Vagif S. GULIYEV, Yagub Y. MAMMADOV

SOME ESTIMATIONS FOR RIESZ POTENTIALS IN TERMS MAXIMAL AND FRACTIONAL MAXIMAL FUNCTIONS ASSOCIATED WITH THE DUNKL OPERATOR ON THE REAL LINE

Abstract

On the real line, the Dunkl operators are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. In the work by means of the operator of generalized shift, generated by Dunkl operator the maximal functions (Dunkl-type maximal function), fractional-maximal functions (Dunkl-type fractional maximal function) and Riesz potentials (Dunkl-type Riesz potential) are investigated. We proved pointwise and integral estimates for Riesz potentials in terms maximal and fractional maximal functions associated with the Dunkl operator on the real line.

1. Introduction

For a real parameter $\alpha \geq -1/2$, we consider the Dunkl operator, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha(f)(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{x} \left(\frac{f(x) - f(-x)}{2} \right) \qquad (1)$$

Note that $\Lambda_{-1/2} = d/dx$. In the paper we investigate the maximal function, fractional maximal function and Riesz potential using harmonic analysis associated with the Dunkl operator on $\mathbb{R}$. We get pointwise and integral estimates for Riesz potentials in terms maximal and fractional maximal functions associated with the Dunkl operator on the real line.

2. Main result

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1} |x|^{2\alpha+1} dx.$$

For every $1 \leq p \leq \infty$, we denote by $L_p = L_p(d\mu_\alpha)$ the spaces of complex-valued functions f, measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} \equiv \|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p \, d\mu_\alpha(x) \right)^{1/p} < \infty \quad \text{if} \quad p \in [1, \infty),$$

and

$$\|f\|_{L_{\infty,\alpha}} = ess \sup_{x \in \mathbb{R}} |f(x)| \quad \text{if} \quad p = \infty.$$

For $1 \le p < \infty$ we denote by $WL_{p,\alpha}$, the weak $L_{p,\alpha}$ spaces defined as the set of locally integrable functions $f(x)$, $(x) \in \mathbb{R}$ with the finite norm

$$\|f\|_{WL_{p,\alpha}} = \sup_{r>0} r \left(\mu_\alpha \{x \in \mathbb{R} \ : \ |f(x)| > r\}\right)^{1/p}.$$

Note that

$$L_{p,\alpha} \subset WL_{p,\alpha} \quad \text{and} \quad \|f\|_{WL_{p,\alpha}} \le \|f\|_{p,\alpha} \text{ for all } f \in L_{p,\alpha}.$$

For all x, y, $z \in \mathbb{R}$, we put

$$W_\alpha(x,y,z) = (1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x})\Delta_\alpha(x,y,z)$$

where

$$\sigma_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } \ x, \ y \in \mathbb{R} \setminus 0, \\ 0 & \text{otherwice} \end{cases}$$

and Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x,y,z) = \begin{cases} d_\alpha \, \frac{([(|x|+|y|)^2-z^2][z^2-(|x|-|y|)^2])^{\alpha-1/2}}{|xyz|^{2\alpha}} & \text{if } \ |z| \in A_{x,y}, \\ 0 & \text{otherwice,} \end{cases}$$

where $d_\alpha = (\Gamma(\alpha+1))^2/(2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2}))$ and $A_{x,y} = [||x|-|y||, |x|+|y|]$.

In the sequel we consider the signed measure $\nu_{x,y}$, on $\mathbb{R}$, given by

$$\nu_{x,y} = \begin{cases} W_\alpha(x,y,z)\, d\mu_\alpha(z) & \text{if } \ x, \ y \in \mathbb{R} \setminus 0, \\ d\delta_x(z) & \text{if } \ y = 0, \\ d\delta_y(z) & \text{if } \ x = 0. \end{cases}$$

Definition 1. *For x, $y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, we put*

$$\tau_x f(y) = \int_{\mathbb{R}} f(z)\, d\nu_{x,y}(z).$$

The operators τ_x, $x \in \mathbb{R}$, are called Dunkl translation operators on $\mathbb{R}$ and it can be expressed in the following form (see ref. [4])

$$\tau_x f(y) = C_\alpha \int_0^\pi f_e\left(\sqrt{x^2+y^2-2|xy|\cos\theta}\right) h_1(x,y,\theta)(\sin\theta)^{2\alpha}\, d\theta$$

$$+C_\alpha \int_0^\pi f_o\left(\sqrt{x^2+y^2-2|xy|\cos\theta}\right) h_2(x,y,\theta)(\sin\theta)^{2\alpha}\, d\theta,$$

where $f = f_e + f_o$, f_o and f_e being respectively the odd and the even parts of f, with $C_\alpha = \Gamma(\alpha+1)/(\sqrt{\pi}\,\Gamma(\alpha+1/2))$,

$$h_1(x,y,\theta) = 1 - sgn(xy)\cos\theta \ \text{ and } \ h_2(x,y,\theta) = \begin{cases} \frac{(x+y)\,[1-sgn(xy)\cos\theta]}{\sqrt{x^2+y^2-2|xy|\cos\theta}} & \text{if } \ xy \ne 0, \\ 0 & \text{if } \ xy = 0. \end{cases}$$

Let $B(x,t) = \{y \in \mathbb{R} : |y| \in]\max\{0,|x|-t\}, |x|+t[\ \}$ and $t > 0$. Then $B(0,t) =]-t,t[$ and $\mu_\alpha(]-t,t[) = \left(2^{\alpha+1}(\alpha+1)\,\Gamma(\alpha+1)\right)^{-1} t^{2\alpha+2}$.

We define the fractional maximal function associated with the Dunkl operator by

$$M_\beta f(x) = \sup_{r>0} \big(\mu_\alpha B(0,r)\big)^{\frac{\beta}{2\alpha+2}-1} \int_{B(0,r)} \tau_x |f|(y)\, d\mu_\alpha(y), \quad 0 \le \beta < 2\alpha+2.$$

If $\beta = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator associated with the Dunkl operator (see [1]).

In [1] was proved the following theorem.

Theorem 1. *[1] 1. If $f \in L_{1,\alpha}(\mathbb{R})$, then for every $\beta > 0$*

$$\mu_\alpha \{x \in \mathbb{R} : Mf(x) > \beta\} \le \frac{C}{\beta} \int_{\mathbb{R}} |f(x)|\, d\mu_\alpha(x),$$

where $C > 0$ is independent of f.

2. If $f \in L_{p,\alpha}(\mathbb{R})$, $\quad 1 < p \le \infty$, then $Mf \in L_{p,\alpha}(\mathbb{R})$ and

$$\|Mf\|_{p,\alpha} \le C \|f\|_{p,\alpha},$$

where $C > 0$ is independent of f.

For the fractional maximal operator associated with the Dunkl operator M_β the following theorem is valid.

Theorem 2. *[3] Let $0 \le \beta < 2\alpha+2$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2}$, $1 \le p \le \frac{2\alpha+2}{\beta}$.*

1)If $f \in L_{1,\alpha}(\mathbb{R})$, then for all $\theta > 0$

$$\int\limits_{\{x\in\mathbb{R}:M_\beta f(x)>\theta\}} d\mu_\alpha(x) \le \left(\frac{C}{\theta} \int_{\mathbb{R}} |f(x)| d\mu_\alpha(x) \right)^q, \tag{2}$$

where C is independent of f.

2) Let $1 < p < \frac{2\alpha+2}{\beta}$, $f \in L_{p,\alpha}(\mathbb{R})$, then $M_\beta f \in L_{q,\alpha}(\mathbb{R})$ and

$$\left(\int_{\mathbb{R}} (M_\beta f(x))^q d\mu_\alpha(x) \right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x) \right)^{\frac{1}{p}}. \tag{3}$$

where C is independent of f.

3) Let $p = \frac{2\alpha+2}{\beta}$, $f \in L_{p,\alpha}(\mathbb{R})$, then $M_\beta f \in L_\infty(\mathbb{R})$ and

$$\sup_{x\in\mathbb{R}} M_\beta f(x) \le C \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x) \right)^{\frac{1}{p}}, \tag{4}$$

where C is independent of f.

Now we define the Riesz potential associated with the Dunkl operator by

$$I_\beta f(x) = \sup_{r>0} \big(\mu_\alpha B(0,r)\big)^{\frac{\beta}{2\alpha+2}-1} \int_{B(0,r)} \tau_x |y|^{\beta-2\alpha-2} f(y)\, d\mu_\alpha(y), \quad 0 < \beta < 2\alpha+2.$$

Theorem 3. *[2] Let* $0 < \beta < 2\alpha + 2$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2}$, $1 \le p < \frac{2\alpha+2}{\beta}$.

1)If $f \in L_{1,\alpha}(\mathbb{R})$, *then for all* $\theta > 0$

$$\int\limits_{\{x\in\mathbb{R}: I_\beta f(x)>\theta\}} d\mu_\alpha(x) \le \left(\frac{C}{\theta}\int_{\mathbb{R}} |f(x)| d\mu_\alpha(x)\right)^q, \tag{5}$$

where C *is independent of* f.

2) Let $1 < p < \frac{2\alpha+2}{\beta}$, $f \in L_{p,\alpha}(\mathbb{R})$, *then* $I_\beta f \in L_{q,\alpha}(\mathbb{R})$ *and*

$$\left(\int_{\mathbb{R}} (I_\beta f(x))^q d\mu_\alpha(x)\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)\right)^{\frac{1}{p}}, \tag{6}$$

where C *is independent of* f.

The following theorems is our main result in which we obtain pointwise and integral estimates for Dunkl-type Riesz potentials in terms Dunkl-type maximal and fractional maximal functions.

Theorem 4. *Let* $0 < \beta < 2\alpha + 2$, $1 \le p < \frac{\lambda}{\beta}$. *Then for any locally summable functions* f *exists the positive numbers* C_1 *and* C_2, *such that for every* $r > 0$ *and* $x \in \mathbb{R}$ *the following inequality is valid:*

$$(I_\beta|f|)(x) \le C_1 r^\beta (Mf)(x) + C_2 r^{\beta-\frac{\lambda}{p}} (M_{\frac{\lambda}{p}} f)(x). \tag{7}$$

Theorem 5. *Let* $0 < \beta < \lambda$, $1 < p < \frac{\lambda}{\beta}$, $1 \le r \le \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{\lambda} + \frac{\beta p}{\lambda r}$. *Then for any functions* $f \in L_{p,\alpha}(\mathbb{R})$ *and* $M_{\frac{\lambda}{p}} f \in L_{r,\alpha}(\mathbb{R})$ *the following estimations is valid:*

$$\|I_\beta f\|_{L_{q,\alpha}} \le C_4 \, \|M_{\frac{\lambda}{p}} f\|_{L_{r,\alpha}}^{\frac{\beta p}{\lambda}} \ \|f\|_{L_{p,\alpha}}^{1-\frac{\beta p}{\lambda}}, \tag{8}$$

where $C > 0$ *is independent of function* f.

3. Proof of the Theorem 4 and 5

Proof of Theorem 4. Let r be an arbitrary positive real number. We write the integral as the sum of two integrals:

$$I_\beta|f|(x) = \int_{B(0,r)} |y|^{\beta-2\alpha-2} \tau_y |f(x)| d\mu_\alpha(y)$$

$$+ \int_{\complement_{B(0,r)}} |y|^{\beta-2\alpha-2} \tau_y |f(x)| d\mu_\alpha(y) = J_1(x,r) + J_2(x,r).$$

First we shall estimate $J_1(x,r)$. Summarizing on all $k \ge 0$ we have

$$J_1(x,r) = \int_{B(0,r)} |y|^{\beta-2\alpha-2} \tau_y |f(x)| d\mu_\alpha(y)$$

$$= \sum_{k=0}^{\infty} \int\limits_{B(0,2^{-k}r)\setminus B(0,2^{-k-1}r)} |y|^{\beta-2\alpha-2} \tau_y |f(x)| d\mu_\alpha(y)$$

$$\leq \sum_{k=0}^{\infty} \left(2^{-k-1}r\right)^{\beta-2\alpha-2} \int_{B(0,2^{-k}r)} \tau_y |f(x)| d\mu_\alpha(y)$$

$$= 2^{2\alpha+2-\beta} r^\beta \sum_{k=0}^{\infty} 2^{-k\beta} \left(2^{-k}r\right)^{-2\alpha-2} \int_{B(0,2^{-k}r)} \tau_y |f(x)| d\mu_\alpha(y)$$

$$\leq 2^{2\alpha+2-\beta} r^\beta Mf(x) \sum_{k=0}^{\infty} 2^{-k\beta} = C_1 r^\beta Mf(x),$$

where

$$C_1 = \frac{2^{2\alpha+2-\beta}}{1-2^{-\beta}}.$$

On the other hand, for $J_2(x,r)$ we have

$$J_2(x,r) = \int_{\complement B(0,r)} |y|^{\beta-2\alpha-2} \tau_y f(x) d\mu_\alpha(y)$$

$$= \sum_{k=0}^{\infty} \int\limits_{B(0,2^{k+1}r)\setminus B(0,2^k r)} |y|^{\beta-2\alpha-2} \tau_y f(x) d\mu_\alpha(y)$$

$$\leq \sum_{k=0}^{\infty} \left(2^k r\right)^{\beta-2\alpha-2} \int_{B(0,2^{k+1}r)} \tau_y f(x) d\mu_\alpha(y)$$

$$= 2^{2\alpha+2-\frac{\lambda}{p}} r^{\beta-\frac{\lambda}{p}} \sum_{k=0}^{\infty} 2^{-k(\frac{\lambda}{p}-\beta)} \left(2^{k+1}r\right)^{\frac{\lambda}{p}-2\alpha-2} \int_{B(0,2^{k+1}r)} \tau_y f(x) d\mu_\alpha(y)$$

$$\leq 2^{2\alpha+2-\frac{\lambda}{p}} r^{\beta-\frac{\lambda}{p}} M_{\frac{\lambda}{p}} f(x) \sum_{k=0}^{\infty} 2^{-k(\frac{\lambda}{p}-\beta)} \leq C_2 r^{\beta-\frac{\lambda}{p}} M_{\frac{\lambda}{p}} f(x),$$

as under our assumption $\beta - \frac{\lambda}{p} < 0$, where

$$C_2 = \frac{2^{2\alpha+2-\frac{\lambda}{p}}}{1-2^{\beta-\frac{\lambda}{p}}}.$$

Now the statement of a Theorem directly follows from these two estimations for $J_1(x,r)$ and $J_2(x,r)$.

Thus the proof of Theorem 4 is completed.

Proof of Theorem 5.

Taking

$$r = r(x) = \left(\frac{M_{\frac{\lambda}{p}} f(x)}{Mf(x)}\right)^{\frac{p}{\lambda}}$$

in (7) we have

$$|I_\beta f(x)| \leq C_3 \left(M_{\frac{\lambda}{p}} f(x)\right)^{\frac{\beta p}{\lambda}} (Mf(x))^{1-\frac{\beta p}{\lambda}} \tag{9}$$

for every $x \in \mathbb{R}$.

Then we have

$$\int_{\mathbb{R}} |I_\beta f(x)|^q \, d\mu_\alpha(x) \le C_3^q \int_{\mathbb{R}} \left(M_{\frac{\lambda}{p}} f(x) \right)^{\frac{\beta pq}{\lambda}} (Mf(x))^{q-\frac{\beta pq}{\lambda}} \, d\mu_\alpha(x) \le$$

$$\le C_3^q \left(\int_{\mathbb{R}} \left(M_{\frac{\lambda}{p}} f(x) \right)^{\frac{\beta pqs'}{\lambda}} d\mu_\alpha(x) \right)^{1/s'} \left(\int_{\mathbb{R}} (Mf(x))^{(q-\frac{\beta pq}{\lambda})s} \, d\mu_\alpha(x) \right)^{1/s},$$

where $(q - \frac{\beta pq}{\lambda})s = p$, $s' = \frac{s}{s-1} = \frac{\lambda r}{\beta pq}$, $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{\lambda} + \frac{\beta p}{\lambda r}$.

Therefore

$$\left(\int_{\mathbb{R}} |I_\beta f(x)|^q \, d\mu_\alpha(x) \right)^{\frac{1}{q}} C_3 \left(\int_{\mathbb{R}} (Mf(x))^p \, d\mu_\alpha(x) \right)^{\frac{1}{sq}} \left(\int_{\mathbb{R}} \left(M_{\frac{\lambda}{p}} f(x) \right)^r d\mu_\alpha(x) \right)^{\frac{\beta pq}{\lambda r}}$$

$$\le C_4 \left(\int_{\mathbb{R}} |f(x)|^p \, d\mu_\alpha(x) \right)^{\frac{1}{sq}} \left(\int_{\mathbb{R}} \left(M_{\frac{\lambda}{p}} f(x) \right)^r d\mu_\alpha(x) \right)^{\frac{\beta pq}{\lambda r}}$$

or

$$\|I_\beta f\|_{L_{q,\alpha}} \le C_4 \, \|f\|_{L_{p,\alpha}}^{\frac{1}{sq}} \left\| M_{\frac{\lambda}{p}} f \right\|_{L_{r,\alpha}}^{\frac{\beta p}{\lambda}} \le C_4 \left\| M_{\frac{\lambda}{p}} f \right\|_{L_{r,\alpha}}^{\frac{\beta p}{\lambda}} \|f\|_{L_{p,\alpha}}^{1-\frac{\beta p}{\lambda}}.$$

Thus the proof of Theorem 5 is completed.

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Vagif S. Guliyev
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F. Agayev str., AZ-1141, Baku, Azerbaijan.
Tel: (99412) 465 63 67
E-mail : vagif@guliyev.com

Yagub Y. Mammadov
Nakhcivan State University, Nakhcivan, Azerbaijan,
Nakhchivan Teacher-Training Institute, Azerbaijan,
E-mail: yagub52@yandex.ru

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