Tunjay S. GUSEYNOV

ON BASICITY OF A UNITARY SYSTEM OF DEGENERATE COEFFICIENTS EXPONENTS

Abstract

In the paper we consider a problem on basicity of unitary system of exponents with complex-valued coefficients and degenerations in Lebesgue space of functions. Hilbert case $p = 2$ is considered separately.

We consider the following unitary system of exponents

$$\vartheta_n^{\pm}(t) \equiv a(t)\,\rho^{+}(t)\,e^{int} \pm b(t)\,\rho^{-}(t)\,e^{int}, \quad n \geq 1, \tag{1}$$

with complex-valued coefficients $a(t) \equiv |a(t)|\,e^{i\arg a(t)}$ and $b(t) \equiv |b(t)|\,e^{i\arg b(t)}$ on the segment $[0,\pi]$ where degenerate coefficients $\rho^{\pm}(t)$ are determined by the formula

$$\rho^{\pm}(t) \equiv \prod_{i=1}^{l^{\pm}} \left|t - \tau_i^{\pm}\right|^{\beta_i^{\pm}},$$

$\left\{\tau_i^{\pm}\right\}_{i=1}^{\pm}$: $0 < \tau_1^{\pm} < \tau_2^{\pm} < ... < \tau_{l^{\pm}}^{\pm} < \pi$; $\left\{\beta_i^{\pm}\right\}_1^{l^{\pm}} \subset R$ is some set.

Earlier the basicity of binary system of exponents

$$\left\{A(t)\,\mu^{+}(t)\,e^{int}; B(t)\,\mu^{-}(t)\,e^{-int}\right\}_{n\geq 1},$$

with complex-valued coefficients $A(t)$; $B(t)$ with degenerations $\mu^{\pm}(t)$ was considered in the S.G.Veliyev's paper [1]. We'll essentially use these results.

We make the following assumptions:

1) $\arg a(t)$, $\arg b(t)$ are piecewise-holder functions on $[0,\pi]$:$\{s_i\}_1^r$:$0 < s_i < ... < \pi$ are discontinuity points of the function $\theta_0(t) \equiv \arg a(t) - \arg b(t)$ on $(0,\pi)$;

2) $|a(t)|$, $|b(t)|$ are measurable on $(0,\pi)$ and it holds

$$\sup_{(0,\pi)} wrai \left\{|a(t)|^{\pm}; |b(t)|^{\pm}\right\} < +\infty;$$

3) The sets $T_0^{\pm} \equiv \left\{\tau_i^{\pm}\right\}_1^{l\pm}$ don't intersect: $T_0^{+} \cap T_0^{-} = \{\varnothing\}$.

So, we consider system (1). Define:

$$A(t) \equiv \begin{cases} a(t), & t \in [0,\pi], \\ b(-t), & t \in [-0,\pi]; \end{cases}, \quad B(t) \equiv A(-t).$$

$$\nu^{+}(t) \equiv \begin{cases} \rho^{+}(t), & t \in [0,\pi], \\ \rho^{-}(-t), & t \in [-0,\pi]; \end{cases}, \quad \nu^{-}(t) \equiv \nu^{+}(t),\ t \in [-\pi,\pi].$$

Alongside with (1) we consider the binary system

$$\left\{A(t)\,\nu^{+}(t)\,e^{int}; B(t)\,\nu^{-}(t)\,e^{-int}\right\}_{n\geq 0}. \tag{2}$$

When there is no degeneration, i.e. $\rho^{\pm}(t) \equiv 1$, in the paper [2] B.T.Bilalov established relation between basis properties (completeness, minimality, basicity) of the systems (1) and (2) in the spaces $L_p(0,\pi)$ and $L_p(-\pi,\pi)$, $1 \leq p < +\infty$, respectively. The following lemma is proved in the similar way.

Lemma 1. *Let $\beta_i^{\pm} > -\frac{1}{p}$, $\forall i = \overline{1, l^{\pm}}$. System (2) forms a basis in $L_p(-\pi,\pi)$ only if the systems $\left\{\vartheta_n^{+}(t)\right\}_{n\geq 0}$ and $\left\{\vartheta_n^{-}(t)\right\}_{n\geq 0}$ form bases in $L_p(0,\pi)$, $1 \leq p < +\infty$.*

We represent system (2) in the form

$$\left\{A_1^{+}(t)\,\nu^{+}(t)\,e^{int}; A_1^{-}(t)\,\nu^{-}(t)\,e^{-int}\right\}_{n\geq 0, k\geq 1},$$

where $A_1^{+}(t) = e^{it}A(t)$; $A_1^{-}(t) \equiv B(t)$, $t \in [-\pi,\pi]$.

Apply the results of the paper [1] to the basicity of system (3). Following the results of this paper we'll find corresponding quantities. It obvious that the degeneration points of the function $\nu^{+}(t)$ are $T^{+} \equiv \left\{\tau_i^{+}\right\}_{i=1}^{l+} \cup \left\{-\tau_i^{-}\right\}_{i=1}^{l-}$, and degeneration orders at these points equal: $\tau_i^{+} \rightarrow \beta_i^{+}$, $i = \overline{1, l^{+}}$; $-\tau_i^{-} \rightarrow \beta_i^{-}$, $i = \overline{1, l^{-}}$. Similarly, for the function $\nu^{-}(t)$ these quantities are: $T^{-} \equiv \left\{-\tau_i^{+}\right\}_{i=1}^{l+} \cup \left\{\tau_i^{-}\right\}_{i=1}^{l-}$, $-\tau_i^{+} \rightarrow \beta_i^{+}$, $i = \overline{1, l^{+}}$; and $\tau_i^{-} \rightarrow \beta_i^{-}$, $i = \overline{1, l^{-}}$. Denote

$$\omega^{+}(t) \equiv \rho^{+}(t) \cdot \rho^{-}(-t), \quad t \in [-\pi,\pi];$$

and

$$\omega^{-}(t) \equiv \omega^{+}(-t), \quad t \in [-\pi,\pi];$$

Let

$$\mu^{+}(t) = \begin{cases} \dfrac{1}{\rho^{-}(-t)}, & t \in (0,\pi] \\[2ex] \dfrac{1}{\rho^{+}(-t)}, & t \in [-\pi, 0) \end{cases}$$

$$\mu^{-}(t) \equiv \mu^{+}(-t), \quad t \in [-\pi,\pi].$$

Alongside with system (3) we consider its equivalent system

$$\left\{A^{+}(t)\,\omega^{+}(t)\,e^{int}; A^{-}(t)\,\omega^{-}(t)\,e^{-int}\right\}_{n\geq 0, k\geq 1}, \tag{4}$$

where $A^{+}(t) \equiv A_1^{+}(t)\,\mu^{+}(t)$; $A^{-}(t) \equiv A_1^{-}(t)\,\mu^{-}(t)$.

It is obvious that:

$$\sup_{[-\pi,\pi]} wrai \left|\mu^{+}(t)\right|^{\pm 1} < +\infty;$$

$$\sup_{[-\pi,\pi]} wrai \left|\mu^{-}(t)\right|^{\pm 1} < +\infty.$$

are fulfilled.

Redenote:

$$T^{+}=\left\{t_{i}^{+}\right\}_{i=1}^{m^{+}}; \qquad T^{-}=\left\{t_{i}^{-}\right\}_{i=1}^{m^{-}}.$$

Thus, the degeneration order β_k^+ corresponds to the point t_i^+, if $t_i^+=\tau_k^+$; or order β_i^-, if $t_i^+=-\tau_k^-$ for some k. In a similar way, the order β_k^+ corresponds to the point t_i^-, if $t_i^-=-\tau_k^+$; or order β_i^-, if $t_i^-=\tau_k^-$ for some k.

Let $\alpha(t)\equiv\arg A^{+}(t)$; $\beta(t)\equiv\arg A^{-}(t)$ and $\theta(t)\equiv\beta(t)-\alpha(t)$. It is clear that the discontinuity points of the function $\theta(t)$ are $\{s_i\}_{i=1}^r\cup\{-s_i\}_{i=1}^r\cup\{0\}$. Let's find jumps $\{h(\pm s_i);h(0)\}_{i=1}^r$ of the function $\theta(t)$ at these points, i.e. $h(\pm s_i)=\theta(\pm s_i+0)-\theta(\pm s_i-0)$; $h(0)=\theta(+0)-\theta(-0)$.

We have:

$$h(s_i)=\beta(s_i+0)-\alpha(s_i+0)-\beta(s_i-0)+\alpha(s_i-0)=$$
$$=\arg b(s_i+0)-\arg a(s_i+0)-\arg b(s_i-0)+\arg a(s_i-0)=$$
$$=\theta_0(s_i-0)-\theta_0(s_i+0);$$
$$h(-s_i)=\beta(-s_i+0)-\alpha(-s_i+0)-\beta(-s_i-0)+\alpha(-s_i-0)=$$
$$=\arg a(s_i-0)-\arg b(s_i-0)-\arg a(s_i+0)+\arg b(s_i+0)=h(s_i);$$
$$h(0)=\theta(+0)-\theta(-0)=\beta(+0)-\alpha(+0)-\beta(-0)+\alpha(-0)=$$
$$=\arg b(+0)-\arg a(+0)-\arg a(+0)+\arg b(+0)=$$
$$=2\left[\arg b(+0)-\arg a(+0)\right]=-2\theta_0(+0).$$

Let

$$\{\sigma_i\}_1^l\equiv\{\pm s_i\}_{i=1}^r\cup\{\pm t_i\}_{i=1}^{m^+}\cap\{0\},$$

moreover: $\pi<\sigma_1<\sigma_2<...<\sigma_l<\pi$.

Following the results of the paper [1] we form single-valued congruences:

$$\pm s_i\rightarrow\frac{h(\pm s_i)}{2\pi}; \qquad t_i^+\rightarrow\begin{cases}\beta_k^+, & \text{if } t_i^+=\tau_k^+;\\ \beta_k^-, & \text{if } t_i^+=-\tau_k^-;\end{cases}$$

$$t_i^-\rightarrow\begin{cases}\beta_k^+, & \text{if } t_i^-=-\tau_k^+;\\ \beta_k^-, & \text{if } t_i^-=\tau_k^-.\end{cases}$$

Determine the quantities $\lambda_i^{\pm}$, $\lambda_i(\cdot)$ and ν_i, $i=\overline{1,l}$; from the following expressions:

$$\lambda_i\left(T^+\right)=\begin{cases}\dfrac{\beta_k^+}{2}, & \text{if } \{\sigma_i\}\cap T^+=\tau_k^+;\\[2mm] \dfrac{\beta_k^-}{2}, & \text{if } \{\sigma_i\}\cap T^+=-\tau_k^-;\\[2mm] 0, & \text{if } \{\sigma_i\}\cap T^+=\{\varnothing\};\end{cases} \qquad i=\overline{1,l}; \tag{5}$$

$$\lambda_i\left(T^-\right)=\begin{cases}\dfrac{\beta_k^+}{2}, & \text{if}\quad \{\sigma_i\}\cap T^-=-\tau_k^+;\\[2ex] \dfrac{\beta_k^-}{2}, & \text{if}\quad \{\sigma_i\}\cap T^-=\tau_k^-;\\[2ex] 0, & \text{if}\quad \{\sigma_i\}\cap T^-=\{\varnothing\};\end{cases}\qquad i=\overline{1,l}; \tag{6}$$

$$\lambda_i^{\pm}=\begin{cases}\dfrac{h(\pm s)}{2\pi}, & \text{if}\quad \{\sigma_i\}\cap\{\{\pm s_i\}_{i=1}^r\}=\pm s_k;\\[2ex] 0, & \text{if}\quad \{\sigma_i\}\cap\{\{\pm s_i\}_{i=1}^r\}=\{\varnothing\};\end{cases}\qquad i=\overline{1,l}; \tag{7}$$

Let

$$\nu_i=\lambda_i^++\lambda_i^++\lambda_i\left(T^+\right)+\lambda_i\left(T^-\right),\quad i=\overline{1,l}. \tag{8}$$

Calculate (take into account that $\theta(t)=\beta_0(t)-\alpha_0(t)-t$; where $\beta_0\equiv\arg B(t)$, $\alpha_0(t)\equiv\arg a(t)$):

$$h_\pi=\theta(-\pi+0)-\theta(\pi-0)=\beta(-\pi)-\alpha(-\pi)+\alpha(\pi)=$$

$$\arg a(\pi)-\arg b(\pi)-\arg b(\pi)+\arg a(\pi)+2\pi=$$

$$=2\left[\arg a(\pi)-\arg b(\pi)\right]+2\pi.$$

Now, we require the fulfilment of the following inequalities: $\dfrac{1}{p}+\dfrac{1}{q}=1$

$$\left.\begin{aligned}&-\frac{1}{p}<\beta_i^{\pm}<\frac{1}{q},\quad i=\overline{1,l^{\pm}};\\ &-\frac{1}{p}<\frac{\arg b(0)-\arg a(0)}{\pi}<\frac{1}{p};\\ &-\frac{1}{q}-1<\frac{\arg b(\pi)-\arg a(\pi)}{\pi}<-\frac{1}{p};\\ &-\frac{1}{p}<\nu_i<\frac{1}{q}.\end{aligned}\right\} \tag{9}$$

while fulfilling conditions (9) as it follows from the results of the paper [1], system (4) and consequently system (2) form a basis in $L_p(-\pi,\pi)$, $1<p<+\infty$.

Then, by lemma 1, each of the systems $\left\{\vartheta_n^+(t)\right\}_{n\ge1}$ and $\left\{\vartheta_n^-(t)\right\}_{n\ge1}$ forms a base is in $L_p(0,\pi)$. Thus we arrive at the following conclusion.

Theorem 1. *Let conditions 1)-3) be fulfilled for the functions $a(t)$ and $b(t)$. The quantities $\lambda_i(T^{\pm})$, $\lambda_i^{\pm}$, ν_i, $i=\overline{1,l}$; be determined from the relations (5)-(8). If inequalities (9) are fulfilled, the systems $\left\{\vartheta_n^+(t)\right\}_{n\ge1}$ and $\left\{\vartheta_n^-(t)\right\}_{n\ge1}$ determined by the expression (1) form bases in $L_p(0,\pi)$, $1\le p<+\infty$.*

Now, let's consider the Hilbert case, i.e. let $p = 2$. Again, if inequalities (9) are fulfilled for $p = 2$, by the results of the paper [1] system (2) forms a basis in $L_2(-\pi,\pi)$. As the result, by the above-mentioned reason the systems $\left\{\vartheta_n^+(t)\right\}_{n\geq 0}$ and $\left\{\vartheta_n^-(t)\right\}_{n\geq 0}$ form bases in $L_2(0,\pi)$. Let's consider Riesz basicity of these systems. The following lemma is easily proved.

Lemma 2. *Let conditions 1)-3) be fulfilled and $\beta_i^\pm < -\frac{1}{2}$. Then system (2) forms a Riesz basis in $L_2(0,\pi)$ only if the systems $\left\{\vartheta_n^+(t)\right\}_{n\geq 1}$ and $\left\{\vartheta_n^-(t)\right\}_{n\geq 1}$ form a Riesz basis in $L_2(0,\pi)$.*

If conditions 1)-3) are observed and the inequalities

$$\left.\begin{array}{l} -\frac{1}{2} < \beta_i^\pm, \quad \nu_k < \frac{1}{2}, \quad i = \overline{1,l^\pm}; k = \overline{1,l} \\ \\ -\frac{1}{2} < \frac{\arg b(0) - \arg a(0)}{\pi} < \frac{1}{2}; \\ \\ -\frac{3}{2} < \frac{\arg b(\pi) - \arg a(\pi)}{\pi} < -\frac{1}{2}, \end{array}\right\} \qquad (10)$$

are fulfilled, then by the results of the paper [1] system (2) forms a Riesz basis in $L_2(-\pi,\pi)$ only for $\beta_i^\pm = 0$, $i = \overline{1,l^\pm}$. If for some i_0: $\beta_{i_0} \neq 0$, where $\beta_{i_0} = \beta_{i_0}^+$, or $\beta_{i_0} = \beta_{i_0}^-$, then by lemma 2 one of the systems $\left\{\vartheta_n^+(t)\right\}_{n\geq 1}$ and $\left\{\vartheta_n^-(t)\right\}_{n\geq 1}$ doesn't form a Riesz basis in $L_2(0,\pi)$. Consequently, we have the following theorem.

Theorem 2. *Let conditions 1)-3) be fulfilled and inequalities (10) hold. If therewith $\beta_i^\pm = 0$, $i = \overline{1,l^\pm}$; the systems $\left\{\vartheta_n^+(t)\right\}_{n\geq 1}$ and $\left\{\vartheta_n^-(t)\right\}_{n\geq 1}$ form Riesz bases in $L_2(0,\pi)$. But if $\exists i_0$: $\beta_{i_0} \neq 0$, where eihter $\beta_{i_0} = \beta_{i_0}^+$, $\beta_{i_0} = \beta_{i_0}^-$, then even if one of the systems $\left\{\vartheta_n^+(t)\right\}_{n\geq 1}$ and $\left\{\vartheta_n^-(t)\right\}_{n\geq 1}$ doesn't form a Riesz basis in $L_2(0,\pi)$.*

Remark 1. In theorem 2 depending on the signs of $\beta_i^\pm$, $i = \overline{1,l^\pm}$ system (2) may posses Hilbert or Bessel system property in $L_2(-\pi,\pi)$. Similar results in this case may be obtained for the systems $\left\{\vartheta_n^+(t)\right\}_{n\geq 1}$ and $\left\{\vartheta_n^-(t)\right\}_{n\geq 1}$ in $L_2(0,\pi)$.

Remark 2. Following the method applied in [2] the similar results may be obtained for the systems $1 \cap \left\{\vartheta_n^+(t)\right\}_{n\geq 1}$ and $\left\{\vartheta_n^-(t)\right\}_{n\geq 1}$ in $L_2(0,\pi)$.

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Tunjay S. Guseynov
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F. Agayev str., AZ-1141, Baku, Azerbaijan.
Tel: (99412) 439 47 20 (off.)

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