\tau)\,d\tau\Phi^{-1}(t)\,,\Delta_{\widetilde{u}}f(x,u,t)\right\rangle dt-$$

$$-\int\limits_T\Delta_{\widetilde{u}}H(\psi,x,u,t)\,dt+\eta(u,\Delta u)\,,$$

here $\eta(u,\Delta u)$ is an expression that linearly depends on $\eta_i(t)$ $i=1-6$. (Because of its awkwardness we don't cite this expression).

3. Necessary optimality conditions. Let's consider an increment formula of the aim functional on needle-shaped variation of admissible control. As variation parametes we choose the point $\theta \in (t_0, t_1]$, the number $\varepsilon \in (0, \theta - t_0]$, the vector $v \in V$. Variation interval $(\theta - \varepsilon, \theta]$ wholly lies in T. We give the needle-shaped control variation $u = u(t)$ in the form:

$$\widetilde{u}(t) = u_\varepsilon(t) = \begin{cases} v \in V, & t \in (\theta - \varepsilon, \theta] \subset T, \ \varepsilon > 0 \\ u(t), & t \in T / (\theta - \varepsilon, \theta] \end{cases} \tag{3.1}$$

By we denote $x_\varepsilon(t)$ the solution of problem $(1.1), (1.2)$ corresponding to the control $u_\varepsilon(t) \in U$, $t \in T$. Let $\Delta x_\varepsilon(t) = x(t, u_\varepsilon) - x(t, u)$. Provided $\det(A + B) \neq 0$ we can prove that for the solution of the boundary value problem $(2.1), (2.2)$ the estimation is valid:

$$\|\Delta x_\varepsilon(t)\| \leq K\varepsilon, \quad t \in T, \quad K = const > 0. \tag{3.2}$$

Hence, it follows

$$\Delta_\varepsilon x(t) = x(t, u_\varepsilon) - x(t, u) \sim \varepsilon. \tag{3.3}$$

Taking into account (3.1) and the fact $\eta(u, \Delta u_\varepsilon) = o(\varepsilon^2)$, from formula (2.30) we get

$$\Delta_\varepsilon J(u) = J(u_\varepsilon) - J(u) = -\Delta_v H(\psi, x, u, \theta)\,\varepsilon -$$

$$-\varepsilon^2 \left[\frac{1}{2} \left\langle \Delta_v' f(x(\theta), u(\theta), \theta) R(\theta, \theta),\ \Delta_v f(x(\theta), u(\theta), \theta) \right\rangle + \right.$$

$$\left. + \left\langle \Delta_v \frac{\partial H(\psi(\theta), x(\theta), u(\theta), \theta)}{\partial x}, \Delta_v f(x(\theta), u(\theta), \theta) \right\rangle \right] + o(\varepsilon^2) \tag{3.4}$$

The following theorem immediately follows from (3.4).

Theorem 1. (Pontryagin's maximum principle). *Let $\{u^*, x^*\}$ be an optimal process in the problem* $(1.1) - (1.4)$. *Then this process satisfies almost everywhere on T the condition*

$$\Delta_v H(\psi^*(\theta), x^*(\theta), u^*(\theta), \theta) \leq 0 \tag{3.5}$$

for all $v \in V$, $\theta \in [t_0, t_1]$ is an arbitrary tame point of the control $u(t)$.

Inequality (3.5) is Pontryagin's maximum principle for the optimal control problem (1.1) − (1.4) and is the first order necessary optimality condition. This condition gives restricted information on controls that are suspicious for optimality. There are cases when condition (3.5) is fulfilled in a trivial way, i.e. it degenerates. In these cases it is desirable to have new optimality conditions allowing to reveal non-optimality of those admissible controls for which the Pontryagin's maximum principle degenerates.

Definition. *The admissible control $u(t)$ is said to be singular in the Pontryagin's maximum principle sense, if for all $v \in V$, $\theta \in [t_0, t_1]$*

$$\Delta_v H(\psi(\theta), x(\theta), u(\theta), \theta) = 0 \tag{3.6}$$

Fulfilment of (3.6) makes necessary to get second order optimality conditions.

Now let's get second order necessary optimality condition when Pontryagin's maximum principle degenerates.

Theorem 2. *For the optimality of the singular control* $u(t)$ *in the problem* $(1.1-1.4)$ *the inequality should be fulfilled*

$$\frac{1}{2}\left\langle \Delta_v' f\left(x\left(\theta\right), u\left(\theta\right), \theta\right) R\left(\theta, \theta\right),\ \Delta_v f\left(x\left(\theta\right), u\left(\theta\right), \theta\right)\right\rangle +$$

$$+\left\langle \Delta_v \frac{\partial H\left(\psi\left(\theta\right), x\left(\theta\right), u\left(\theta\right), \theta\right)'}{\partial x}\Phi\left(\theta\right)\left[A+B\Phi\left(t_1\right)\right]^{-1}\times\right.$$

$$\left.\times B\Phi\left(t_1\right)\Phi^{-1}\left(\theta\right), \Delta_v f\left(x\left(\theta\right), u\left(\theta\right), \theta\right)\right\rangle +$$

$$+\left\langle \Delta_v \frac{\partial H\left(\psi\left(\theta\right), x\left(\theta\right), u\left(\theta\right), \theta\right)}{\partial x}, \Delta_v f\left(x\left(\theta\right), u\left(\theta\right), \theta\right)\right\rangle \leq 0$$

for all $v \in V$, $\theta \in [t_0, t_1]$.

The proof of the theorem follows from the equality (3.4), following $[2-4]$.

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