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ON STANCU TYPE GENERALIZATION OF THE SZASZ OPERATOR IN COMPLEX DOMAIN

Abstract

The convergence of generalization Szasz operators of Stancu type in complex domain is studied in this paper.

Introduction and preliminaries. Otto Szasz [3] gave a generalization of the well-known Bernstein polynomials to the infinite interval as

$$S_n(x,f)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right)\qquad n>0. \tag{1}$$

He proved that if $f\in C[0,\infty)$ and $f(x)=O(x^A)$ (or more generally, if $f(x)=O(e^{Ax})$ [Favard 11]) where A is a positive constant, then $S_n(x,f)\to f(x)$ as $n\to\infty$ uniformly on any finite interval $[0,a]$. There have been many studies of Bernstein type polynomials in the infinite interval and we refer to the papers [5, 7, 8, 9].

The other generalization of Bernstein polynomials was given by Stancu [1] in the form

$$B_{n,\alpha,\beta}(x,f)=\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k+\alpha}{n+\beta}\right)x^k(1-x)^{n-k}, \tag{2}$$

where $\alpha,\ \beta$ are real numbers, $0\le\alpha\le\beta$.

Note that this type generalization of Bernstein-Chlodowsky polynomials was studied in [6]. In 1963 J.J.Geregen, F.G.Dressel and H. Purcell studied the convergence of classical Szasz operators (1) in some parabolic set $p(d)=\left\{z:|z|<x+2d^2\right\}$, where d is a positive number in complex domain and $x=\mathrm{Re}\,z$. In this work, we consider the Szasz operators of Stancu type

$$S_{n,\alpha,\beta}(z,f)=e^{-nz}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!}f\left(\frac{k+\alpha}{n+\beta}\right) \tag{3}$$

in complex domain.

Definition. *A function $f(z)$ defined on $p(d)$ is said posses property B in $p(d)$ if for each b, $0<b<d$, there exists a positive number $B(b)$ such that, for every $z\in p(b)$,*

$$|f(z)|<B(b)\exp\left[\frac{1}{2}x-|x|^{1/2}\left[b^2-\frac{1}{2}(|z|-x)\right]^{1/2}\right]. \tag{4}$$

Note that property (4) were introduced by Otto Szasz and Nelson Yeardley ([10]) which gives a necessary and sufficient condition for representation of an analytic function by general Laguerre series.

As usual the functions $\{f_k(z)\}$, $k \geq 0$ defined on $p(d)$ are said to have the property B uniformly in $p(d)$ if for each b, $0 < b < d$, the constant $B(b)$ in (4) does not depend on $f_k(z)$.

Main results.

The following two theorems are our main results.

Theorem 1. *Let $f(z)$ be analytic and has property B in $p(d)$ and $e_{n,\beta} = \exp\left[\frac{1}{2(n+\beta)}\right]$. Then the function $\{S_{n,\alpha,\beta}(z/e_{n,\beta}; f)\}$ has property B uniformly on $p(d)$.*

Theorem 2. *Let $f(z)$ be analytic and has property B on $p(d)$. Then*

1. *$S_{n,\alpha,\beta}(z, f)$ is an entire function for every n and every fixed α, β;*
2. *$S_{n,\alpha,\beta}(z, f) \to f(z)$ as $n \to \infty$ in $p(d)$ for every fixed α, β;*
3. *The convergence in b) is uniform on each compact subset of $p(d)$.*

Auxiliary lemmas to prove the main results.

We assume that z is an arbitrary complex number and $n \geq 1$.

Lemma 1. *If $f(z)$ is a polynomial, then $S_{n,\alpha,\beta}(z, f)$ is a polynomial of degree equal to the degree of $f(z)$.*

Proof. Let $f(z) = z^m$, where m is a natural number. Then

$$e^{-z}\sum_{\lambda=0}^{\infty}\frac{z^{\lambda}}{\lambda!}\lambda^n = e^{-z}\left(z\frac{d}{dz}\right)^n e^z = \sum_{j=0}^{n} C_j^{(n)} z^j, \tag{5}$$

where $C_j^{(n)}$ are constants. Now by using (5) for $S_{n,\alpha,\beta}(z, f)$ we have

$$S_{n,\alpha,\beta}(z, f) = e^{-nz}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!} f\left(\frac{k+\alpha}{n+\beta}\right) =$$

$$= \frac{e^{-nz}}{(n+\beta)^m}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!}(k+\alpha)^m = \frac{e^{-nz}}{(n+\beta)^m}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!}\sum_{v=0}^{m}k^v\alpha^{m-v}\begin{pmatrix} m \\ v \end{pmatrix} =$$

$$= \sum_{v=0}^{m}\begin{pmatrix} m \\ v \end{pmatrix}\alpha^{m-v}\frac{e^{-nz}}{(n+\beta)^m}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!}k^v = \begin{pmatrix} m \\ 0 \end{pmatrix}\frac{\alpha^m e^{-nz}}{(n+\beta)^m}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!} +$$

$$+ \begin{pmatrix} m \\ 1 \end{pmatrix}\frac{\alpha^{m-1}e^{-nz}}{(n+\beta)^m}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!}k + \begin{pmatrix} m \\ 2 \end{pmatrix}\frac{\alpha^{m-2}e^{-nz}}{(n+\beta)^m}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!}k^2 + ... +$$

$$+ \begin{pmatrix} m \\ m \end{pmatrix}\frac{e^{-nz}}{(n+\beta)^m}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!}k^m \tag{6}$$

On the other hand,

$$e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!} \Rightarrow \frac{d}{dz}e^z = \sum_{k=0}^{\infty}\frac{z^{k-1}}{k!}k \Rightarrow z\frac{d}{dz}e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!}k$$

$$ze^{z}=\sum_{k=0}^{\infty}\frac{z^{k}}{k!}k\Rightarrow(1+z)e^{z}=\sum_{k=0}^{\infty}\frac{z^{k-1}}{k!}k^{2}\Rightarrow(z+z^{2})e^{z}=$$

$$=\sum_{k=0}^{\infty}\frac{z^{k}}{k!}k^{2}\Rightarrow...\Rightarrow p(z)e^{z}=\sum_{k=0}^{\infty}\frac{z^{k}}{k!}k^{n},$$

$$p(z)=e^{-z}\sum_{k=0}^{\infty}\frac{z^{k}}{k!}k^{n}, \tag{7}$$

where $p(z)$ is a polynomial of degree n and we can easily see by induction that the coefficient of z^n in $p(z)$ is one. Now by considering (7) in (6) we have

$$S_{n,\alpha,\beta}(z,f)=\begin{pmatrix} m\\ 0\end{pmatrix}\frac{\alpha^{m}}{(n+\beta)^{m}}+\begin{pmatrix} m\\ 1\end{pmatrix}\frac{\alpha^{m-1}(nz)}{(n+\beta)^{m}}+$$

$$+\begin{pmatrix} m\\ 2\end{pmatrix}\frac{\alpha^{m-2}((nz)^{2}+(nz))}{(n+\beta)^{m}}+...+$$

$$+\begin{pmatrix} m\\ m\end{pmatrix}\frac{[(nz)^{m}+a_{1}(nz)^{m-1}+...+a_{m-1}]}{(n+\beta)^{m}}.$$

It is clear that all the terms of the above sum except the last summand tend to zero as $n\to\infty$ and the last summand tends to z^m. Then it follows that $S_{n,\alpha,\beta}\to z^m$ as $n\to\infty$ for every z and the convergence is uniform in every compact set.

Lemma 2. *Let $Q^{(m)}_{n,\alpha,\beta}(z)=S_{n,\alpha,\beta}(z,L_m)$, $m=0,1,2,...$ be the polynomials where L_m is the m-th Laguerre polynomial of order 0, then*

$$\left|Q^{(m)}_{n,\alpha,\beta}(z)\right|\le(e_{n,\beta})^{\alpha}\exp\left(-nx+n\left|z\right|e_{n,\beta}\right) \tag{8}$$

and

$$\sum_{m=0}^{\infty}Q^{(m)}_{n,\alpha,\beta}(z)\,\omega^{m}=\frac{e^{-\frac{\alpha\omega}{n+\beta}}}{1-\omega}\exp\left\{-nz+nz\exp\left[\frac{-\omega}{(n+\beta)(1-\omega)}\right]\right\}. \tag{9}$$

Proof. Using the known inequality

$$\left|L_{n}(x)\right|\le\exp\left(\frac{1}{2}x\right)\qquad 0\le x,\quad n=1,2,... \tag{10}$$

(See [4 -p.164]), we can write

$$\left|Q^{(m)}_{n,\alpha,\beta}(z)\right|=\left|e^{-nz}\sum_{k=0}^{\infty}\frac{(nz)^{k}}{k!}L_{m}\left(\frac{k+\alpha}{n+\beta}\right)\right|\le\left|e^{-nz}\right|\sum_{k=0}^{\infty}\left|\frac{(nz)^{k}}{k!}\right|e^{\frac{(k+\alpha)}{2(n+\beta)}}=$$

$$=e^{\frac{\alpha}{2(n+\beta)}}\left|e^{-nz}\right|\sum_{k=0}^{\infty}\left|\frac{(n\left|z\right|)^{k}}{k!}\right|\left(e^{\frac{1}{2(n+\beta)}}\right)^{k}=(e_{n,\beta})^{\alpha}\exp\left(-nx+n\left|z\right|e_{n,\beta}\right)$$

which gives the inequality (8).

To prove (9) we use equality (see [4 - p. 110])

$$\sum_{n=0}^{\infty} L_n(z)\,\omega^n = \frac{1}{1-\omega}\exp\left(\frac{-z\omega}{\omega-1}\right), \qquad |\omega|<1, \tag{11}$$

then

$$e^{-nz}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!}\sum_{m=0}^{\infty}L_m\left(\frac{k+\alpha}{n+\beta}\right)\omega^m = \frac{e^{-nz}}{1-\omega}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!}\exp\left(\frac{-\frac{k+\alpha}{n+\beta}\omega}{1-\omega}\right)=$$

$$=\frac{e^{-\frac{\alpha\omega}{n+\beta}}\cdot e^{-nz}}{1-\omega}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!}\exp\left(\frac{-k\omega}{(n+\beta)(1-\omega)}\right)=$$

$$=\frac{e^{-\frac{\alpha\omega}{n+\beta}}\cdot e^{-nz}}{1-\omega}\sum_{k=0}^{\infty}\frac{\left(nz\exp\left(\frac{-\omega}{(n+\beta)(1-\omega)}\right)\right)^k}{k!}=$$

$$=\frac{e^{-\frac{\alpha\omega}{n+\beta}}}{1-\omega}\exp\left\{-nz+nz\exp\left(\frac{-\omega}{(n+\beta)(1-\omega)}\right)\right\}$$

for fixed z, n, ω, $|\omega|<1$, the left double series is absolutely convergent. By changing the order of summation, we can get (9).

Lemma 3. *Let*

$$K_n(z,\omega)=\Re\left\{-nz+nz\exp\left(\frac{-\omega}{(n+\beta)(1-\omega)}\right)\right\}$$

then

$$K_n(z,\omega)\le vr\frac{(|z|-rx)}{1-r^2}, \quad |\omega|=r<1, \tag{12}$$

where $v=v(r,n)=\exp\left\{\frac{r}{(n+\beta)(r+1)}\right\}$.

Proof. The left side of inequality (12) for cases $z=0$ and $\omega=0$ is zero i.e.

$$K_n(0,\omega)=K_n(z,0)=0.$$

We suppose then $|z|$, $|\omega|$, n fixed such that $z\neq 0$, $0<r=|\omega|<1$. we consider

$$z=|z|\,e^{i\phi}, \rho=\frac{r}{1-r^2}, \;\; e^{i\theta}=\frac{\omega(1-\bar{\omega})}{r(1-\omega)}, \;\; a=\frac{1}{k+\beta}, \;\; \Phi=\phi-a\rho\sin\theta.$$

Then we have,

$$e^{i\theta}=\frac{\omega(1-\bar{\omega})}{r(1-\omega)}\Rightarrow\; r+e^{i\theta}=r+\frac{\omega(1-\bar{\omega})}{r(1-\omega)}=\frac{r^2(1-\omega)+\omega-\omega\bar{\omega}}{r(1-\omega)}\;\Rightarrow$$

$$\Rightarrow\; r+e^{i\theta}=\frac{\omega\left(1-r^2\right)}{r(1-\omega)}\;\Rightarrow\;\left(\frac{r}{1-r^2}\right)\left(r+e^{i\theta}\right)=$$

$$=\frac{\omega}{1-\omega}\;\Rightarrow\;\rho\left(r+e^{i\theta}\right)=\frac{\omega}{1-\omega}$$

then for

$$K_n(z,\omega) \le \nu r \frac{(|z|-rx)}{1-r^2}$$

or

$$\Re\left\{-nz + nz\exp\left(\frac{-\omega}{(n+\beta)(1-\omega)}\right)\right\} \le \frac{\nu r(|z|-rx)}{1-r^2}. \qquad (13)$$

On the other hand,

$$-nz + nz\exp\left[\frac{-\omega}{(n+\beta)(1-\omega)}\right] = -n|z|e^{i\phi} + n|z|e^{i\phi}e^{-a\rho\left(r+e^{i\theta}\right)} =$$

$$= -n|z|e^{i\phi} + n|z|e^{i\phi}e^{-a\rho(r+\cos\theta+i\sin\theta)} =$$

$$= -n|z|e^{i\phi} + n|z|e^{i\phi}e^{-a\rho(r+\cos\theta)}e^{-ia\rho\sin\theta} =$$

$$= -n|z|e^{i\phi} + n|z|e^{i(\phi-a\rho\sin\theta)}e^{-a\rho(r+\cos\theta)}$$

$$\Rightarrow \Re\left\{-n|z|e^{i\phi} + n|z|e^{i(\phi-a\rho\sin\theta)}e^{-a\rho(r+\cos\theta)}\right\} =$$

$$= -n|z|\cos\phi + n|z|\cos(\phi-a\rho\sin\theta)\,e^{-a\rho(r+\cos\theta)} =$$

$$= n|z|\left(-\cos\phi + \cos\Phi e^{-a\rho(r+\cos\theta)}\right). \qquad (14)$$

(13), (14) $\Rightarrow$ $n|z|\left(-\cos\phi + \cos\Phi e^{-a\rho(r+\cos\theta)}\right) \le \nu\rho(|z|-rx)$

$$-\cos\phi + \cos\Phi e^{-a\rho(r+\cos\theta)} \le \frac{1}{k}\nu\rho(|z|-rx)$$

$$x+iy = z = |z|e^{i\phi} = |z|\cos\phi + i|z|\sin\phi \;\Rightarrow\; \frac{x}{|z|} = \cos\phi$$

$$\Rightarrow\; -\cos\phi + \cos\Phi e^{-a\rho(r+\cos\theta)} \le \frac{a\nu\rho}{1-a\beta}(1-r\cos\phi)$$

$$-\cos\phi + \cos\Phi e^{-a\rho(r+\cos\theta)} \le \frac{a\nu\rho}{1-a\beta} - \frac{a v\rho r\cos\phi}{1-a\beta}$$

$$\frac{avpr\cos\phi}{1-a\beta} - \cos\phi + \cos\Phi e^{-a\rho(r+\cos\theta)} \le \frac{a\nu\rho}{1-a\beta}$$

$$av\rho r\cos\phi + (1-a\beta)\left(-\cos\phi + \cos\Phi e^{-ap(r+\cos\theta)}\right) \le av\rho$$

$$\cos\phi(av\rho r - (1-a\beta)) + (1-a\beta)\cos\Phi e^{-ap(r+\cos\theta)} \le av\rho \qquad (15)$$

It is enough to show that (15) holds for $0 \le \theta \le \pi$, $|\phi| \le \pi$.

For this purpose we study the inequality

$$T(\theta,\phi) = (av\rho r - (1-a\beta))\cos\phi + (1-a\beta)\cos\Phi e^{-a\rho(r+\cos\theta)} \le av\rho$$

in $R:\; 0 \le \theta \le \pi,\; |\phi| \le \pi \quad ((\theta,\phi) \in R)$.

Suppose that $avr\rho \ge 1$, (we will use the fact $e^t \le 1 + te^t$)

$$T(\theta,\phi) = (av\rho r - (1-a\beta))\cos\phi + (1-a\beta)\cos\Phi e^{-a\rho(r+\cos\theta)} \le$$

$$\leq \left(av\rho r-(1-a\beta)\right)+(1-a\beta)\,e^{-a\rho(r-1)}=$$

$$=av\rho r-1+a\beta+(1-a\beta)\left(1-a\rho\,(r-1)\right)e^{-a\rho(r-1)}=$$

$$=av\rho r-1+a\beta+1-a\rho\,(r-1)\,e^{-a\rho(r-1)}-a\beta+a\beta a\rho\,(r-1)\,e^{-a\rho(r-1)}=$$

$$=av\rho r-a\rho\,(r-1)\,e^{-a\rho(r-1)}\,(1-a\beta)=av\rho r+a\rho\,(1-r)\,e^{-a\frac{r(r-1)}{1-r^2}}\,(1-a\beta)=$$

$$=av\rho r+a\rho\,(1-r)\,e^{\frac{ar}{1+r}}\,(1-a\beta)=av\rho r+(a\rho v-a\rho vr)\,(1-a\beta)=$$

$$=av\rho r+a\rho v-a\rho vr-a\rho va\beta+a\rho vra\beta=a\rho v-a\rho va\beta\,(1-r)\leq a\rho v$$

Suppose then that $avr\rho\leq 1$. Let (θ,ϕ) is maximum point for T- in R. We discuss three cases

1) $\theta=0$ 2)$\theta=\pi$ 3) $0<\theta<\pi$.

If $\theta=0$ then,

$$T=\left(avr\rho-(1-a\beta)\right)\cos\phi+(1-a\beta)\left(\cos\phi e^{-a\rho(r+1)}\right)=$$

$$=\cos\phi\left(avr\rho-(1-a\beta)+(1-a\beta)\,e^{-a\rho(v+1)}\right)=$$

$$=\cos\phi\,(avr\rho+(1-a\beta)\left(-1+e^{-a\rho(v+1)}\right).$$

For this we have two cases;

i) $avr\rho+(1-a\beta)\left(-1+e^{-a\rho(r+1)}\right)\geq 0$

$$T<avr\rho+(1-a\beta)\left(-1+e^{-a\rho(r+1)}\right)=$$

$$=avr\rho+(1-a\beta)\left(-1+e^{\frac{ar}{r-1}}\right)<avr\rho<av\rho$$

ii) $avr\rho+(1-a\beta)\left(-1+e^{-a\rho(r+1)}\right)\leq 0$

$$T\leq\left(1-e^{-a\rho(r+1)}\right)(1-a\beta)-avr\rho=\left(1-e^{\frac{ar}{r-1}}\right)(1-a\beta)-avr\rho\leq$$

$$\leq-\left(e^{\frac{ar}{r-1}}-1\right)(1-a\beta)-avr\rho\leq e^{\frac{ar}{r-1}}\left(e^{\frac{ar}{r-1}}-1\right)(1-a\beta)-avr\rho\leq$$

$$\leq e^{\frac{ar}{r-1}}\left(1+\frac{ar}{r-1}e^{\frac{ar}{r-1}}-1\right)(1-a\beta)-avr\rho\leq$$

$$\leq e^{\frac{ar}{r-1}}\left(\frac{ar}{r-1}e^{\frac{ar}{r-1}}\right)(1-a\beta)-avr\rho\leq$$

$$\leq\frac{ar}{r-1}\,(1-a\beta)-avr\rho\leq\frac{ar}{r-1}\,(1-a\beta)-avr\rho\leq$$

$$\leq\frac{ar}{r-1}avr\rho<avr\rho<av\rho.$$

2) If $\theta=\pi$ then

$$T=\left(avr\rho-(1-a\beta)\right)\cos\phi+(1-a\beta)\cos\phi e^{-a\rho(r-1)}$$

$$T=\left[av r\rho-(1-a\beta)+(1-a\beta)\,e^{-a\rho(r-1)}\right]\cos\phi$$

$$T=\left[\left(av r\rho-(1-a\beta)+(1-a\beta)\,e^{\frac{ar}{1+r}}\right)\right]\cos\phi=$$

$$=\left[av r\rho+(1-a\beta)\,(-1+v)\right]\cos\phi\leq$$

$$\leq av r\rho+(-1+v)\,(1-a\beta)\leq av r\rho-1+v=av r\rho-1+e^{\frac{ar}{1+r}}=$$

$$=av r\rho+\frac{ar}{1+r}e^{\frac{ar}{1+r}}=av r\rho+\frac{arv\,(1-r)}{(1+r)\,(1-r)}=$$

$$=av r\rho+a\rho v\,(1-r)=av r\rho+a\rho v-a\rho r v=a\rho v$$

for completing the proof it remains to consider the case $0<\theta<\pi$. Both of the first partial derivatives of T at point (θ,ϕ) are zero.

$$T_\theta=e^{-a\rho(r+\cos\theta)}\,(a\rho\sin\theta)\cos\left(\phi-a\rho\sin\theta\right)(1-a\beta)+$$

$$+a\rho\,\cos\theta\,\sin\Phi e^{-ap(r+\cos\theta)}\,(1-a\beta)=0$$

$$T_\theta=e^{-a\rho(r+\cos\theta)}a\rho\,(1-a\beta)\left[\sin\theta\cos\Phi+\cos\theta\sin\Phi\right]=0\Rightarrow\sin\left(\theta+\Phi\right)=0 \quad (16)$$

$$T_\phi=-\left(av r\rho-(1-a\beta)\right)\sin\phi-(1-a\beta)\sin\Phi e^{-a\rho(r+\cos\theta)}=0$$

$$\left(av r\rho-(1-a\beta)\right)\sin\phi+(1-a\beta)\sin\,\Phi e^{-a\rho(r+\cos\theta)}=0 \quad (17)$$

from definition of T and (16), (17) relations we have

$$T\sin\theta=\left(av r\rho-(1-a\beta)\right)\cos\phi\sin\theta+(1-a\beta)\left(e^{-a\rho(r+\cos\theta)}\cos\Phi\sin\theta\right)=$$

$$=\left(avrp-(1-a\beta)\right)\cos\phi\sin\theta-(1-a\beta)\,e^{-ap(r+\cos\theta)}\cos\theta\,\sin\Phi=$$

$$=\left(avrp-(1-a\beta)\right)\cos\phi\sin\theta+\left(avrp-(1-a\beta)\right)\sin\phi\cos\theta=$$

$$=\left(avrp-(1-a\beta)\right)\sin\left(\theta+\phi\right)$$

from (16) one can write

$$\sin\left(\theta+\Phi\right)=0\;\Rightarrow\;\theta+\Phi=n\pi,\;\;n=0,\pm1,\pm2,,....$$

Thus

$$\theta+\phi=\theta+\Phi+ap\sin\theta=n\pi+a\rho\sin\theta,$$

and

$$T\sin\theta=\left(avrp-(1-a\beta)\right)\sin\left(n\pi+ap\sin\theta\right). \quad (18)$$

By (18) and $a\alpha\,r\rho<1,\;\;0<\theta<\pi$, we have

$$T\sin\theta\leq\left(1-a\beta-av r\rho\right)a\rho\sin\theta\leq a\rho\sin\theta,$$

$$T\leq a\rho v.$$

For completing of the proof Theorems 1 and 2, we will use the following lemmas proved in [2].

Lemma 4. *Let $\alpha_1, \beta_1, \gamma_1$ be positive constant numbers such that $\alpha_1 < \beta_1$ and define $U(t) = \frac{4\alpha_1^2}{t} + \frac{t\beta_1^2}{4+t}$, then following inequality is satisfied*

$$I(\alpha_1, \beta_1, \gamma_1) = \int_0^\infty \frac{1}{1-e^{-t}} \frac{1}{t^{\frac{3}{2}}} \exp\left[-U(t) - \frac{4\gamma_1^2}{t}\right] dt \leq$$

$$\leq M_1(\gamma) \exp\left(\alpha_1^2 - 2\alpha_1\beta_1\right),$$

$$M_1(\gamma) = \frac{e\left[2 + \frac{\sqrt{\pi}}{16\alpha_1^2}\right]}{e-1}.$$

Lemma 5. *If $0 < b < c$ and*

$$J(b,c,z) = \int_0^\infty \frac{1}{1-e^{-t}} \frac{1}{t^{\frac{3}{2}}} \exp\left[-\frac{4c^2}{t} + \frac{2e^{-\frac{t}{2}}}{1-e^{-t}}(|z| - xe^{\frac{t}{2}})\right] dt$$

then the following inequality is satisfied

$$J(b,c,z) \leq M_2(b,c) \exp\left\{x - 2|x|^{\frac{1}{2}} \left[b^2 - \frac{1}{2}(|z| - x)\right]^{\frac{1}{2}}\right\},$$

where $z \in p(b)$ and $M_2(b,c) = M_1\left((c^2 - b^2)^{\frac{3}{2}}\right)$.

Lemma 6. *Let $0 < b < c$. Then*

$$\sum_{m=0}^{\infty} \left|Q_{n,\alpha,\beta}^{(m)}(z/e_{n,\beta})\right|^2 \exp(-4c\sqrt{m}) \leq$$

$$\leq M_3(b,c) \exp\left\{x - 2|x|^{\frac{1}{2}} \left[b^2 - \frac{1}{2}(|z| - x)\right]^{\frac{1}{2}}\right\}$$

for $z \in p(b)$, such that $M_3(b,c) = (2c\sqrt{\pi})M_2(b,c)$.

Proof. Let $Cr, 0 < r < 1$ is a circle with radius r in the ω-plane. By lemma 2 and lemma 3 and the classical integral formula we can write

$$\sum_{m=0}^{\infty} \left|Q_{n,\alpha,\beta}^{(m)}(z)\right|^2 r^{2m} =$$

$$= \frac{1}{2\pi r} \int_{C_r} \frac{\left|e^{-\frac{\alpha\omega}{n+\beta}}\right|^2}{|1-\omega|^2} \left|\exp\left\{-nz + nz\exp\left(\frac{-\omega}{(n+\beta)(1-\omega)}\right)\right\}\right|^2 |d\omega| =$$

$$= \frac{1}{2\pi r} \int_{C_r} \frac{\left|e^{-\frac{\alpha\omega}{n+\beta}}\right|^2}{|1-\omega|^2} \exp[2K_n(z,\omega)]\, |d\omega| \leq$$

by lemma 3 we can use this fact $K_n(z,\omega) \le e_{n,\beta}\frac{r(|z|-rx)}{1-r^2}$ then

$$\le \frac{1}{2\pi r}\int\limits_{C_r} \frac{1}{|1-\omega|^2} \exp\left[2e_{n,\beta} r\left(\frac{|z|-rx}{1-r^2}\right)\right] = \frac{1}{1-r^2}\exp\left[2e_{n,\beta} r\left(\frac{|z|-rx}{1-r^2}\right)\right].$$

If $t>0$, then

$$\sum_{m=0}^{\infty}\left|Q_{n,\alpha,\beta}^{(m)}(z/e_{n,\beta})\right|^2 e^{-mt} \le \left[\frac{1}{1-e^{-t}}\right]\exp\left\{2e^{-\frac{t}{2}}\left(\frac{|z|-xe^{-\frac{t}{2}}}{1-e^{-t}}\right)\right\}.$$

On the other hand,

$$\exp(-4c\sqrt{m}) = \left(\frac{2c}{\sqrt{\pi}}\right)\int\limits_0^{\infty} t^{-\frac{3}{2}}\exp\left(-mt-\frac{4c^2}{t}\right)dt.$$

Now we have

$$\sum_{m=0}^{\infty}\left|Q_{n,\alpha,\beta}^{(m)}(z/e_{n,\beta})\right|^2 \exp(-4c\sqrt{m}) =$$

$$= \left(\frac{2c}{\sqrt{\pi}}\right)\sum_{m=0}^{\infty}\left|Q_{n,\alpha,\beta}^{(m)}(z/e_{n,\beta})\right|^2\int\limits_0^{\infty} t^{-\frac{3}{2}}\exp\left(-mt-\frac{4c^2}{t}\right)dt =$$

$$= \left(\frac{2c}{\sqrt{\pi}}\right)\int\limits_0^{\infty} t^{-\frac{3}{2}}\exp\left(\frac{-4c^2}{t}\right)\left[\sum_{m=0}^{\infty}\left|Q_{n,\alpha,\beta}^{(m)}\left(z/e_{n,\beta}\right)\right|^2\exp(-mt)\right]dt \le$$

$$\le \left(\frac{2c}{\sqrt{\pi}}\right)\int\limits_0^{\infty}\frac{t^{-\frac{3}{2}}}{1-e^{-t}}\exp\left[-\frac{4c^2}{t}+\frac{2e^{-\frac{t}{2}}}{1-e^{-t}}\left(|z|-xe^{-\frac{t}{2}}\right)\right]dt \le$$

and by applying lemma 5,

$$\le \left(\frac{2c}{\sqrt{\pi}}\right)M_2(b,c)\exp\{x-2|x|^{\frac{1}{2}}[b^2-\frac{1}{2}(|z|-x)]^{\frac{1}{2}}\}$$

for $z\in p(d)$.

Proof of Theorems

Proof of Theorem 1.

The conditions of Theorem allows us to write Lagurre series in $p(d)$. Then

$$f(z)=\sum_{m=o}^{\infty} a_m L_m(z)\,,\ z\in p(d)\,;\ a_m=\int\limits_{o}^{\infty} e^{-x}L_m(x)f(x)dx \tag{19}$$

Since series (19) is convergent, we follow

$$|a_m| \le A_\varepsilon \exp\left[2m\left(-d+\varepsilon\right)\right] \quad m=1,2,... \tag{20}$$

For an arbitrary positive ε and a suitably choose constant A_ε.

From (20) we obtain

$$\sum_{m=o}^{\infty}|a_m|<\infty\ ,\ M(c\,,f)=\sum_{m=o}^{\infty}|a_m|^2\exp\left(4c\sqrt{m}\right)<\infty \tag{21}$$

such that $0<c<d$.

Now consider $S_{n\,,\alpha\,,\beta}(z,f)$. We have

$$S_{n,\alpha,\beta}(z,f)=e^{-nz}\sum_{k=o}^{\infty}\frac{(nz)^k}{k!}\sum_{m=o}^{\infty}a_mL_m\left(\frac{k+\alpha}{n+\beta}\right)$$

$$=\sum_{m=0}^{\infty}a_m\left[e^{-nz}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!}L_m\left(\frac{k+\alpha}{n+\beta}\right)\right]=\sum_{m=0}^{\infty}a_mQ_{n,\alpha,\beta}^{(m)}(z). \tag{22}$$

By (10) and the first inequality in (21) we see that $S_{n,\alpha,\beta}(z,f)$ converges absolutely for arbitrary $z,n,0<n$.

From (22) we obtain

$$|S_{n,\alpha,\beta}\left(z,f\right)|^2\leq\sum_{m=0}^{\infty}|a_m|^2\exp\left(4c\sqrt{m}\right)\sum_{m=0}^{\infty}\left|Q_{n,\alpha,\beta}^{(m)}\left(z\right)\right|^2\exp\left(-4c\sqrt{m}\right)$$

then, by applying Lemma 6, if $0<b<c<d$, we have

$$|S_{n,\alpha,\beta}\left(z/e_{n,\beta},f\right)|^2\leq M(c,f)\,M_3(b,c)\exp\left\{x-2\,|x|^{\frac{1}{2}}\left[b^2-\frac{1}{2}\left(|z|-x\right)\right]^{\frac{1}{2}}\right\}$$

for $z\in p(b)$. For fixed $b,0<b<d$ and by putting

$$B(b)=\left[M\left(c\ ,\ f\right)M_3\left(b\,,\ c\right)\right]^{\frac{1}{2}},\quad c=\frac{1}{2}\left(b+d\right)$$

the proof is complete.

Proof of Theorem 2. Note that the hypothesis of Theorem allows writing the inequality $|f(x)|<Ae^{\frac{x}{2}},\ 0\leq x$ for some positive constant A. Then the series

$$S_{n,\alpha,\beta}\left(z,f\right)=e^{-nz}\sum_{k=0}^{\infty}\frac{(nz)^k}{k!}f\left(\frac{k+\alpha}{n+\beta}\right),\qquad 0<n$$

for arbitrary z,n converges, therefore a) holds.

For completing the proof it is sufficient to show that, if S be compact subset of $p(d)$ then $S_{n,\alpha,\beta}\left(z,f\right)\underset{n\to\infty}{\to}f\left(z\right)$ uniformly convergence on S.

For $b>0,\ x_0>0$ let $U\left(b,x_0\right)=\left\{z\,||z|<x+2d^2,x<x_0\right\}$. Choose b_1,b_2,b_3x_1,x_2,x_3 such that $0<b_1<b_2<b_3<d,0<x_1<x_2<x_3$ and $S\subset U\left(b_1,x_1\right)$, by theorem1 there exists a constant M^* such that

$$|S_{n,\alpha,\beta}\left(z/e_{n,\beta},f\right)|\leq M^*,\qquad z\in U\left(b_3,x_3\right).$$

Choose $n_0 = \max\left\{\left[4 \cdot Ln\left(\frac{b_3}{b_2}\right)\right]^{-1}, \left[2 \cdot Ln\left(\frac{x_3}{x_2}\right)\right]^{-1}\right\}$. Then for a sufficiently large n, $n_0 < n$ and for $z \in U(b_2, x_2)$ we have $ze_{n,\beta} \in U(b_3, x_3)$. Therefore

$$|S_{n,\alpha,\beta}(z,f)| = |S_{n,\alpha,\beta}(ze_{n,\beta}/e_{n,\beta}, f)| \leq M^* n_0 < n, z \in U(b_2, x_2).$$

Also by Korovkin's theorem [12] we can write $S_{n,\alpha,\beta}(x,f) \underset{n\to\infty}{\to} f(x)$ for $0 < x < x_2$ and by Vitali's theorem, $\{S_{n,\alpha,\beta}(z,f)\}$ converges uniformly on $U(b_1, x_1)$ to a function F(z) that analytic on $U(b_1, x_1)$. On the other hand f(z) is analytic on $U(b_1, x_1)$ and $F(x) = f(x)$ for $0 < x < x_1$, it follows $F(z) = f(z)$ throughout $U(b_1, x_1)$ and the theorem has been proved.

Acknowledgment. The author is grateful to academician A.D.Gadjiev for the problem statement and for useful discussions.

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Received May 24, 2007; Revised September 20, 2007.