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ON BASICITY OF A PART OF A SYSTEMS WITH INFINITE DEFECT

Abstract

In the paper we consider a system $\Phi \cup \Psi$ in a Banach space $B = B_1 \oplus B_2$, where B_1, and B_2 are infinite dimensional subspaces, $\Phi \equiv \{\varphi_n\}_{n\in N}$, $\Psi \equiv \{\psi_n\}_{n\epsilon N}$. We find the conditions under which the system $P_{rB_2}\Psi$, where P_{rB_2} is a projection operator from B to sub-space B_2, forms a basis of the space B_2.

Application of some known methods (e.g. the Fourier method, Laplace transformation and etc.) to the solution of many problems of mathematical physics and mechanics requires to study spectral problems of the form

$$L(x,\lambda)u = \frac{d^m}{dx^m}u + \sum_{i=1}^{m} a_i(x,\lambda)\frac{d^{m-i}}{dx^{m-i}}u = 0,$$

$$U_\nu(\lambda,u) = 0, \quad \nu = \overline{1,l};$$

where U_ν are boundary operators.

History of such problems is sufficiently rich originating with fundamental results of Birkhoff G.D. [1] and Tamarkin Ya.D. [2]. In general, in these papers completeness of root vectors or expansion of functions from the domain of definition of an operator, are obtained. Further development in this direction and in this connection the problems of the basicity of root elements is related with names of many well-known mathematicians [3].

In sequel, M.V. Keldysh [4] noticed that in the case of pencil the root elements are so much that they form n-fold basis or they are n-fold complete. So such papers we can refer the papers [5-7], as well. Then, there appeared the papers (eg. [8]) where the basicity or completeness of some part of root elements in the considered space is proved. Dividing into parts the root elements in eigen-values, the Riesz basicity of the appropriate part was proved in [9].

As a result we see that in the case of a pencil or when spectral parameter enters into boundary conditions, defect systems appear. When a spectral parameter enters into the boundary conditions polynomially, a finite defect arises. The paper [10] and the next papers in this direction refer to this case. Naturally, there arises a question on selection of a basis from the root elements of the considered operator. Possibility of such selection of a basis in the case of finite defect is shown in the paper [10], its constructive method is in [12].

Naturally, there arises a question on possibility of selection of a basis in the case of infinite defect. The present paper is devoted to this problem.

At first we give some arguments of general character.

1. Let B_i, $i = 1,2$ be sub-spaces of Banach space B and $\left\{e_k^i\right\}_{k\in N}$, $i = 1,2$ be appropriate bases. Let the spaces of coefficients of these bases coincide and we denote it as B^k, and the norm $\|\cdot\|_{B^K}$. On a linear span $L_i \equiv L_i\left[\left\{e_k^i\right\}_{k\in N}\right]$ we define an operator T: $L_1 \to L_2$: for $\forall u = \sum_{k=1}^{m} a_k e_k^1$ we assume

$$Tu = \sum_{k=1}^{m} a_k e_k^2.$$

We have
$$\|Tu\|_B = \left\| \sum_{k=1}^{m} a_k e_k^2 \right\|_B \leq M_1 \left\| \{a_k\}_1^m \right\|_{B^k} \leq M_2 \|u\|_B .$$

Continuing the operator T by continuity we have $T \in L(B_1, B_2)$. From $Tu = 0 \Longrightarrow \sum_{k=1}^{m} a_k e_k^2 = 0$, where $u = \sum_k a_k e_k^1$. Consequently $a_k = 0$, $\forall k \in N$ and as a result $u = 0$. Then we take $\forall f \in B_2$, let $f = \sum_k b_k e_k^2$. Consider $u = \sum_k b_k e_k^1$. Clearly, $Tu = f$. Then by the Banach theorem T is boundedly invertible.

Now, let $\Phi \cup \Psi$ be a basis in B, where $\Phi \equiv \{\varphi_n\}_{n\in N}$, $\Psi \equiv \{\psi_n\}_{n\epsilon N}$. Everywhere $\overline{M}$ denotes a closure of a linear span M in B. Thus,
$$B = \overline{\Phi} + \overline{\Psi},$$
where $\overline{\Phi}$ and $\overline{\Psi}$ are sub-spaces in B, moreover $\overline{\Phi} \cap \overline{\Psi} = \{0\}$.

Let $T \in L(\overline{\Phi}; B_1)$ be boundedly invertible. If $B_1 \cap \overline{\Psi} = \{0\}$, then the system $E \cup \Psi$ forms a basis in $B_1 + \overline{\Psi}$, where $\{e_n\}_{n\in N} \equiv E$, $e_n = T\varphi_n$, $n \in N$ and $B_1 \equiv \overline{E} \subset B$ is a sub-space. Indeed, we introduce an operator $\widetilde{T}$:$\widetilde{T}_{\overline{\Phi}} = T$, $\widetilde{T}_{\overline{\Psi}} = I$, $I \in L(\overline{\Psi}, \overline{\Psi})$ is an identity operator, $\widetilde{T}_M$ is contraction of the operator $\widetilde{T}$ on M.

Thus, if $u = u_1 + u_2$, where $u_1 \in \overline{\Phi}$, $u_2 \in \overline{\Psi}$, then
$$\widetilde{T}u = Tu_1 + u_2.$$

Let $\widetilde{T}u = 0 \Longrightarrow Tu_1 + u_2 = 0$. From $B_1 \cap \overline{\Psi} = \{0\}$ it follows that $u_2 = 0$ and $\widetilde{T}u_1 = 0 \Longrightarrow u_1 = 0 \Longrightarrow u = 0$, i.e. $Ker\widetilde{T} = \{0\}$. And now, let $f \in B_1 + \overline{\Psi}$, i.e. $f = f_1 + f_2$, $f_1 \in B_1$, $f_2 \in \overline{\Psi}$, and such a representation is unique. Let $u = T^{-1}f_1 + f_2$. Clearly $\widetilde{T}u = f_1 + f_2 = f$. Then, again by the Banach theorem $\widetilde{T}$ is boundedly invertible. In fact, it suffices to prove the boundedness of the operator $\widetilde{T}$.

Let's prove the continuity of the operator $\widetilde{T}$. Let $u_n \to u$ in B $u_n = u_n^1 + u_n^2$, $u = u^1 + u^2$, where u_n^1, $u^1 \in \overline{\Phi}$, u_n^2, $u^2 \in \overline{\Psi}$. Then $\widetilde{T}u_n = Tu_n^1 + u_n^2 \to Tu^1 + u^2 = Tu$. Thus, $\widetilde{T}$ is bounded. Basicity of $E \cup \Psi$ in $B_1 + \overline{\Psi}$ is proved.Obviously, if $E \cup \Psi$ is complete in B, it forms a basis therein.

Now, let's consider such possibility. Let $\overline{E} = \overline{\Phi}$ and the operator T: $\overline{\Phi} \to \overline{E}$ have the inverse operator, i.e. $Ker\widetilde{T} = \{0\}$. Obviously, if E is a base sequence, then $E \cup \Psi$ forms a basis in B. Really, we take $\forall u \in B \Longrightarrow u = u_1 + u_2$, where $u_1 \in \overline{E}$, $u_2 \in \overline{\Psi}$.Since E is a base sequence, then $\exists \{c_n\}_{n\in N}$: $u_1 = \sum_n c_n e_n$.

Show that this representation is unique. Let $u_1 = \sum_n d_n e_n$.

Consider
$$\widetilde{u}_1 = \sum_n (c_n - d_n)\varphi_n,$$
$$T\widetilde{u}_1 = \sum_n (c_n - d_n)e_n = 0.$$

From $KerT = \{0\} \Longrightarrow \widetilde{u}_1 = 0$, i.e.
$$\sum_n (c_n - d_n)\varphi_n = 0 \Longrightarrow c_n = d_n.$$

Uniqueness of expansion is proved. The remaining is obvious.

2. Now let $A = (a_{ij})_{i,j=\overline{1,\infty}}$ be some matrix and $\Phi \equiv \{\varphi_n\}_{n\in N} \subset B$ be some system. Under $A\Phi \equiv \Psi \equiv \{\psi_n\}_{n\in N}$ we'll understand:

$$\psi_n = \sum_{k=1}^{\infty} a_{nk}\varphi_k, \quad \forall n \in N. \tag{1}$$

We'll assume that $\{\widetilde{\varphi}_n\}_{n\in N}$ is some basis in B and by $K_{\widetilde{\Phi}}$ we'll denote appropriate Banach space of coefficients:

$$K_{\widetilde{\Phi}} \equiv \left\{ \{\lambda_n\}_{n\in N} \subset C : \sum_n \lambda_n \widetilde{\varphi}_n \in B \right\},$$

$$\left\| \{\lambda_n\}_{n\in N} \right\|_{K_{\widetilde{\Phi}}} = \sup_m \left\| \sum_{n=1}^{m} \lambda_k \widetilde{\varphi}_n \right\|.$$

Let $\{e_n\}_{n\in N} \subset B_1$ be also a basis in the sub-space B_1, with the same space of coefficients $K_{\widetilde{\Phi}}$. Assume that the matrix A maps $K_{\widetilde{\Phi}}$ onto $K_{\widetilde{\Phi}}$ one–to-one , i.e. $K_{\widetilde{\Phi}} \to K_{\widetilde{\Phi}}$ is invertible, and moreover

$$E = A\Phi + B\Psi, \tag{2}$$

where $E \equiv \{e_n\}_{n\in N}$; $\widetilde{\Phi} = \Phi \cup \Psi$, $\Phi \equiv \{\varphi_n\}_{n\in N}$, $\Psi \equiv \{\psi_n\}_{n\in N}$, $B = (b_{ij})_{i,j=\overline{1,\infty}}$: B: $K_{\widetilde{\Phi}} \to K_{\widetilde{\Phi}}$ is a matrix.

Since $A^{-1} \in L(K_{\widetilde{\Phi}})$, from (2) we have:

$$\Phi = A^{-1}E - A^{-1}B\Psi. \tag{3}$$

Let $x \in B$ be an arbitrary element, then

$$x = \sum_n \lambda_n \varphi_n + \sum_n \mu_n \psi_n.$$

Using (3) we have:

$$x = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\infty} c_{kn} e_k + \sum_{n=1}^{\infty} \left[\mu_n - \sum_k d_{kn}\lambda_n \right] \psi_k, \tag{4}$$

where $A^{-1} \equiv (c_{ij})_{i,j=\overline{1,\infty}}$; $A^{-1}B \equiv (d_{ij})_{i,j=\overline{1,\infty}}$.

The completeness of the system $E \cup \Psi$ in B follows directly from (4).Moreover, any element of the space is expanded in series by this system. Show that such expansion is unique. Let

$$\sum_n a_n e_n + \sum_n b_n \psi_n = 0. \tag{5}$$

From (2) we have:

$$\sum_n a_n \sum_k a_{nk}\varphi_k + \sum_n a_n \sum_k b_{nk}\psi_k + \sum_n b_n \psi_n = 0,$$

i.e.

$$\sum_k \left(\sum_n a_n a_{nk} \right) \varphi_k + \sum_k \left[b_k + \sum_n a_n b_{nk} \right] \psi_k = 0.$$

It follows from the basicity of $\Phi \cup \Psi$ that

$$\sum_n a_n a_{nk} = 0, \;\; \forall k \in N,$$

i.e. $A'a = 0$, where $a = (a_1, a_2, ...)$, A' in a transposed matrix.

Consequently, $a = 0$, and at the same time from (5) we have

$$\sum_n b_n \psi_n = 0 \Longrightarrow b_n = 0, \;\; \forall n \in N.$$

Basicity of $E \cup \Psi$ in B is proved.

Summarizing we arrive at the following conclusion.

Theorem 1. *Let $B = B_1 \oplus B_2$, $\widetilde{\Phi} \equiv \Phi \cup \Psi$ be a basis in B, and E be a basis in B_1, moreover $K_E \equiv K_{\widetilde{\Phi}}$. Let $\widetilde{E} \equiv \{\widetilde{e}_n\}_{n \in N}$, $\widetilde{e}_n = \binom{e_n}{0}$, $\forall n \in N$ have an expansion*

$$\widetilde{E} = A\Phi + B\Psi,$$

where A, $B \in L(K_{\widetilde{\Phi}})$, moreover A is invertible . Then $\widetilde{E} \cup \Psi$ also forms a basis in B.

Now, let's take $\forall u \in B_2$ and expand the element $\widehat{u} = \binom{0}{u}$ in basis $\widetilde{E} \cup \Psi$.

$$\widehat{u} = \begin{pmatrix} 0 \\ u \end{pmatrix} = \sum_n u_n^{'} \widetilde{e}_n + \sum_n u_n^2 \psi_n = \begin{pmatrix} \sum_n u_n^{'} \widetilde{e}_n + \sum_n u_n^2 P_{rB_2} \psi_n \\ \sum_n u_n^2 P_{rB_2} \psi_n \end{pmatrix}.$$

Consequently

$$u = \sum_n u_n^2 P_{rB_2} \psi_n,$$

where P_{rB_i}: $B \to B_i$ is a projection operator generated by expansion.

We get that the system $\left\{P_{rB_2} \psi_n\right\}_{n \in N}$ forms a base sequence in B_2 and it is complete in B_2. Show that it is minimal in B_2. Let it not be so. Then $\exists n^0 \in N$: $P_{rB_2} \psi_{n^0} \in \overline{L\left[\left\{P_{rB_2} \psi_n\right\}_{n \neq n^0}\right]}$, where $\overline{L[M]}$ is a closure of a linear span M. Then it is easy to show that the system $\widetilde{E} \cup \{\psi_n\}_{n \neq n^0}$ is complete in B. Indeed, otherwise there exists a nonzero functional $\widehat{u}^* = \binom{u^*}{\vartheta^*} \in B^*$, annihilating the system $\widetilde{E} \cup \{\psi_n\}_{n \neq n^0}$: $\widehat{u}^*(\widetilde{e}_n) = 0$, $\widehat{u}^*(\psi_k) = 0$, $\forall n \in N$, $\forall k \neq n^0$. It follows from $\widehat{u}^*(\widetilde{e}_n) = 0$, $\forall n \in N$, that $u^* = 0$. Thus,

$$\widehat{u}^*(\psi_k) = \vartheta^* \left(P_{rB_2} \psi_k\right) = 0, \quad \forall k \neq n^0.$$

It follows from the completeness of $\left\{P_{rB_2} \psi_k\right\}_{k \neq n^0}$ in B_2 that $\vartheta^* = 0$. Thus, the system $\widetilde{E} \cup \{\psi_n\}_{n \neq n^0}$ is complete in B, and this contradicts the basicity of the system $\widetilde{E} \cup \Psi$ in B. As a result, we have the basicity of the system $\left\{P_{rB_2} \psi_n\right\}_{n \in N}$ in B_2. Thus, it is valid

Theorem 2. *Let all the conditions of theorem 1 be satisfied and $\widetilde{E} \cup \Psi$ form a basis in B. Then, the system $P_{rB_2} \psi$ forms a basis in B_2.*

3. Now we pass to concrete realization of the theorems obtained above.

So, let again $\widehat{B} = B_1 \oplus B_2$ be a direct sum of Banach spaces B_i, $i = 1, 2$. Assume that $\left\{\widetilde{\widehat{e}}_n\right\}_{n \in N} \equiv \widehat{E} \subset \widehat{B}$ is some basis in $\widehat{B}$ with a space of coefficients $K_{\widehat{E}}$. In B_1 we take an arbitrary for the present, basis $\Phi \equiv \{\varphi_n\}_{n \in N}$. By $\overline{L[M]}$ we denote a closure of a linear span of the set M in an appropriate space. Let

$$L_k \equiv L[\{\varphi_n\}_{1 \le n \le k}], \quad B^k \equiv \overline{L\left[\{\varphi_n\}_{n \ge k+1}\right]}.$$

Obviously $B_1 = L_1 \oplus B^1$, consequently

$$\widehat{B} = L_1 \oplus B^1 \oplus B_2 = L_1 \oplus \widehat{B}^1, \tag{6}$$

where $\widehat{B} \equiv B^1 \oplus B_2$.

Further we pay attention to the following fact. Let a Banach space B be a direct sum of Banach spaces B_i, $i = \overline{1, 3}$:

$$B = B_1 \oplus B_2 \oplus B_3, \tag{7}$$

and $P_r : B \to B_i$ be a projection operator generated by the expansion (7).

Then

$$P_{r(B_k \oplus B_n)} = P_{rB_k} + P_{rB_n}.$$

By the results of the paper [11], from (6) it follows that $\exists n_1 \in N$:such that the system $\left\{P_{r\widehat{B}_1} \widehat{e}_n\right\}_{n \in N \backslash N'}$ forms a basis in $\widehat{B}_1$, where $N^k \equiv \{n_1; n_2; ...; n_k\}$. Continuing this process we have $\exists \{n_l\}_{l=1}^k \subset N$, $n_i \neq n_j$ for $i \neq j$ such that the system $\left\{P_{r\widehat{B}_k} \widehat{e}_n\right\}_{n \in N \backslash N^k}$ forms a basis in $\widehat{B}_k$, where $\widehat{B}_k = B^k \oplus B_2$. Thus, we get a sequence of natural numbers $N^\infty \equiv \left\{n_k, \, k = \overline{1, \infty}: n_i \neq n_j \ for \ i \neq j\right\}$ such that for $\forall k \in N$ the system $\left\{P_{r\widehat{B}_k} \widehat{e}_n\right\}_{n \in N \backslash N^k}$ forms a basis in $\widehat{B}_k$.

Now, let's rewrite the system $\{\widehat{e}_n\}_{n \in N}$ in the form $\{\widehat{e}_n^-, \, \widehat{e}_n^+\}_{n \in N}$, where $\widehat{e}_k^- = \widehat{e}_{nk}^-$ and $\widehat{e}_k^+ = \widehat{e}_k$ for $k \in N \backslash N^\infty$. Assume that the system $\Phi \cup \Psi$ also forms a basis in $\widehat{B}$. This holds, for example, when the system $\widehat{E}$ forms unconditional basis in $\widehat{B}$. Let $\{g_n^-, \, g_n^+\}_{n \in N}$ be a system conjugated to $\{\widehat{e}_n^-, \, \widehat{e}_n^+\}_{n \in N}$. It holds the expansion:

$$\varphi_k = \sum_{n=1}^{\infty} g_n^-(\varphi_k) \widehat{e}_n^- + \sum_{n=1}^{\infty} g_n^+(\varphi_k) \, \widehat{e}_n^+, \quad k \in N.$$

From the results of [13] in the case of finite defect, it follows that for $\forall k \in N$:

$$\Delta_k = \det \left(g_i^- \left(\varphi_j\right)\right)_{i,j=\overline{1,k}} \neq 0.$$

Assume $K_E \equiv K_\Phi$ and moreover $G \in L(K_E)$ is invertible, where $G \equiv \left(g_i^- \left(\varphi_j\right)\right)_{i,j=\overline{1,\infty}}$. Then, it follows from the previous results that in this case the system $\left\{P_{r\widehat{B}_k} \widehat{e}_n^+\right\}_{n \in N}$ forms a basis in B_2.

Let's consider a Hilbert case. Let $H = H_1 \oplus H_2$ where H_i, $i = 1, 2$ are Hilbert sub-spaces of H. Following [14], by $n(A)$ we'll denote the lower bound of a self-adjoint operator A. Assume that the system E forms a Riesz basis in H. Let's take an arbitrary Riesz basis Φ in H and G be an appropriate matrix. By the results of [14, p.300], if $n(G^* G) > 0$, then G is invertible and thus $\left\{P_{r\widehat{H}_2} \widehat{e}_n^+\right\}_{n \in N}$ forms a basis in H_2.

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