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ON RIEMANN PROBLEM IN VARIABLE SUMMABILITY INDEX HARDY CLASSES AND COMPLETENESS OF A SYSTEM OF EXPONENTS

Abstract

Some problems of the theory of conjugation problems in Hardy classes with variable summability index and their applications to the establishment of completeness of a system of elements with complex-valued coefficients are studied in appropriate Lebesgue spaces of functions.

Today there is a great interest to studying these or other problems in variable summability index Lebesgue spaces in connection with applications in different fields of mechanics and mathematical physics. The papers [1-4] consider such problems. Consideration of variable index Lebesgue spaces enables to introduce appropriate spaces of functions harmonic and analytic interior to a unit circle. Some aspects of these problems are considered in [5]. At the consequent stage it is necessary to study the problems of solvability, in other words Neother property of these or other conjunction problems of the theory of analytic functions in these spaces. Not introducing these spaces, the Neother property of the Riemann problem in the class of analytic functions represented by the Cauchy type integral with density from variable summability index Lebesgue space was shown in the paper [6].

In the present paper we'll determine correct statement of the Riemann problem in variable summability index Hardy classes and show the equivalence of completeness of exponents system and only trivial solvability of Riemann one-dimensional problem in appropriate spaces.

1. Give some introduction to the theory of variable summability index Lebesgue spaces. Let ρ: $[-\pi,\pi] \to [1,+\infty)$ be some measurable function. By $\mathcal{L}_0$ we denote a class of functions measurable on $[-\pi,\pi]$ (with respect to Lebesgue measure). For we accept $f \in \mathcal{L}_0$ we accept

$$I_p(f) \equiv \int_{-\pi}^{\pi} |f(x)|^{p(x)}\, dx$$

Let

$$\mathcal{L} \equiv \{f \in \mathcal{L}_0 :\ I_p(f) < +\infty\}.$$

For ordinary linear operations of addition of functions and multiplication by the number, $\mathcal{L}$ turns into a linear space. Let

$$\|f\|_{p_t} \equiv \inf\left\{\lambda > 0 :\ I_p\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

For the norm $\|\cdot\|_{p_t}$ $\mathcal{L}$ is a Banach space and we denote it by L_{p_t}. This space was first considered by W. Orlicz [1] and with norm $\|\cdot\|_{p_t}$- by H.Nakano [2].

Weight variant of this space is introduced as ordinarily, i.e. let ρ: $[-\pi,\pi] \to$ $\to [1,+\infty)$ be some weight function. Denote

$$\|f\|_{p_t,\rho_t} \equiv \|\rho f\|_{p_t}.$$

So, the linear space

$$L_{p_t,\rho_t} \equiv \left\{ f \in \mathcal{L}_0 : \ \|f\|_{p_t,\rho_t} < +\infty \right\}$$

for the norm $\|\cdot\|_{p_t,\rho_t}$ is a Banach space.

Moreover, for further statement we'll need the following class of functions determined on the segment $[-\pi,\pi]$:

$$\mathcal{L}_n \equiv \Big\{ f : \exists A > 0, \ \ \forall x_1, x_2 \in [-\pi,\pi], \ \ |x_1 - x_2| \le \tfrac{1}{2} \Longrightarrow \\ \Longrightarrow f(x_1) - f(x_2) \le \frac{A}{-\ln|x_1 - x_2|} \Big\}.$$

Everywhere $q(t)$ denotes a function conjugated to $p(t)$ in the sense $\dfrac{1}{p(t)} + \dfrac{1}{q(t)} \equiv 1$.

It should be noted that if $1 < p^- \le p^+ < +\infty$, $p^- \equiv ess \inf\limits_{(-\pi,\pi)} p(t)$, $p^+ \equiv ess \sup\limits_{(-\pi,\pi)} p(t)$, the space L_{p_t} coincides with the space of functions [3]:

$$\left\{ \{f \in \mathcal{L}_0 :\} \left| \int_{-\pi}^{\pi} f(t) g(t) dt \right| < +\infty, \forall g \in L_{q_t} \right\}$$

with equivalent norm

$$\|f\|_{p_t} \sim \sup_{\|g\|_{q_t} \le 1} |F(g)|,$$

where

$$F(g) = \int_{-\pi}^{\pi} f(t) g(t) dt$$

is a continuous functional on L_{p_t}. It holds Hölder generalized inequality

$$\int_{-\pi}^{\pi} |f(t)| \cdot |g(t)| \, dt \le C \|f\|_{p_t} \cdot \|g\|_{q_t},$$

where $C = 1 + \dfrac{1}{p^-} + \dfrac{1}{p^+}$. Moreover

$$(2\pi)^{\frac{1}{p^+}} \le \|1\|_{p_t} \le (2\pi)^{\frac{1}{p^-}}.$$

2. Variable summability index Hardy classes. Variable summability index Hardy spaces and Riemann boundary value problems of the theory of analytic functions in these spaces play an important role by investigating bases properties (completeness, minimality, basicity) of systems of exponents with complex-valued coefficients. Therefore, following the paper [5] we determine these spaces and cite some facts concerning these spaces.

Let $U \equiv \{z : |z| < 1\}$ be a unit ball on a complex plane and $\Gamma = \partial U$ be a unit circle. Assume that $U(z)$ is a function harmonic in U and accept

$$\|U\|_{h_{p_t}} \equiv \sup_r \left\|U(re^{it})\right\|_{p_t},$$

where $p(\cdot)$: $[-\pi, \pi] \to [1, +\infty)$ is a measurable function. Denote

$$h_{p_t} \equiv \left\{u : \triangle u = 0 \;\; in\ U, \quad \|u\|_{h_{p_t}} < +\infty\right\}.$$

For ordinary operations and the norm $\|\cdot\|_{h_{p_t}}$ the space h_{p_t} is a Banach space.

The inclusion

$$h_{p^+} \hookrightarrow h_{p_t} \hookrightarrow h_{p^-}$$

is obvious.

In the paper [7] it is proved the following

Theorem [7]. *Let $p(\theta)$: $1 < p^- \leq p^+ < +\infty$ be a measurable function. If $u \in h_{p_t}$, $\exists f \in L_{p_t}$:*

$$U(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(t) dt, \tag{1}$$

where $P_r(\alpha) = \dfrac{1 - r^2}{1 + r^2 - 2r\cos\alpha}$ is a Poisson kernel.

On the contrary, if $f \in L_{p_t}$ and $p \in \mathcal{L}_n$ the function (1) belongs to the space h_{p_t}.

The Banach space of functions $H^+_{p_t}$ analytic in U:

$$H^+_{pt} \equiv \left\{f : f \ anal.in\ U; \ \|f\|_{H^+_{p_t}} < \infty\right\},$$

where

$$\|f\|_{H^+_{p_t}} = \sup_{0<r<1} \left\|f(re^{it})\right\|_{p_t}$$

is introduced in a similar way.

Let $f = u + iv$. It is obvious that $f \in H^+_{p_t} \Leftrightarrow u;\ v \in h_{p_t}$. It is easy to prove the following refined variant of theorem 5 of paper [5].

Theorem 1. *Let $p \in \mathcal{L}_n$, $1 < p^- \leq p^+ < +\infty$. $F \in H^+_{p_t} \Leftrightarrow \exists f \in L_{p_t}$:*

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} f(t) dt}{e^{it} - z}.$$

In future, we'll need the following generalization of Smirnov's well known theorem:

Theorem [5]. *Let $p_i(t)$: $0 < p_i^- \leq p_i^+ < +\infty$ be measurable functions, $F \in H^+_{p_{1_t}}$ and $p_2 \in \mathcal{L}_n$, $p_2^- > 1$. Then $F \in H^+_{p_{2_t}}$.*

Following the classic case we determine a class of functions ${}_mH^-_{p_t}$, analytic outside of a circle, having order $\leq m$ at infinity.

Thus, let $f(z)$ be an analytic on $C\backslash\overline{U}$ (C is a complex plane, $U = U \cup \Gamma$) function having order $\leq m$, at infinity, i.e. $f(z) \equiv f_1(z) + f_2(z)$, where $f_1(z)$ is a polynomial of degree $\leq m$, $f_2(z)$ is tame part of expansion of $f(z)$ in Lorentz series in the vicinity of the point at infinity. If the function $\varphi(z) \equiv \overline{f_2\left(\frac{1}{\bar{z}}\right)}$ ($(\bar{\cdot})$ is a complex conjucation) belongs to the class $H^+_{p_t}$ we'll say that the function $f(z)$ belongs to the class ${}_mH^-_{p_t}$: $f \in_m H^-_{p_t}$.

3. Riemann problem in the classes $H^{\pm}_{p_t}$. Determine Riemann problem's statement in the classes $H^{\pm}_{p_t}$. Let a complex-valued on a segment $[-\pi, \pi]$ function satisfy $G(t)$ the following conditions:

1) The function $|G(t)|$ belongs to the space L_{r_t} for some r: $0 < r^- \leq r^+ < < +\infty$ and $|G(t)|^{-1} \in L_\omega$ for ω: $0 < \omega^- \leq \omega^+ < +\infty$.

2) The argument $\theta(t) \equiv \arg G(t)$ has an expansion of the form

$$\theta(t) = \theta_0(t) + \theta_1(t) + \theta_2(t),$$

where $\theta_0(t)$ is a function continuous on $[-\pi, \pi]$; $\theta_1(t)$ is a bounded variation function of $[-\pi, \pi]$; $\theta_2(t)$ is a measurable part on $[-\pi, \pi]$.

It is necessary to find a piece-wise analytic function $F^{\pm}(z)$ on a complex plane with section Γ satisfying the following conditions:

a) $F^+(z) \in H^+_{p_t}$: $0 < p^- \leq p^+ < +\infty$;

b) $F^-(z) \in_m H^+_{\nu_t}$: $0 < \nu^- \leq \nu^+ < +\infty$;

c) Non-tangential boundary values on a unit circle a.e. satisfy the relation:

$$F^+(e^{it}) - G(t)F^-(e^{it}) = g(t), \ a.e. \ t \in [-\pi, \pi],$$

where $g \in L_{p_t}$: $0 < \rho^- \leq \rho^+ < +\infty$ is some given function.

It should be noted that when summability indices are constants, the theory of such problems has been studied well. As regards these problems we can consider the monograph [8].

4. Reduction of completeness of a system of exponents with complex-valued coefficients to boundary value problems. Let's consider the following system of exponents

$$\left\{A(t)e^{int}; \ B(t)e^{-i(n+1)t}\right\}_{n\geq 0}, \tag{2}$$

where $A(t) \equiv |A(t)| e^{i\alpha(t)}$; $B(t) \equiv |B(t)| e^{i\beta(t)}$ are the complex-valued functions on $[-\pi, \pi]$. We'll consider the completeness of system (2) in the space

$$L_{p_t} : 1 < p^- \le p^+ < +\infty.$$

In this case, it is known that [4] the space conjugated to L_{p_t} is isometrically isomorphic to the space L_{q_t}: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Therefore the completeness of system (2) in L_{p_t} is equivalent to the equality of zero of any function $f(t)$ from the space L_{q_t}, for which there hold the relations:

$$\int_{-\pi}^{\pi} A(t) e^{int} \overline{f(t)} dt = 0; \quad \int_{-\pi}^{\pi} B(t) e^{-i(n+1)t} \overline{f(t)} dt = 0, \quad \forall n \ge 0 \tag{3}$$

Assume that the following main condition:

$$\operatorname*{ess\,sup}_{[-\pi,\pi]} \left\{ |A(t)|^{\pm 1}; \quad |B(t)|^{\pm 1} \right\} < +\infty \tag{4}$$

is fulfilled.

From the first equation of (3) we have:

$$\int_{-\pi}^{\pi} A(t) e^{int} \overline{f}(t) dt = \frac{1}{i} \int_{\Gamma} (\tau) \tau^n d\tau = 0, \quad \forall n \ge 0, \tag{5}$$

where

$$f^+(\tau) \equiv A(\arg \tau) \cdot \overline{f}(\arg \tau) \cdot \overline{\tau}, \quad \tau \in \Gamma.$$

It is obvious that $f^+(\tau) \in L_1(\Gamma)$. Then, it is known well that [9] (see p.205) conditions (5) equivalent to the existence of the function $F^+(z)$ from H_1^+, for which tangential boundary values on Γ coincide a.e. with $f^+(\tau)$: $F^+(\tau) =$
$= f^+(\tau)$, a.e. on Γ.

Similarly, from the second equality of (3) we have:

$$\int_{-\pi}^{\pi} \overline{B(t)} e^{-i(n+1)t} f(t) dt = \frac{1}{i} \int_{\Gamma} (\tau) \tau^n d\tau = 0, \quad \forall n \ge 0; \tag{6}$$

where

$$f^-(\tau) = \overline{B(\arg z)} \cdot f(\arg \tau), \quad \tau \in \Gamma.$$

By the above-stated reason equalities (6) are equivalent to the existence of the function $\Phi^+(z) \in H_1^+$ for which non-tangential boundary values $\Phi^+(z)$ on Γ a.e. coincides with $f^-(\tau)$: $\Phi^+(\tau) = f^-(\tau)$, a.e. on Γ. Obviously, $F^+(\tau)$; $\Phi^+(\tau) \in L_{q_t}(\Gamma)$. Consequently, if additionally we require $p(t) \in \mathcal{L}_n$ to hold, then from theorem [5] the inclusion $F^+(z);\ \Phi^+(z) \in H^+_{qt}$ follows. Expressing $f(t)$ by $F^+(\tau)$ and $\Phi^+(\tau)$ we get the following conjugation problem

$$F^+(\tau) - \frac{A(\arg \tau)}{B(\arg \tau)}\overline{\tau \cdot \Phi^+(\tau)} = 0. \quad \tau \in \Gamma.$$

Determine the following function $F^-(z)$ analytic outside of a unit circle:

$$F^-(z) = \frac{1}{z} \cdot \overline{\Phi^+\left(\frac{1}{\overline{z}}\right)}, \quad |z| > 1.$$

Obviously, $F^-(\infty) = 0$. Moreover $F^-(\tau) = \overline{\tau} \cdot \overline{\Phi^+(\tau)}$, $\tau \in \Gamma$. As a result we arrive at the following Riemann problem

$$\begin{cases} F^+(\tau) - G(\tau)F^-(\tau) = 0, & \tau \in \Gamma \\ F^-(\infty) = 0. \end{cases} \tag{7}$$

where $G(z) \equiv \dfrac{A(\arg \tau)}{B(\arg \tau)}$, $\tau \in \Gamma$. By definition we have $F^-(z) \in_1 H^-_{qt}$. Consequently, if system (2) is incomplete in L_{p_t}, the Riemann problem (7) is non-trivially solvable in the classes $(H^+_{qt};\ {}_1H^-_{qt})$.

Now, let's assume that problem (7) is non-trivially solvable in the classes $(H^+_{qt};\ {}_1H^-_{qt})$, i.e. $F^+(z) \in H^+_{qt};\ F^-(z) \in_1 H^-_{qt}$.

Determine: $\Phi^+_1(z) \equiv F^-\left(\dfrac{1}{z}\right)$ for $|z| < 1$.

We have: $F^-(\tau) = \overline{\Phi^+_1(\tau)}$, $\tau \in \Gamma$ and $\Phi^+(0) = 0$.

Then, clearly, the function $\Phi^+(z) = z^{-1}\Phi^+_1(z)$ will be analytic for $|z| < 1$, moreover $\Phi^+(z) \in H^+_{qt}$. Thus

$$F^+(\tau) - G(\tau)\overline{\tau \cdot \Phi^+(\tau)} = 0, \quad \tau \in \Gamma,$$

or

$$\frac{F^+(\tau)}{A(\arg \tau)\overline{\tau}} = \frac{\overline{\Phi^+(\tau)}}{B(\arg \tau)}, \quad \tau \in \Gamma.$$

Denote

$$f(t) = \frac{\overline{F^+(e^{it})}}{\overline{A(t)} \cdot e^{it}} = \frac{\Phi^+(e^{it})}{B(t)}.$$

Obviously, $f(t) \in L_{q_t}$. From $F^+(z);\ \Phi^+(z) \in H^+_1$ there follow the equalities

$$\int_\Gamma F^+(\tau)\,\tau^{-n}d\tau = 0; \quad \int_\Gamma \Phi^+(\tau)\,\tau^n d\tau = 0, \quad \forall n \geq 0.$$

Expressing $F^+(\tau)$ and $\Phi^+(\tau)$ by $f(\arg \tau)$ for $\tau \in \Gamma$ we have:

$$\int_{\Gamma} A(t)e^{-it}\overline{f(t)}e^{int}de^{it} = i\int_{-\pi}^{\pi} A(t)e^{int}\overline{f}(t)dt = 0, \quad \forall n \geq 0;$$

$$\int_{\Gamma} \overline{B(t)}f(t)e^{int}de^{it} = i\int_{-\pi}^{\pi} \overline{B(t)}e^{i(n+1)t}f(t)dt = 0, \quad \forall n \geq 0; \ i.e.$$

$$\int_{-\pi}^{\pi} B(t)e^{-ikt}\overline{f(t)}dt = 0, \quad \forall k \geq 0.$$

Obviously, $f(t) \neq 0$ on $[-\pi, \pi]$. Then these relations show that system (2) is incomplete in L_{p_t}. As a result we have the following theorem:

Theorem 2. *Let $p : 1 < p^- \leq p^+ < +\infty$, $p(t) \in \mathcal{L}_n$ and the complex-valued coefficients $A(t)$; $B(t)$ satisfy condition (4). Then the system of exponents (2) is complete in L_{p_t} only in the case if the Riemann problem (7) is only trivially solvable in the classes $(H^+_{q_t};\ {}_1H^-_{q_t})$.*

5. General solution of Riemann homogeneous problem in the classes $(H^+_{q_t};\ {}_mH^-_{q_t})$:

$$\begin{cases} F^+(\tau) - G(\tau)F^-(\tau) = 0, & \tau \in \Gamma, \\ F^+(z) \in H^+_{q_t}; & F^-(z) \in_m H^-_{q_t} \end{cases} \tag{8}$$

We'll investigate problem (8) by the method worked out in the monograph of I.I. Danilynk [8]. Introduce the following function $X^{\pm}_i(z)$ analytic inside (sign "+") and outside (sign "-") of a unit circle:

$$X^{\pm}_i(z) \equiv \exp\left\{\pm\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\left|G(e^{it})\right|\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X^{\pm}_2(z) \equiv \exp\left\{\pm\frac{1}{4\pi}\int_{-\pi}^{\pi} \theta(t)\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

where $\theta(t) \equiv \arg G(e^{it})$.

Determine:

$$Z_i(z) \equiv \begin{cases} X^{\pm}_i(z), & |z| < 1, \\ X^-_i(z)^{-1}, & |z| > 1. \end{cases}$$

The Sokhotskii-Plamel relations give:

$$\left|G(e^{it})\right| = \frac{Z^+_1(e^{it})}{Z^-_1(e^{it})}; \quad e^{i\theta(t)} = \frac{Z^+_2(e^{it})}{Z^-_2(e^{it})}.$$

Introducing the denotation $Z^{\pm}(z) \equiv Z^{\pm}_1(z)\cdot Z^{\pm}_2(z)$, we have

$$Z^+(\tau) - G(\tau)Z^-(\tau) = 0, \quad \tau \in \Gamma. \tag{9}$$

Following the classical case, we'll call the function $Z(z)$ a canonical solution of problem (8), where

$$Z(z) \equiv \begin{cases} Z^+(z), & |z| < 1, \\ Z^-(z), & |z| > 1. \end{cases}$$

Substituting the values of $G(\tau)$ from (9) to (8), we have:

$$\frac{F^+(\tau)}{Z^+(\tau)} = \frac{F^-(\tau)}{Z^-(\tau)}, \quad \tau \in \Gamma.$$

Let $\Phi^+(z) \equiv \dfrac{F^+(z)}{Z^+(z)}$ and

$$\Phi(z) \equiv \begin{cases} \Phi^+(z), & |z| < 1, \\ \Phi^-(z), & |z| > 1. \end{cases}$$

It is easy to notice that the function $Z(z)$ has no zeros and poles for $Z \notin \Gamma$. Therefore the functions $\Phi(z)$ and $F(z)$ have the same order at infinity. From the results of the monograph it directly follows that the function $\Phi(z)$ belongs to the Hardy class $H^{\pm}_{\delta}$ for sufficiently small $\delta > 0$. Show that $\Phi^{\pm}(\tau) \in L_1(\Gamma)$. The further one follows directly from Smirnov's theorem.

Obviously $F^-(e^{it}) \in L_{q_t}$. Therefore, in order to establish the inclusion $\Phi^-(\tau) \in L_1$, it suffices to show that $[Z^-(\tau)]^{-1} \in L_{p_t}$. Naturally, it doesn't always hold. Further we'll assume that the measurable part $\theta_2(t)$ of the function $\theta(t)$ equals zero and $\theta_1(t)$ has a finite number of discontinuity points on $[-\pi, \pi]$. Let

$\{s_k\}_1^r$: $-\pi < s_1 <...< s_r < \pi$ be discontinuity points and

$\{h_k\}_1^r$: $h_k = \theta(s_k + 0) - \theta(s_k - 0)$, $k = \overline{1, r}$; be appropriate jumps of the function $\theta(t)$ at these points. Denote

$$h_0 = \theta(-\pi) - \theta(\pi); \quad h_0^{(0)} = \theta_0(\pi) - \theta_0(-\pi).$$

Let

$$u_0(t) \equiv \left\{ \sin\left|\frac{t-\pi}{2}\right| \right\}^{-\frac{h_0^{(0)}}{2\pi}} \exp\left\{ -\frac{1}{4\pi} \int_{-\pi}^{\pi} \theta_0(\tau) ctg \frac{t-\tau}{2} d\tau \right\}.$$

Divide the set $\{h_k\}$ into two sets: a positive part $\{h_k^+\}$ and absolute values of the negative part $\{h_k^-\}$. Denote

$$U^{\pm}(t) = \prod_k . \left\{ \sin\left|\frac{t - s_k^{\pm}}{2}\right| \right\}^{-\frac{h_k^{\pm}}{2\pi}}.$$

As it follows from the results of the paper , $\left|Z_2^-(\tau)\right|$ is expressed by the relation

$$\left|Z_2^-(e^{it})\right| = U_0(t) \left[U^+(t)\right]^{-1} \cdot U^-(t) \cdot \left\{ \sin\left|\frac{t-\pi}{2}\right| \right\}^{-\frac{h_0}{2\pi}}.$$

It directly follows from Sokhotskii-Plemel relation, that

$$\sup_{(-\pi,\pi)} vrai \left\{ \left| Z_2^-(e^{it})^{\pm 1} \right| \right\} < +\infty.$$

Thus, for $\left|Z^-(e^{it})^{-1}\right|$ we have the representation:

$$\left|Z^-(e^{it})^{-1}\right| = \left|Z_2^-(e^{it})^{-1}\right| \cdot |U_0(t)|^{-1} \cdot \left|U^+(t)\right| \cdot \left|U^-(t)\right|^{-1} \cdot \left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{\frac{h_0}{2\pi}} . \quad (10)$$

In sequel we'll need the following properties of the space L_{p_t}.

Property A. Let $|f(t)| \leq |g(t)|$ a.e. on $(-\pi, \pi)$. Then $\|f(t)\|_{p_t} \leq \|g(t)\|_{p_t}$.

It directly follows from the definition of L_{p_t}.

It is easy to establish the validity of the statement:

Lemma 1. *Let $p(t) \in C[-\pi, \pi]$ and $p(t) > 0$, $\forall t \in [-\pi, \pi]$. Then the function $\omega(t) = |t - t_0|^\alpha$ belongs to the space L_{p_t}, if $\alpha > -\frac{1}{p(t_0)}$, where $t_0 \in \in [-\pi, \pi]$ and $\alpha \in R$ is some number.*

Thus, a final product $\prod\limits_k |t - t_0|^{\alpha_k}$ belongs to L_{p_t}, if the inequalities $\alpha_k > -\frac{1}{p(t_k)}$, are fulfilled; where $p(t)$ satisfies the conditions of lemma 1.

Now, pay attention to relation (10). By the results of the paper it holds the relation

$$\sup_{(-\pi,\pi)} vrai \left|U_0(t)^{\pm 1}\right| < +\infty.$$

Then, from the property A, it follows that product (10), i.e. the function $\left|Z^-(e^{it})^{-1}\right|$ belongs to the space L_{p_t}, if the inequalities

$$\left.\begin{array}{l} -\frac{h_k^-}{2\pi} > -\frac{1}{p(s_k^-)}, \quad \forall k; \\ \frac{h_0}{2\pi} > -\frac{1}{p(\pi)}. \end{array}\right\} \quad (11)$$

are fulfilled.

As a result we get that by fulfilling inequalities (11) the function $\Phi^\pm(e^{it})$ belongs to the space L_1 and consequently $\Phi(z) \in H_1^\pm$. Then, by a uniqueness theorem (lemma 19.1, p.194) is a polynomial $P_m(z)$ of degree $\leq m$,, and so

$$F(z) \equiv Z(z) \cdot P_m(z).$$

Clarify the conditions under which the boundary value $F^-(e^{it})$ belongs to the space L_{q_t}. Clearly $F^-(e^{it}) = Z^-(e^{it}) \cdot P_m(e^{it})$. But again, if we pay attention to relation (10) we get that by fulfilling the inequalities

$$\left.\begin{array}{l} -\frac{h_k^-}{2\pi} > -\frac{1}{q(s_k^+)}, \quad \forall k; \\ -\frac{h_0}{2\pi} > -\frac{1}{q(\pi)}. \end{array}\right\}$$

$F^-(e^{it})$ belongs to the space L_{q_t}. Obviously, $F(z) \in H_1^\pm$. Then from theorem it follows that $F^\pm(z) \in \left(H_{q_t}^+;\ {}_mH_{q_t}^-\right)$. Thus, we arrive at the following conclusion:

Theorem 3. *Let for the coefficient $G(e^{it})$ of problem (8) conditions 1), 2) for $r(t) \equiv \omega(t) \equiv q(t) \in \mathcal{L}_n$, $1 < q^- \leq q^+ < +\infty$ be fulfilled and the jumps $\{h_k\}_0^r$ of the function $\arg G(e^{it})$ satisfy the inequalities*

$$\left.\begin{array}{l} -\dfrac{1}{p(s_k)} < \dfrac{h_k}{2\pi} < \dfrac{1}{q(s_k)}, \quad k = \overline{1,r}; \\ -\dfrac{1}{p(\pi)} < \dfrac{h_0}{2\pi} < \dfrac{1}{q(\pi)}. \end{array}\right\} \tag{12}$$

Then a general solution of the Riemann problem (8) in the classes $\left(H_{qt}^+;\ {}_mH_{qt}^-\right)$ is of the form:

$$F(z) \equiv Z(z) \cdot P_m(z),$$

where $Z(z)$ is a canonical solution of a homogeneous problem, $P_m(z)$ is an arbitrary polynomial of degree $\leq m$.

The following Corollary follows directly from this theorem:

Corollary 1. *Let all the requirements of theorem 3 be fulfilled. Then provided $F^-(\infty) = 0$ the Riemann homogeneous (8) in the classes $\left(H_{qt}^+;\ H_{qt}^-\right)$ has only trivial solution $F^\pm(z) \equiv 0$.*

6. Completeness of a system of exponents (2) in the space L_{p_t}. Return to the completeness of system (2). Assume that the coefficients $A(t)$ and $B(t)$ satisfy conditions 1) and 2), where $G(e^{it}) \equiv \dfrac{A(t)}{B(t)}$. Besides, it is necessary to fulfill the conditions:

3) $\dfrac{h}{2\pi} + \dfrac{1}{p(s_k)} \notin Z$ (a set of integers), where h_k is a jump of the function $\theta(t) \equiv \alpha(t) - \beta(t)$ at the discontinuity point s_k. $k = \overline{0,r}$; $s_0 = \pi$.

Determine integers n_i, $i = \overline{1,r}$ from the inequalities:

$$\left\{\begin{array}{l} -\dfrac{1}{p(s_k)} < \dfrac{h_k}{2\pi} + n_k - n_{k-1} < \dfrac{1}{q(s_k)}, \quad k = \overline{1,r}; \\ n_0 = 0. \end{array}\right. \tag{13}$$

Let

$$\omega_r = \frac{1}{2\pi}\left[\alpha(-\pi) - \alpha(\pi) + \beta(\pi) - \beta(-\pi)\right] + n_r.$$

Using the scheme of the paper we easily prove the following final theorem on the completeness of system (2) in L_{p_t}.

Theorem 4. *Let the coefficients $A(t)$, $B(t)$ of system (2) satisfy conditions 1)–3), where $G(e^{it}) \equiv \dfrac{A(t)}{B(t)}$, the integer is determined from relations (13); $p(t) \in \in \mathcal{L}_n$. $1 < p^- \leq p^+ < +\infty$. Then, if the inequality $\omega_r > -\dfrac{1}{p(\pi)}$ holds, system (2) is complete in the space L_{p_t}.*

In fact, by the scheme of the paper the completeness problem is reduced to the case $n_i = 0$, $i = \overline{0,r}$. The further one follows from Theorem 2 having applied Corollary 1 to boundary value problem (7) in the classes $\left(H_{qt}^+;\ {}_0H_{qt}^-\right)$.

Apply the obtained theorem to the special case

$$\left\{e^{i[n+\alpha signn]t}\right\}_{n\in Z}, \tag{14}$$

where $\alpha \in C$ is a complex parameter. Base properties of system (14) in the spaces L_p have been studied well (see e.g.). From Theorem 4 we have

Corollary 2. *Let* $p(t) \in \mathcal{L}_n,\ 1 < p^- \leq p^+ < +\infty$ *and* $\operatorname{Re}\alpha \notin Z$ *be fulfilled. If* $\operatorname{Re}\alpha < \dfrac{1}{2p(\pi)}$, *system (14) is complete in* L_{p_t}.

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