MECHANICS

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ON A MODEL OF DELAYED FRACTURE OF INTERNAL PRESSURE ANNULAR DOMAIN

Abstract

In the paper the process of diffuse fracture of thick tube under internal pressure in the presence of damageability property of tube material on the basis of experimental observations, is modeled as two-phase. Two fracture fronts sequentially moving one after another are formed correspondigly to this representation. A system of equations describing motion of these fronts is constructed. Numerical illustration has been considered.

Numerous problems in studying mechanical processes in rock masses surrounding mining generation are reduced to the investigations problems of stressed-deformed state of an annular domain for various rheological models of a material of this domain, and also researches of its load-carrying ability. Presence of axial symmetry in rock mass results to axially symmetric problem, rock masses in the vicinity of vertical borehole of circular cross-section may serve as an example. In the presence of planar symmetry in rock masses the plane problem is considered under plane deformation, that is typical for mechanical processes research in the vicinity of horizontal driftings.

The problems of delayed resistance of thick pipes under action of internal pressure are also reduced to the problems of load-carrying capacity of annular domain.

As usual, the material of annular domain in loading process tends to formation and accumulation of various sort of defects united by one term - damageability. It initiates the process of stage-by-stage fracture connected with partial or full failure and crack formation. In this case, around the stress concentrator, what is the internal hole, the postlimit and full fracture domains are formed . At this, the process of transition from prelimit in postlimit state develops in time as the mechanical and strength characteristics decrease at the long-term application of load. So for example, under consideration of rock mass surrounding mining generation, rock destruction occurs in process of development of displacements to the side of mined-out space, that it is formed the domain of postlimit deformation, where load-carrying ability of rock mass decreases from maximal up to some residual load-carrying ability. Otherwise, the following three domains are formed in surrounding rock mass: the domain of ruin fracture adjoining to a workingcontour, where rocks are in limiting equilibrium state subject to their residual strength; following it deep into the mass the domain of postlimit deformation; other part of mass is the domain being in prelimit deformation state. Some authors, for example, consider the fourth domain of a limiting state being intermediate between the domain of ruin fracture and postlimit deformation.

In the paper the process of fracture of annular domain assumes sequential formation of three domains (fig. 1).

At the first stage two domains are formed: the domain of postlimit deformation, $R_c < R < R_2$ and the domain of postlimit deformation $R_0 < R < R_c$. In the domain of prelimit deformations it takes place the process of damageability with accompanying continuous redistribution of pressures that brings to size increase of postlimit deformation domain with residual strength. It is connected with movement of interface of the domain $R = R_c$ which we'll name the fracture front.The material behind of fracture front, also tests damageability which, however, can be neglected in view of its insignificance.

From some time the formation of the third domain with the minimal residual strength is possible. From this time there hold three domains of deformations (fig. 2), these are the domain of prelimit deformations, $R_c < R < R_2$; the domain of postlimit deformation with intermediate residual strength, $R_c < R < R_1$, and the fracture domain with the minimal residual strength, $R_0 < R < R_1$.

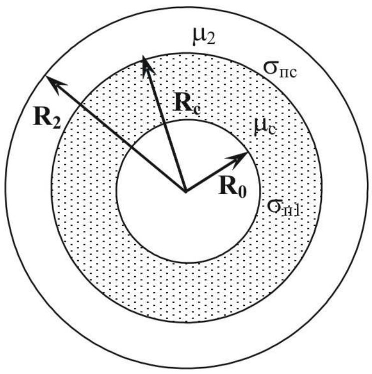

Fig.1.

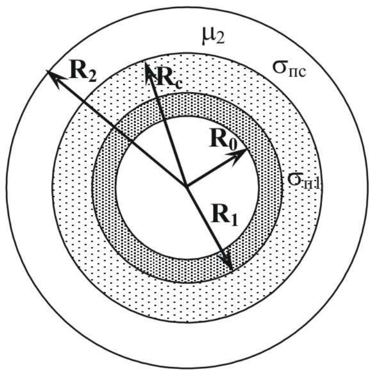

Fig.2.

Realization of two-region fracture (fig. 1) and of possible transition to three-regional fracture should be determined by corresponding conditions which should be specified. For the present paper they will be formulated below as the conditions imposed on a stress state level.

In connection with necessity of definition of stress state it is necessary to specify concrete deformationally strength model of medium. Here we accept as a basis the simplified variant of model of hereditary damaging body [1]. The determining equations of this model has the form:

$$\ni_{ij}= \frac{1}{\mu}(1 + M^*)S_{ij},\ \varepsilon = \frac{\sigma}{3K}, \tag{1}$$

where $\ni_{ij}$ and S_{ij} are deviators of deformation and stress tensors, and ε and σ are their spherical parts; μ is modulus of rigidity; K is modulus of dilatation; M^* is integral operator of damageability, which for continuously steadily changing mode of deformation has the form of usual integral operator of viscoelasticity

$$M^* f(t) = \int\limits_0^t M(t-\tau) f(\tau) d\tau. \tag{2}$$

Similar representation strongly simplifies problem of definition of stress state since enables to use for this Volterr-Rabotnov correspondence principle .

The fracture criterion has the form:

$$(1 + M^*)\,\sigma_u = \sigma_n, \tag{3}$$

where σ_u is stress intensity, σ_n is ultimate strength.

Let's pass to the formulation of mathematical problems for the both of the above mentioned stages.

Problem of the first stage. The annular domain $R_0 < R < R_2$ of hereditary damaged material satisfying deformation equations (1) and fracture criterion (2) is given. Thus on internal contour there the uniformly distributed pressure P_0 operates, and the external contour is free from efforts. Then there takes place the axially symmetric case and stresses (for plane deformation), on the basis of Volterra-Rabotnov correspondence principle, are represented in the form [2]:

$$\begin{cases} \sigma_r = \dfrac{R_0^2 P_0}{R_2^2 - R_0^2}\left(1 - \dfrac{R_2^2}{R^2}\right); \\[2ex] \sigma_\theta = \dfrac{R_0^2 P_0}{R_2^2 - R_0^2}\left(1 + \dfrac{R_2^2}{R^2}\right); \\[2ex] \sigma_z = \dfrac{R_0^2 P_0}{R_2^2 - R_0^2}. \end{cases} \tag{4}$$

Stress intensity defined by the formula

$$\sigma_u = \frac{1}{\sqrt{2}}\sqrt{(\sigma_r - \sigma_\theta)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_\theta - \sigma_z)^2} \tag{5}$$

will be

$$\sigma_u = \sqrt{3}\frac{R_0^2 R_2^2}{R_2^2 - R_0^2}\frac{P_0}{R^2} \tag{6}$$

The maximal value of stress intensity is achieved on internal contour where $R = R_0$, where the process of fracture will occur for the first time. Time of this fracture t_0 determining the incubatory period of latent fracture, we find from fracture criterion (3) represented in the form

$$(1 + M^*)\,\sigma_{u,\max} = \sigma_n. \tag{7}$$

Allowing for representations (2) and (6) in (7), we get:

$$\int_0^{t_0} M(\tau)\,d\tau = \frac{\sigma_n}{\sqrt{3}P_0}\left[1 - \left(\frac{R_0}{R_2}\right)^2\right] - 1. \tag{8}$$

Introducing dimensionless quantities:

$$P = \frac{\sqrt{3}P_0}{\sigma_n}; \qquad \frac{R_0}{R_2} = \beta_0 \tag{9}$$

we find

$$\int_0^{t_0} M(\tau)\,d\tau = \frac{1-\beta_0^2}{P} - 1. \tag{10}$$

In view of positivity of the left part, and according to this and the right part of formula (10) we have the following restriction on quantity of external load at which the scattered type of fracture takes place

$$P < P_*; \quad P_* = 1-\beta_0^2. \tag{11}$$

Otherwise, the fracture will take place instantly in the application of load.

Concretizing a kind of operator kernel of damageability, for example, supposing that $M(\tau) = m\tau^{-\alpha}$, $0 < \alpha < 1$ for t_0, we'll find

$$t_0 = \left[\frac{1-\alpha}{m}\left(\frac{1-\beta_0^2}{P} - 1\right)\right]^{\frac{1}{1-\alpha}} \tag{12}$$

For $t > t_0$ the extending destroyed annularlayer $R_0 < R < R_c$ or the domain of postlimit deformation with residual strength (fig. 1) is formed. On variable border between the domain of prelimit deformations and the domain of postlimit deformation the conditions:

$$u^{(2)} = u^{(c)}, \quad \text{for} \quad R = R_c, \tag{13}$$

$$(1+M^*)\,\sigma_u^{(2)} = \sigma_n, \quad \text{for} \quad R = R_c. \tag{14}$$

hold

Suppose that in the both of domains the state equations (1) and criterion of fracture (3) are true. At that, they have different kernels of damageability operators, instantaneous modulus of elasticity and ultimate strength.

In the domain of postlimit deformation we denote through μ_c the modulus of elasticity, $\sigma_{\Pi 1}$ the ultimate strength, $M_c(t)$ the kernel of the damageability operator. Let also q_c be contact pressure at the front of fracture $R = R_c$. Then for radial displacements and stresses in both domains, according to [2], we have:

$$\begin{cases} \sigma_r = \dfrac{R_c^2 q_c}{R_2^2 - R_c^2}\left(1 - \dfrac{R_2^2}{R^2}\right); \quad \sigma_\theta = \dfrac{R_c^2 q_c}{R_2^2 - R_c^2}\left(1 - \dfrac{R_2^2}{R^2}\right); \\ \sigma_z = \dfrac{R_c^2 q_c}{R_2^2 - R_c^2}; \quad R_c < R < R_2. \end{cases} \tag{15}$$

$$u_r = \frac{1}{2\mu}(1+M^*)\,\frac{R_c^2 R_2^2}{R_2^2 - R_c^2}\,\frac{q}{R}; \quad R_c < R < R_2 \tag{16}$$

$$\begin{cases} \sigma_r = \dfrac{R_0^2 P_0 - R_c^2 q_c}{R_c^2 - R_0^2} + \dfrac{R_0^2 (q_c - P_0)}{R_c^2 - R_0^2}\dfrac{R_c^2}{R^2}; \\ \sigma_\theta = \dfrac{R_0^2 P_0 - R_c^2 q_c}{R_c^2 - R_0^2} - \dfrac{R_0^2 (q_c - P_0)}{R_c^2 - R_0^2}\dfrac{R_c^2}{R^2}; \\ \sigma_z = \dfrac{R_0^2 P_0 - R_c^2 q_c}{R_c^2 - R_0^2}; \quad R_0 < R < R_c. \end{cases} \tag{17}$$

$$u_r = \frac{1}{2\mu}\left(1 + M^*\right)\frac{R_0^2 R_c^2}{R_c^2 - R_0^2}\frac{P_0 - q_c}{R}; \quad R_0 < R < R_c \tag{18}$$

Stress intensities according to (5) will be the following

$$\sigma_u = \sqrt{3}\frac{R_c^2 R_2^2}{R_2^2 - R_c^2}\frac{q_c}{R^2}; \quad \text{for} \quad R_c < R < R_2 \tag{19}$$

$$\sigma_u = \sqrt{3}\frac{R_0^2 R_c^2}{R_c^2 - R_0^2}\frac{P_0 - q_c}{R^2}; \quad \text{for} \quad R_0 < R < R_c \tag{20}$$

Taking into account representation for the radial displacements u (16) and (18) in condition (13), and also expressions for the stress intensities σ_u (19) in condition (14) we obtain the following system of two integral equations with regard to the unknown functions: $q_c(t)$ is the pressure at the fracture front, and $R_c(t)$ is the radial coordinate of fracture front :

$$\frac{1}{2\mu_c}\left\{\frac{R_0^2\left(P_0 - q_c(t)\right)R_c^2(t)}{R_c^2(t) - R_0^2} + \int_0^t M_c(t-\tau)\frac{R_0^2 R_c^2(\tau)}{R_c^2(\tau) - R_0^2}\frac{P_0 - q_c(\tau)}{R_c(t)}d\tau\right\} =$$

$$= \frac{1}{2\mu_2}\left\{\frac{R_c(t) R_2^2 q_c(t)}{R_2^2 - R_c^2(t)} + \int_0^t M(t-\tau)\frac{R_2^2 R_c^2(\tau)}{R_2^2 - R_c^2(\tau)}\frac{q_c(\tau)}{R_c(\tau)}d\tau\right\}; \tag{21}$$

$$\frac{R_2^2 q_c(t)}{R_2^2 - R_c^2(t)} + \int_0^t M(t-\tau)\frac{R_2^2 R_c^2(\tau)}{R_2^2 - R_c^2(\tau)}\frac{q_c(\tau)}{R_c^2(t)}d\tau = \frac{\sigma_n}{\sqrt{3}}. \tag{22}$$

Introduce the dimensionless quantities:

$$\chi_c = \frac{\mu_c}{\mu}; \quad \beta = \frac{R_c}{R_2}; \quad Q_c = \frac{q_c}{P_0}. \tag{23}$$

Then system of equations (21) has the form:

$$\frac{\beta_0^2}{\chi_c}\left\{\frac{1 - Q_c(t)}{\beta^2(t) - \beta_0^2} + \frac{1}{\beta^2(t)}\int_0^t M_c(t-\tau)\frac{\beta^2(\tau)}{\beta^2(\tau) - \beta_0^2}\left(1 - Q_c(\tau)\right)d\tau\right\} =$$

$$= \frac{Q_c(t)}{1 - \beta^2(t)} + \frac{1}{\beta^2(t)}\int_0^t M(t-\tau)\frac{\beta^2(\tau)}{1 - \beta^2(\tau)}Q_c(\tau)\,d\tau; \tag{24}$$

$$\frac{Q_c(t)}{1 - \beta^2(t)} + \frac{1}{\beta^2(t)}\int_0^t M(t-\tau)\frac{\beta^2(\tau)}{1 - \beta^2(\tau)}Q_c(\tau)\,d\tau = \frac{1}{P}. \tag{25}$$

Since the right hand side of (24) equals the left hand side of (25), equation (24) can be substituted by the following integral equation:

$$\frac{1 - Q_c(t)}{\beta^2(t) - \beta_0^2} + \frac{1}{\beta^2(t)}\int_0^t M_c(t-\tau)\frac{\beta^2(\tau)}{\beta^2(\tau) - \beta_0^2}\left(1 - Q_c(\tau)\right)d\tau = \frac{\chi_c}{\beta_0^2 P}. \tag{26}$$

Thus, in the first stage Volterra's two integral equations of second order (25) and (26) are simultaneously solved with respect to the two unknown functions $\beta(t)$ and $Q_c(t)$, the structures of which are defined in the following way:

$$\beta(t) = \begin{cases} \beta_0; \quad t \le t_0 \\ \beta(t) > \beta_0; \quad t > t_0 \end{cases}, \qquad Q_c(t) = \begin{cases} 1; \quad t \le t_0 \\ Q_c(t) < 1; \quad t > t_0 \end{cases} \tag{27}$$

where the time t_0 is defined by formula (10).

The time t_1 of first stage completion occurs when fracture criterion (3) is first held in the domain of postlimit deformation $R_0 < R < R_c$. Since the largest value of the stress intensities σ_u according to (20), and taking into account, that $q_c < P_0$, are reached on the internal countour $R = R_0$, then it holds for the first time. Taking into account accepted designations (9) and (23) fracture criterion (3) has the form:

$$\frac{\beta^2(t_1)}{\beta^2(t_1) - \beta_0^2}(1 - Q_c(t_1)) + \int_{t_0}^{t_1} M_c(t-\tau)\frac{\beta^2(\tau)}{\beta^2(\tau) - \beta_0^2}(1 - Q_c(\tau))\,d\tau = \frac{\sigma_{\Pi_1}}{\sqrt{3}P_0}, \tag{28}$$

where σ_{Π_1} is ultimate stress in the domain of postlimit deforamtion, residual strength, which is less than the ultimate stress σ_Π in the domain of prelimit deformation, therefore

$$\lambda = \frac{\sigma_\Pi}{\sigma_{\Pi_1}} > 1. \tag{29}$$

The right hand side of (28) will be

$$\frac{\sigma_{\Pi_1}}{\sqrt{3}P_c} = \frac{\sigma_\Pi}{\sqrt{3}P_0}\frac{\sigma_{\Pi_1}}{\sigma_\Pi} = \frac{1}{\lambda P}.$$

Equation (28) serves for determining the time t_1, and this equation is algebraic as the functions $\beta(t)$ and $Q(t)$ figuring there are found beforehand as solutions of the system of equations (25), (26).

The physically conscious conditions

$$1 > \beta(t) < \beta_0 \quad \text{and} \quad 0 < Q_c(t) < 1 \tag{30}$$

impose constraints on the values of initial parameters. In other words, the solution of the system of integral equations (25), (26) exists not for any combinations of problem's parameters but for such parameters which provide satisfiability of conditions (30).

On the first stage, when conditions (30) hold the calculation stops for the time $t = t_1$ determined according to (28), or when one of the conditions $\beta(t) \ge 1$ or $Q_c(t) \le 0$ holds. The last condition means apperance of arc microcrack. At this, if one of the conditions $\beta(t) \ge 1$ or $Q_c(t) \le 0$ holds for $t < t_1$ the second stage isn't realized.

For more clear understanding we consider the particular case, when we can neglect damageability in postlimit domain, i.e. we suppose $M_c(t-\tau) = 0$. Then from (26) we get:

$$Q_c(t) = \begin{cases} 1; \quad \text{for} \quad t \le t_0 \\ 1 - \dfrac{\chi_c}{P}\left[\dfrac{\beta^2(t)}{\beta_0^2} - 1\right] \quad \text{for } t_0 < t \le t_1 \end{cases} \tag{31}$$

In turn from (28) we find:

$$Q_c(t_1) = 1 - \frac{1}{\lambda P}\left(1 - \frac{\beta_0^2}{\beta^2(t_1)}\right) \qquad (32)$$

Comparing this expression with the expression of formula (31), we find the value of the dimensionless coordinate $\beta(t_1)$ determining position of fracture front at the end of first stage

$$\beta^2(t_1) = \frac{\beta_0^2}{\lambda \chi_c}. \qquad (33)$$

Naturally that for the second stage realization it is necessary $\beta(t_1) < 1$, and besides according to the structure of the function $\beta(t)$ there takes place $\beta(t_1) > \beta_0$. Then from formula (33) we have the following limitation on load and strength characteristics:

$$\beta_0^2 < \lambda \chi_c < 1. \qquad (34)$$

If $\lambda \chi_c < \beta_0^2$, the process of fracture consists only of the first stage.

Thus, for $M_c(t - \tau) = 0$ the problem consists of solution of integral equation (25), where the function $Q_c(t)$ is defined according to (31). Introducing (31) in (25) allowing for the denotation

$$y(t) = \beta^2(t), \qquad (34a)$$

we obtain the explicit solving equation:

$$\frac{1}{1 - y(t)}\left[1 - \frac{\chi_c}{P}\left(\frac{y(t)}{\beta_0^2} - 1\right)\right] + \frac{1}{y(t)}\int_0^t M(t - \tau)\frac{y(\tau)}{1 - y(\tau)} \times$$

$$\times \left[1 - \frac{\chi_c}{P}\left(\frac{y(\tau)}{\beta_0^2} - 1\right)\right] d\tau = \frac{1}{P}. \qquad (34b)$$

Denote

$$\varphi(y(t); y(\tau)) = \frac{y(\tau)}{y(t)(1 - y(\tau))}\left[1 - \frac{\chi_c}{P}\left(\frac{y(\tau)}{\beta_0^2} - 1\right)\right]. \qquad (34c)$$

The equation (34b) is written in the standard form

$$f(y(t)) + \int_0^t M(t - \tau)\varphi(y(t); y(\tau))\, d\tau = g, \qquad (34d)$$

where

$$f(y(t)) = \varphi(y(t); y(t)); \quad g = \frac{1}{P}, \qquad (34e)$$

to which we apply the numerical method of solution [3].

For fulfilment of qualitative picture of process we assume $M(t - \tau) = m = const$ and we consider the time t_0 dimensionless with regard to parameter m. Let's write equation (34b) in the following way:

$$\frac{y(t)}{1 - y(t)}\left[1 - \frac{\chi_c}{P}\left(\frac{y(t)}{\beta_0^2} - 1\right)\right] +$$

$$+\int_0^t \frac{y(\tau)}{1-y(\tau)}\left[1-\frac{\chi_c}{P}\left(\frac{y(\tau)}{\beta_0^2}-1\right)\right]d\tau=\frac{y(t)}{P}. \qquad (34f)$$

Differentiate this equation with respect to time and resolve the obtained one relative to derivative, then we get:

$$\frac{dy}{dt}=\frac{p(1-y)^2\left(\left(1+\frac{\chi_c}{p}\right)y-\frac{\chi_c}{py_0}y^2\right)}{p\left(\left(1+\frac{\chi_c}{p}\right)-2\frac{\chi_c}{py_0}y+\frac{\chi_c}{py_0}y^2\right)-(1-y)^2} \qquad (34g)$$

The initial condition for this differential equation follows from (12) for $\alpha=0$

$$t_0=\frac{1-y_0}{P}-1, \qquad (34h)$$

where $y_0=\beta_0^2$ is a specified quantity.

The solution $y(t)$ has the structure

$$y(t)=\begin{cases} y_0, & \text{for } t\le t_0 \\ y(t)>y_0, & \text{for } t>t_0 \end{cases} \qquad (34i)$$

The first order differential equation (34g) under initial condition (34h) and structure of solution (34i) is solved by the Runge-Kutta method. The solution holds untill the instant time t_1, for which

$$y(t_1)=\frac{y_0}{\lambda\chi_c}. \qquad (34j)$$

Problem of the second stage. On this stage after the beginning of fulfillement at the time t_1 fracture criterion (3) in the domain of postlimit deformation near the internal boundary there begins to be formed and extend the domain of fracture with residual ultimate strength (fig.2), $R_0<R<R_1$. In this case the process of fracture of annular domain consists of occurrence of contracting domain of prelimit deformation and two expanding alternating domains of postlimit deformation.

Let μ_1 be rigidity modulus of newly formed domain of postlimit deformation $R_0<R<R_1$, and $q_1(t)$ be pressure on second front of fracture $R=R_1$. We suppose that the damageablity process in the domain $R_0<R<R_1$ is exhausted and its material is pure elastic.

Stresses in tree domains mentioned in fig.2, stress intensities and radial displacements have the form [2]:

$$\begin{cases} \sigma_r=\frac{R_0^2P_0-R_1^2q_1}{R_1^2-R_0^2}+\frac{R_0^2(q_1-P_0)}{R_1^2-R_0^2}\frac{R_1^2}{R^2} \\ \sigma_\theta=\frac{R_0^2P_0-R_1^2q_1}{R_1^2-R_0^2}-\frac{R_0^2(q_1-P_0)}{R_1^2-R_0^2}\frac{R_1^2}{R^2} \\ \sigma_z=\frac{R_0^2P_0-R_1^2q_1}{R_1^2-R_0^2} \\ u_r=\frac{1}{2\mu_1}\frac{R_0^2(P_0-q_1)}{R_1^2-R_0^2}\frac{R_1^2}{R^2} \end{cases} \qquad (35)$$

$$\sigma_u = \sqrt{3}\frac{R_0^2 R_1^2}{R_1^2 - R_0^2}\frac{P_0 - q_1}{R^2}; \qquad R_0 < R < R_1.$$

$$\begin{cases} \sigma_r = \dfrac{R_1^2 q_1 - R_c^2 q_c}{R_c^2 - R_1^2} + \dfrac{R_1^2 (q_c - q_1)}{R_c^2 - R_1^2}\dfrac{R_c^2}{R^2}; \quad \sigma_z = \dfrac{R_1^2 q_1 - R_c^2 q_c}{R_c^2 - R_1^2} \\ \sigma_\theta = \dfrac{R_1^2 q_1 - R_c^2 q_c}{R_c^2 - R_1^2} - \dfrac{R_1^2 (q_c - q_1)}{R_c^2 - R_1}\dfrac{R_c^2}{R^2}; \quad \sigma_u = \sqrt{3}\dfrac{R_1^2 R_c^2}{R_c^2 - R_1^2}(q_c - q_1) \\ u_r = \dfrac{1}{2\mu_c}(1 + M_c^*)\dfrac{R_1^2 (q_1 - q_c)}{R_c^2 - R_1^2}\dfrac{R_c^2}{R}; \qquad R_1 < R < R_c \end{cases} \tag{36}$$

$$\begin{cases} \sigma_r = \dfrac{R_c^2 q_1}{R_2^2 - R_r^2}\left(1 - \dfrac{R_2^2}{R^2}\right); \quad \sigma_\theta = \dfrac{R_c^2 q_c}{R_2^2 - R_c^2}\left(1 + \dfrac{R_2^2}{R^2}\right) \\ \sigma_z = \dfrac{R_c^2 q_c}{R_2^2 - R_c^2}; \quad \sigma_u = \sqrt{3}\dfrac{R_c^2 R_2^2}{R_2^2 - R_c^2}\dfrac{q_c}{R^2} \\ u_r = \dfrac{1}{2\mu}(1 + M^*)\dfrac{R_c^2 q_c}{R_2^2 - R_c^2}\dfrac{R_2^2}{R}; \qquad R_c < R < R_2 \end{cases} \tag{37}$$

On the fracture fronts $R = R_c$ and $R = R_1$ the conditions of continuity of radial displacements and fracture criterion are fulfilled.

$$\begin{cases} u^{(2)} = u^c \quad \text{for} \quad R = R_c \\ (1 + M^*)\,\sigma_u^{(2)} = \sigma_\Pi \quad \text{for} \quad R = R_c \\ u^{(c)} = u^{(1)} \quad \text{for} \quad R = R_1 \\ (1 + M_c^*)\,\sigma_u^{(c)} = \sigma_{\Pi_1} \quad \text{for} \quad R = R_1 \end{cases} \tag{38}$$

Allowing for representations (35)-(37) hence we obtain the system of integrla equations with respect to four functions: radial coordinates of fracture fronts $R_c(t)$ and $R_1(t)$, and also the pressures $q_c(t)$ and $q_1(t)$ on them:

$$\frac{1}{2\mu_1}\frac{R_0^2 (P_0 - q_1(t)) R_1(t)}{R_1^2(t) - R_0^2} = \frac{1}{2\mu_c}\left\{\frac{R_1(t) R_c^2(t)[q_1(t) - q_c(t)]}{R_c^2(t) - R_1^2(t)} + \right.$$

$$\left. + \frac{1}{R_1(t)}\int_0^t M_c(t - \tau)\frac{R_1^2(\tau) R_c^2(\tau)[q_1(\tau) - q_c(\tau)]}{R_c^2(\tau) - R_1^2(\tau)}d\tau\right\};$$

$$\frac{1}{2\mu_c}\left\{\frac{R_1^2(t) R_c(t)[q_1(t) - q_c(t)]}{R_c^2(t) - R_1^2(t)} + \frac{1}{R_1(t)}\int_0^t M_c(t - \tau) \times \right.$$

$$\left. \times \frac{R_1^2(\tau) R_c^2(\tau)[q_1(\tau) - q_c(\tau)]}{R_c^2(\tau) - R_1^2(\tau)}d\tau\right\};$$

$$= \frac{1}{2\mu}\left\{\frac{R_c(t) q_c(t) R_2^2}{R_2^2 - R_c^2(t)} + \frac{R_2^2}{R_c(t)}\int_0^t M(t - \tau)\frac{R_c^2(\tau) q_c(\tau)}{R_2^2 - R_c^2(\tau)}\right\}d\tau;$$

$$\frac{R_c^2(t)[q_1(t) - q_c(t)]}{R_c^2(t) - R_1^2(t)} + \frac{1}{R_1^2(t)}\int_0^t M_c(t - \tau)\frac{R_1^2(\tau) R_c^2(\tau)[q_1(\tau) - q_c(\tau)]}{R_c^2(\tau) - R_1^2(\tau)}d\tau = \frac{\sigma_{\Pi_1}}{\sqrt{3}};$$

$$\frac{R_2^2 q_c(t)}{R_2^2 - R_c^2(t)} + \frac{R_2^2}{R_c^2(t)} \int_0^t M(t-\tau) \frac{R_c^2(\tau) q_c(\tau)}{R_2^2 - R_c^2(\tau)} d\tau = \frac{\sigma_\Pi}{\sqrt{3}} \tag{39}$$

Alongside with before intoduced dimensionless quantities we also introduce the following:

$$\chi_1 = \frac{\mu_1}{\mu_c}; \quad Q_1(t) = \frac{q_1(t)}{P}; \quad \alpha(t) = \frac{R_1(t)}{R_2}. \tag{40}$$

Then in dimensionless quantities the system of determining equations (39) has the form:

$$\begin{cases} \frac{1}{\chi_1} \frac{\beta_0^2 (1 - Q_1(t))}{\alpha^2(t) - \beta_0^2} = \frac{\beta^2(t) [Q_1(t) - Q_c(t)]}{\beta^2(t) - \alpha^2(t)} + \\ + \frac{1}{\alpha^2(t)} \int_0^t M_c(t-\tau) \frac{\alpha^2(\tau) \beta^2(\tau)}{\beta^2(\tau) - \alpha^2(\tau)} [Q_1(\tau) - Q_c(\tau)] d\tau; \\ \frac{1}{\chi_c} \left\{ \frac{\alpha^2(t) [Q_1(\tau) - Q_c(\tau)]}{\beta^2(t) - \alpha^2(t)} + \frac{1}{\beta^2(t)} \int_0^t M_c(t-\tau) \times \right. \\ \left. \times \frac{\alpha^2(\tau) \beta^2(\tau) [Q_1(\tau) - Q_c(\tau)]}{\beta^2(\tau) - \alpha^2(\tau)} d\tau \right\} = \\ = \frac{Q_c}{1 - \beta^2(t)} + \frac{1}{\beta^2(t)} \int_0^t M_c(t-\tau) \frac{\beta^2(\tau) Q_c(\tau)}{1 - \beta^2(\tau)} d\tau; \\ \frac{\beta^2(t)}{\beta^2(t) - \alpha^2(t)} [Q_1(\tau) - Q_c(\tau)] + \\ \frac{1}{\alpha^2(t)} \int_0^t M_c(t-\tau) \frac{\alpha^2(\tau) \beta^2(\tau)}{\beta^2(\tau) - \alpha^2(\tau)} [Q_1(\tau) - Q_c(\tau)] d\tau = \frac{1}{\lambda P}; \\ \frac{Q_c(t)}{1 - \beta^2(t)} + \frac{1}{\beta^2(t)} \int_0^t M_c(t-\tau) \frac{\beta^2(\tau) Q_c(\tau)}{1 - \beta^2(\tau)} d\tau = \frac{1}{P}; \end{cases} \tag{41}$$

Substituting the right hand side of first equation of system (41) by equal to it the right hand side of third equation of this system, and the right hand side of second equation by right hand side of fourth equation, the first two equations of system (41) are rewritten in the form:

$$\frac{1}{\chi_1} \frac{\beta_0^2}{\alpha^2(t) - \beta_0^2} [1 - Q_1(t)] = \frac{1}{\lambda P} \tag{42}$$

$$\frac{\alpha^2(t)}{\beta^2(t)} = \lambda \chi_c. \tag{43}$$

From equation (42) we express the pressure $Q_1(t)$ on the second front of fracture in terms of coordinate $\alpha(t)$ of this front:

$$Q_1(t) = 1 - \frac{\chi_1}{\lambda P} \left[\frac{\alpha^2(t)}{\beta_0^2} - 1 \right]. \tag{44}$$

Equation (43) define the connection between the cordinates of fracture fronts

$$\alpha(t) = \sqrt{\lambda\chi_c} \cdot \beta(t). \tag{45}$$

Since $\alpha(t) < \beta(t)$, then

$$\lambda\chi_c < 1, \tag{46}$$

that agrees with condition (34).

Substitute (45) into (44), then we find

$$Q_1(t) = 1 - \frac{\chi_1}{\lambda P}\left[\frac{\lambda\chi_c}{\beta_0^2}\beta^2(t) - 1\right]. \tag{47}$$

Since $\beta_0 \leq \alpha(t) \leq \sqrt{\lambda\chi_c}$, that is possible in view of limitation (34), from condition $Q_1(t) > 0$ it follows the limitation on load P

$$P > P_{**}; \qquad P_{**} = \frac{\chi_1}{\lambda}\left[\frac{\lambda\chi_c}{\beta_0^2} - 1\right]. \tag{48}$$

Possibility of simultaneous fulfilment of conditions (11) and (48) impose constraints on values of initial parameters. Taking into account (45) and (47) in third equation of system (41) we get:

$$\frac{\chi_0\chi_c}{P\beta_0^2}\beta^2(t) + Q_c(t) - \frac{1}{\beta^2(t)}\int_0^t M_c(t-\tau)\beta^2(\tau) \times$$

$$\times\left[\frac{\chi_0\chi_c}{P\beta_0^2}\beta^2(\tau) + Q_c(\tau) - \left(1 + \frac{\chi_1}{\lambda P}\right)\right]d\tau = 1 + \frac{\chi_1 + \lambda\chi_c - 1}{\lambda P}; \tag{49}$$

This equation jointly with fourth equation of system (41) composes the system of two Valterra integral equation of seconf order for determination of two unknown function $\beta(t)$ and $Q_c(t)$ on the second stage under limitations (11), (48) and (34). Here the calculation are carried out from the time t_1 to the time, when either $Q_c \leq 0$, or $Q_1 \leq 0$, or $\beta(t) \geq 1$.

Let's analyse the situation of the second stage for particular case, when we can neglect damageability in the first domain of postlimit deformation, i.e. suppose $M_c(t-\tau) = 0$. Then from (49) we find:

$$Q_c(t) = 1 - \frac{1}{\lambda P}\left\{1 - \chi_1 - \lambda\chi_c + \frac{\lambda\chi_1\chi_c}{\beta_0^2}\beta^2(t)\right\}. \tag{50}$$

Taking into account this expression in the last equation of system (41) we obtain the second order Valterra integral equation for determination of the function $\beta(t)$

$$\frac{1}{1-\beta^2(t)}\left\{1 - \chi_1 - \lambda\chi_c + \frac{\lambda\chi_1\chi_c}{\beta_0^2}\beta^2(t)\right\} +$$

$$+\frac{1}{\beta^2(t)}\int_0^t M_c(t-\tau)\frac{\beta^2(\tau)}{1-\beta^2(\tau)}\times$$

$$\times\left\{1-\frac{1}{\lambda P}\left[1-\chi_1-\lambda\chi_c+\frac{\lambda\chi_1\chi_c}{\beta_0^2}\beta^2(\tau)\right]\right\}d\tau=\frac{1}{P}. \qquad (51)$$

Introduce the denotation $x(t)=\beta^2(t)$, and also denote

$$\varphi[x(t);x(\tau)]=\frac{x(\tau)}{x(t)(1-x(t))}\left\{1-\frac{1}{\lambda P}\left[1-x_1-\lambda x_c+\frac{\lambda\chi_1 x_c}{\beta_0^2}x(\tau)\right]\right\}. \qquad (51a)$$

Then integral equation (51) is represented in form of (34d), (34e).

Knowing $\beta(t)$, next by formulae (47) and (50) we define the pressures on fronts of fracture, and by formula (45) position of secondary fracture front is defined.

For detection of qualitative side of process we suppose, as for the first stage, $M(t-\tau)=m=const$, and also we'll consider that time is nondimensional relative to the parameter m, i.e. $t_{nond}=mt$. However, we continue to denote time as t being condidered already nondimensional.

Equation (51a) is rewritten in the following way:

$$\frac{x(t)}{1-x(t)}\left\{1-\frac{1}{\lambda P}\left[1-\chi_1-\lambda\chi_c+\frac{\lambda\chi_1\chi_c}{\beta_0^2}x(t)\right]\right\}+$$

$$+\int_0^t\frac{x(\tau)}{(1-x(\tau))}\left\{1-\frac{1}{\lambda P}\left[1-\chi_1-\lambda\chi_c+\frac{\lambda\chi_1\chi_c}{\beta_0^2}x(\tau)\right]\right\}d\tau=\frac{x(t)}{P}. \qquad (52)$$

Differentiate this equation with respect to time and solve the obtained with regard to derivative of the function $x(t)$. Then we find:

$$\frac{dx}{dt}=\frac{p(1-x)\left(\frac{\chi_1\chi_c}{py_0}x^2-\left(1-\frac{1}{\lambda p}(1-\chi_1-\lambda\chi_c)\right)x\right)}{p\left(\left(1-\frac{1}{\lambda p}(1-\chi_1-\lambda\chi_c)\right)-2\frac{\lambda\chi_1\chi_c}{py_0}x+\frac{\lambda\chi_1\chi_c}{py_0}x^2\right)-(1-x)^2} \qquad (53)$$

The initial condition for this follows from (33).

$$x|_{t=t_1}=\frac{\beta_0^2}{\lambda\chi_c}, \qquad (54)$$

where t_1 is defined from algebraic equation (28), or for $M_c(t-\tau)$ again from (33), but in left hand side thre is the function $\beta(t)$ is found for the first stage, and wchich is denoted as $\beta^2(t)=y(t)$. Then the time t_1 is found as the solution of implicit algebraic equation:

$$y|_{t=t_1}=\frac{\beta_0^2}{\lambda\chi_c}, \qquad (55)$$

We solve differential equation (54) with initial condition (55) by means of Runge-Kutta method. Calculations holds untill one of the condtions

$$x(t)\geq 1;\quad Q_c(t)\leq 0;\quad Q_1(t)\leq 0, \qquad (56)$$

will be fulfilled; where

$$\begin{cases} Q_c(t) = 1 - \dfrac{1}{\lambda P}\left[1 - \chi_1 - \lambda\chi_c + \dfrac{\lambda\chi_1\chi_c}{\beta_0^2}x(t)\right] \\ Q_1(t) = 1 - \dfrac{\chi_1}{\lambda P}\left(\dfrac{\lambda\chi_c}{\beta_0^2}x(t) - 1\right) \end{cases} \tag{57}$$

Let's accept the following values for the parameters

$$\beta_0 = 0,5; \qquad P = 0,4; \qquad \chi_1 = \chi_c = 0,1; \qquad \lambda = 0,8.$$

Thus calculation consists of the solution on the time interval $[t_0; t_1]$ of differential equation (34g) with boundary condition (34h) at structure of solution (34i) and differential equation (54) with boundary condition (55). The initial time t_1 for the second stage is defined from condition (34j).

Results of calculation defines the function $\beta(t)$ represented as

$$\beta(t) = \begin{cases} y_0; & 0 \le t \le t_0 \\ \sqrt{y(t)}; & t_0 < t \le t_1 \\ \sqrt{x(t)}; & t > t_1 \end{cases} \tag{58}$$

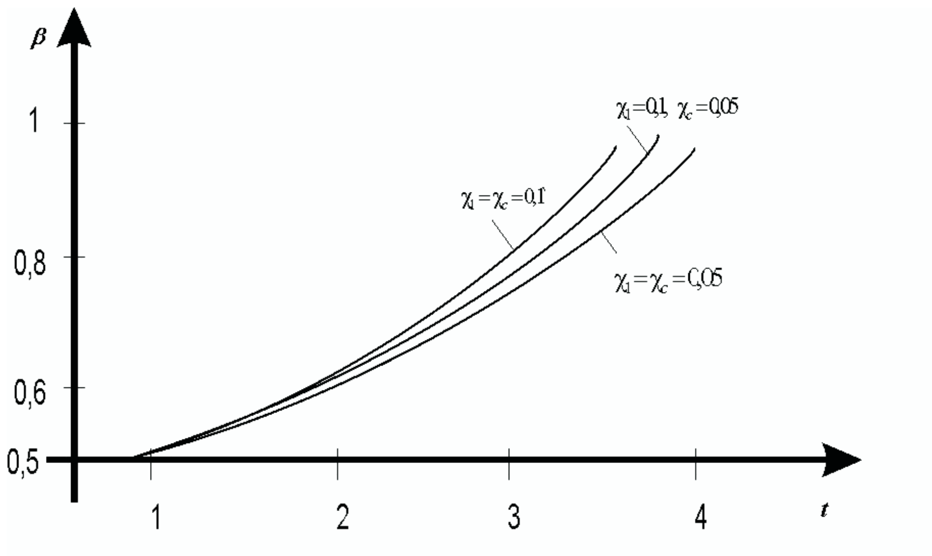

Fig.3.

Results of numerical calculation specifies the necessity to take into account occurrence of residual strength beyond the fracture front.

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