MATHEMATICS

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HAUSDORFF-YOUNG TYPE THEOREM FOR UNITARY SYSTEMS WITH MEASURABLE COEFFICIENTS

Abstract

In the paper we consider unitary systems with measurable coefficients. Hausdorff-Young type theorem is obtained under definite conditions on the coefficients.

Let's consider the following unitary system of exponents

$$\left\{A(t)\,e^{int}-B(t)\,e^{-int}\right\}_{n\geq 1} \tag{1}$$

with measurable coefficients $A(t)\equiv|A(t)|\,e^{i\alpha(t)}$, $B(t)\equiv|B(t)|\,e^{i\beta(t)}$ on $[0,\pi]$. The basicity of the system (1) in L_p with measurable coefficients was earlier studied in B. T. Bilalov's paper [1].

Assume that the following conditions are fulfilled:

1) $A(t);\ B(t)\in L_\infty\equiv L_\infty(0,\pi)$ moreover

$$\left\||A(t)|^{\pm1};\ |B(t)|^{\pm1}\right\|_\infty<+\infty;$$

2) The arguments $\alpha(t)$ and $\beta(t)$ are representable in the form

$$\alpha(t)=\alpha_1(t)+\alpha_2(t);\quad \beta(t)=\beta_1(t)+\beta_2(t),$$

where $\alpha_1(t)$, $\beta_1(t)$ are continuous; $\alpha_2(t)$, $\beta_2(t)$ are measurable parts, moreover

$$\frac{\beta_1(0)-\alpha_1(0)}{2\pi}=\frac{\beta_1(\pi)-\alpha_1(\pi)}{2\pi}\in Z;$$

$\|\beta_2(t)-\alpha_2(t)\|_\infty\leq\nu\pi,\quad 0\leq\nu<\min\left\{\frac{1}{p};\ 1-\frac{1}{p}\right\},\quad p\in(1,+\infty)$ is some number.

Theorem. *Let the coefficients* $A(t)$ *and* $B(t)$ *satisfy the conditions 1), 2). Then there exists an absolute constant* $M>0$ *for which:* $(1<p\leq2)$

a) For $\forall f\in L_p$ *it holds*

$$\left\|\{f_n\}_{n\geq1}\right\|_{l_q}\leq M\cdot\|f\|_p\qquad \frac{1}{p}+\frac{1}{q}=1;$$

where $\{f_n\}_{n\geq1}$ *are biorthogonal coefficients of the function* $f(t)$ *by the system* (1);

b) If the sequence $\{f_n\}_{n\geq1}$ *belongs to the space* l_p *then* $\exists f\in L_q$ *, for which*

$$\|f\|_q\leq M\cdot\left\|\{f_n\}_{n\geq1}\right\|_{l_p}$$

moreover $\{f_n\}_{n\geq1}$ *are biorthogonal coefficients of the function* $f(t)$ *by the system* (1).

Proof. Is carried out similar to the proof of appropriate theorem of the paper [2] . So, let all the conditions of the theorem be fulfilled. Then by the results of the paper [1] the system (1) forms a basis in L_p.

We consider a conjugation problem:

$$\begin{cases} F^+(t) + G(t) F^-(t) = g(t), & |t| = 1, \\ F(\infty) = 0, \end{cases}$$

where

$$G\left(e^{i\theta}\right) = \begin{cases} B(\theta) A^{-1}(\theta), & 0 < \theta < \pi, \\ A(-\theta) B^{-1}(-\theta), & -\pi < \theta < 0, \end{cases}$$

$g \in L_p(-\pi, \pi)$ is an arbitrary function.

As is already known, the index of the problem equals zero and $F^+(0) = 0$ [1] . Thus, biorthogonal coefficients $\{f_n\}_{n\geq 1}$ of the function $f(t)$:

$$f(t) \equiv \begin{cases} A(t) g(t), & 0 < t < \pi, \\ -B(-t) g(-t), & -\pi < t < 0, \end{cases}$$

by the system (1) are the Fourier coefficients of the function $F^+\left(e^{it}\right)$ by the classic system of exponents $\left\{e^{int}\right\}_{-\infty}^{+\infty}$. Given the case a). Obviously

$$\left\| \{f_n\}_{n\geq 1} \right\|_{l_q} \leq M \cdot \left\| F^+\left(e^{it}\right) \right\|_p .$$

Again, using Sokhotskiy-Plemel relation and considering that appropriate singular integral acts boundedly from L_p to L_p, we have:

$$\left\| F^+\left(e^{it}\right) \right\|_p \leq M_1 \left\| g(t) \right\|_p \leq M_2 \left\| f(t) \right\|_p .$$

This proves the case a).

Consider the case b). We take an arbitrary sequence $\{f_n\}_{n\geq 1} \in l_p$. Organize

$$F^+(z) = \sum_{n\geq 1} f_n z^n .$$

By the results of the paper [3] the function $F^+(z)$ belongs to the Hardy class H_q^+ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ in a unit circle. Consequently, boundary value of $F^+\left(e^{it}\right)$ on a unit circle belongs to the space L_q, moreover

$$F^+\left(e^{it}\right) = \sum_{n\geq 1} f_n e^{int} .$$

From similar reasonings we get that the function

$$F^-(z) = \sum_{n\geq 1} f_n z^{-n} .$$

belongs to the space H_q^- outside of a unit circle, and

$$F^-\left(e^{it}\right) = \sum_{n\geq 1} f_n e^{-int} .$$

belongs to L_q. Thus, the function

$$f(t)=A(t)\,F^{+}\left(e^{it}\right)-B(t)\,F^{-}\left(e^{it}\right)$$

belongs to the space L_q, moreover

$$f(t)=\sum_{n\geq 1} f_n\left[A(t)\,e^{int}-B(t)\,e^{-int}\right] \tag{2}$$

It follows from the basicity of the system (1) in L_p that it is minimal in L_q, since $q\geq p$. Then, it follows from representation (2) that $\{f_n\}_{n\geq 1}$ is a sequence of biorthogonal coefficients of the function $f(t)$ by the system (1). Indeed, it follows from $f\in L_q$ that $f\in L_p$. Obviously $\exists M_1>0:$

$$\left\|F^{\pm}\left(e^{it}\right)\right\|_q\leq M_1\left\|\{f_n\}_{n\geq 1}\right\|_{l_q}.$$

Thus,

$$\|f\|_q\leq\left\|A(t)\,F^{+}\left(e^{it}\right)\right\|_q+\left\|B(t)\,F^{-}\left(e^{it}\right)\right\|_q\leq$$
$$\leq\|A(t)\|_\infty\left\|F^{+}\left(e^{it}\right)\right\|_q+\|B(t)\|_\infty\left\|F^{-}\left(e^{it}\right)\right\|_q\leq$$
$$\leq M_1\left(\|A(t)\|_\infty+\|B(t)\|_\infty\right)\left\|\{f_n\}_{n\geq 1}\right\|_{l_p}=M\left\|\{f_n\}_{n\geq 1}\right\|_{l_p}.$$

Clearly, the constant M is independent of the sequence $\{f_n\}_{n\geq 1}$.

The theorem is proved.

We obtain many interesting corollaries from this theorem. We give some of them.

Having taken in (1) $A(t)\equiv e^{i\gamma(t)}$; $B(t)\equiv e^{-i\gamma(t)}$ we get the following.

Corollary 1. *Let* $\gamma(t)\in L_\infty^R$ *and* $\|\gamma(t)\|_\infty\leq\frac{\nu}{2}\pi$; $\nu<\min\left\{\frac{1}{p};\ 1-\frac{1}{p}\right\}$, $p\in(1,2]$ *be some number. Then for a system of sines*

$$\{\sin(nt+\gamma(t))\}_{n\geq 1} \tag{3}$$

the Hausdorff-Young type statements are true, i.e. $\exists M>0$, *for which*

a) for $\forall f\in L_p$ *it holds*

$$\left\|\{f_n\}_{n\geq 1}\right\|_{l_q}\leq M\cdot\|f\|_p \qquad \frac{1}{p}+\frac{1}{q}=1;$$

where $\{f_n\}_{n\geq 1}$ *are biorthogonal coefficients of the function* f *by the system* (3);

b) Let $\{f_n\}_{n\geq 1}\in l_p$. *Then* $\exists f\in L_q$ *for which*

$$\|f\|_q\leq M\cdot\left\|\{f_n\}_{n\geq 1}\right\|_{l_p}$$

moreover $\{f_n\}_{n\geq 1}$ *are biorthogonal coefficients of the function* f *by the system* (3).

In a similar way, having taken $A(t)\equiv e^{i\gamma(t)}$; $B(t)\equiv e^{-i(\gamma(t)+\pi)}$ we arrive at the following conclusion.

Corollary 2. *Let* $\gamma(t)\in L_\infty^R$ *and* $\|2\gamma(t)\pm\pi\|_\infty\leq\nu\pi$; $\nu<\min\left\{\frac{1}{p};\ 1-\frac{1}{p}\right\}$, $p\in(1,2]$ *be some number. Then by a system of cosines*

$$\{\cos(nt+\gamma(t))\}_{n\geq 1}$$

the Hausdorff-Young type statements are true.

Corollary 3. *Let* $\gamma(t) \in L_{\infty}^{R}$ *and* $\|\gamma(t)\|_{\infty} \leq \frac{\nu}{2}\pi$ *;* $\nu < \min\left\{\frac{1}{p};\ 1-\frac{1}{p}\right\}$, $p \in (1,2]$ *be some number. Then for a system of cosines*

$$1 \cup \{\cos(nt + \gamma(t))\}_{n \geq 1}$$

the statements of Corollary 2 hold.

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