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ON BILATERAL ESTIMATES OF UNIFORM APPROXIMATION BY THE PRODUCTS OF FEWER NUMBER VARIABLES FUNCTIONS

Abstract

Bilateral estimates of the best uniform approximation of functions on a multivariate cube by arbitrary products of fewer number variable functions are established in the paper. In particular, one bilateral estimate announced in [2] will be proved. Here we'll use exact annihilator of a class of products of fewer number variable functions constructed in [3].

Establishment of criteria (exact annihilator) for representation of many variable functions by the combinations of fewer number variable functions (by the sums of fewer number variable functions, bilinear forms, quasipolynomials) enabled to solve important problems of approximation of many variable functions by the combinations of fewer number variable functions. Bilateral estimates of the best approximation, establishment of the best approximation order, finding formulae for calculating the best approximation for some classes of approximate functions, construction of extreme functions and etc. (see. e.g.-[1,2]) refer to these problems.

Let's consider a function $f = f(x)$, $x = (x_1,...,x_n)$ determined on a cube $T = I^n$, $I = [0,1]$. Let $u_i \in \{x_1,...,x_n\}$, $i = 1,2,...,m$. Cite a criteria on represent ability of the function f by the product of fewer number variable functions

$$f = \prod_{i=1}^{m} \varphi_i(u_i)\ . \tag{1}$$

It is natural to assume $u_i \not\subset u_j$, $i \neq j$ and $u_i \neq \{x_1,...,x_n\}$.

Clearly, the sets of variables u_i are arbitrary and may have non-empty intersections.

Let Ç be a set of all sub-sets $\overline{m} = \{1,...,m\}$ containing even number elements and an empty set, and H be a set of all sub-sets $\overline{m}$ containing odd number elements.

Take $\alpha \subset \{1,...,m\} = \overline{m}$. By u_α we denote a sub-set of the set $\{t_1,...,t_m\}$ containing groups of variables u_i with indices $i \in \alpha$. We'll need the sets $M, N \subset$Ç with properties $M \cup N =$Ç, $M \cap N = \varnothing$, and also the sets P,$Q \subset H$ with properties $P \cup Q = H,\ P \cap Q = \varnothing$.

Let's take some point $a \in T$ wherein $f(a) \neq 0$. We'll need

Theorem 1[3]. *In order the functions* $f = f(x)$ *be represented in the form* $f = \prod_{i=1}^{m} \varphi_i(u_i)$ *it is necessary and sufficient that*

$$\overset{n}{\nabla} f \overset{def}{=} \begin{vmatrix} \prod\limits_{\alpha\in M} f(x\backslash u_\alpha, a_\alpha) & \prod\limits_{\alpha\in Q} f(x\backslash u_\alpha, a_\alpha) \\ \prod\limits_{\alpha\in P} f(x\backslash u_\alpha, a_\alpha) & \prod\limits_{\alpha\in N} f(x\backslash u_\alpha, a_\alpha) \end{vmatrix} = 0. \tag{2}$$

Let's consider the best uniform approximation of a continuous function f by the products of fewer number variable functions

$$E_f = \inf\sup|f-g|\,,\quad g = \prod_{i=1}^{m} \varphi_i(u_i).$$

It follows from the continuing of f that there is a point $a \in T$ wherein

$$\max |f(x)| = |f(a)| .$$

It is easy to see that if ∇ is an exact annihilator of the set of functions of the form $\prod_{i=1}^{m} \varphi_i(u_i)$, then for an arbitrary constants and the function g differ from zero, the products $C\nabla$ and $g\nabla$ are also exact annihilators of this set.

Theorem 2.[2] *There are two exact annihilators* ∇_1 *and* ∇_2 *allowing to establish bilateral estimates of the best approximation*

$$\|\nabla_2 f\| \leq E_f \quad \leq \|\nabla_1 f\| .$$

Proof. Lower bound. Let the sets M,N,P,Q,Ç,H be the same as in theorem 1. Accept the denotation

$$z = \prod_{\alpha\in K} z(x\backslash u_\alpha, a_\alpha).$$

In sequel M,N,P,Q, will figure as a set K as a function $z-$ the functions f and g. We have

$$\nabla f = \prod_N f\left(\prod_M f - \prod_M g\right) - \prod_Q f\left(\prod_P f - \prod_P g\right) +$$

$$+\prod_M g\left(\prod_N f - \prod_N g\right) - \prod_P g\left(\prod_Q f - \prod_Q g\right) \stackrel{def}{=} I - II + III - IV. \tag{3}$$

Let's estimate each addend of the right hand side of (3).

Each multiplier of the product $\prod_M f = \prod_{\alpha\in M} f(x \setminus u_\alpha, a_\alpha)$ contains a part of the set of variables of the function f. And so the product $\prod_M f$ itself may also contain not all the variables x_1,x_2,...,x_n. Clearly, in all cases it holds the inequality

$$\left\|\prod_M f\right\| \leq \left\|\prod_M f\right\|_{C(T)},$$

where in the left hand side the norm is only in variables participating in the product and therefore in sequel where this fact doesn't give rise to misunderstand, under $\left\|\prod_M f\right\|$ we'll mean $\leq \left\|\prod_M f\right\|_{C(T)}$.

Let's consider the difference $\prod_M f - \prod_M g$ where in each product there are $|M|$ multipliers that correspond to different elements of the set M. Denote the elements of the set M by M_1, M_2,...,M_M.

We have

$$\prod_M f - \prod_M g = f(M_1) \prod_{M\backslash M_1} f - g(M_1) \prod_{M\backslash M_1} g =$$

$$f(M_1) \prod_{M\backslash M_1} f - g(M_1) \prod_{M\backslash M_1} f + g(M_1) \prod_{M\backslash M_1} f - g(M_1) \prod_{M\backslash M_1} g =$$

$$[f(M_1) - g(M_1)] \prod_{M\backslash M_1} f + g(M_1)\left[\prod_{M\backslash M_1} f - \prod_{M\backslash M_1} g\right],$$

that admits to write

$$\left\|\prod_M f-\prod_M g\right\| \le \left\|\prod_{M\setminus M_1} f\right\| \|f-g\| + \|g(M_1)\| \left\|\prod_{M\setminus M_1} f - \prod_{M\setminus M_1} g\right\| \tag{4}$$

Allowing for (4) we get

$$\|I\| \le \left\|\prod_N f\right\| \left\|\prod_M f-\prod_M g\right\| \le$$

$$\|f\|^{|N|}\left\{\|f\|^{|M|-1}\|f-g\| + \|g\| \left\|\prod_{M\setminus M_1} f - \prod_{M\setminus M_1} g\right\|\right\}.$$

Now let's apply this method to $\left\|\prod_{M\setminus M_1} f - \prod_{M\setminus M_1} g\right\|$ considering that it suffices to conduct the approximation by the functions g satisfying the condition $|g| \le 2|f|$, and continuing this process we get

$$\|I\| \le \|f\|^{|N|} \times \left\{\|f\|^{|M|-1}\|f-g\| + \|f\|^{|M|-2}\|g\|\|f-g\| + \|f\|^{|M|-3}\|g\|^2 \times\right.$$

$$\left.\times \left\|\prod_{M\setminus\{M_1,M_2\}} f - \prod_{M\setminus\{M_1,M_2\}} g\right\|\right\} \le \left\{\|f\|^{|M|-1}\|f-g\| + \|f\|^{|M|-2}\times\right.$$

$$\left.\times\|g\|\|f-g\| + \|f\|^{|M|-3}\|g\|^2\|f-g\| + ... + \|g\|^{|M|-1}\|f-g\|\right\} =$$

$$= \|f\|^{|N|}\cdot\|f-g\| \left\{\|f\|^{|M|-1} + \|f\|^{|M|-2}\|g\| + \|f\|^{|M|-3}\times\right.$$

$$\left.\times\|g\|^2 + ... + \|g\|^{|M|-1}\right\} \le \|f\|^{|N|}\|f-g\|\left\{\|f\|^{|M|-1} + 2\|f\|^{|M|-1} +\right.$$

$$\left.+2^2\|f\|^{|M|-1} + ... + 2^{|M|-1}\|f\|^{|M|-1}\right\} = \|f\|^{|N|+|M|-1}\|f-g\|\times$$

$$\left\{1 \quad +2 \quad +2^2 \quad + ... + 2^{|M|-1}\right\} = \left(2^M-1\right)\|f\|^{|N|+|M|-1}\|f-g\|. \tag{5}$$

Acting in a similar way we get

$$\|II\| \le \left(2^{|P|}-1\right)\|f\|^{|P|+|Q|-1}\|f-g\| \tag{6}$$

Further we have

$$III = \prod_M g\left(\prod_N f - \prod_N g\right) \le$$

$$\le \prod_M 2|f| \left|\prod_N f - \prod_N g\right| \le 2^{|M|}|f|^{|M|}\left(2^{|N|}-1\right)\|f\|^{|N|-1} \le$$

$$\le 2^{|M|}\left(2^{|N|}-1\right)\|f\|^{|N|+|M|-1}\|f-g\| \tag{7}$$

$$IY = \prod_P g\left(\prod_Q f - \prod_Q g\right) \le 2^{|P|}\left(2^{|Q|}-1\right)\|f\|^{|P|+|Q|-1}\|f-g\| \tag{8}$$

Using the obtained estimates (5)-(8) in relation (3) we'll have:

$$\|\nabla\| \le \left(2^{|M|}-1\right)\|f\|^{|N|+|M|-1}\|f-g\| + \left(2^{|P|}-1\right)\|f\|^{|P|+|Q|-1}\times$$

$$\times\|f-g\| + 2^{|M|}\left(2^{|N|}-1\right)\|f\|^{|N|+|M|-1}\|f-g\| + 2^{|P|}\left(2^{|Q|}-1\right)\times$$

$$\times\|f\|^{|P|+|Q|-1}\|f-g\| = \|f-g\|\,\|f\|^{|N|+|M|-1}\left[\left(2^{|M|}-1\right)+\right.$$

$$\left.+2^{|M|}\left(2^{|N|}-1\right)\right] + \|f\|^{|P|+|Q|-1}\left[\left(2^{|P|}-1\right)+2^{|P|}\left(2^{|Q|}-1\right)\right] =$$

$$= \|f-g\|\left[\left(2^{|M|+|N|}-1\right)\|f\|^{|M|+|N|-1} + \left(2^{|P|+|Q|}-1\right)\|f\|^{|P|+|Q|-1}\right].$$

Hence we get

$$\frac{\|\nabla\|}{\left(2^{|M|+|N|}-1\right)\|f\|^{|M|+|N|-1} + \left(2^{|P|+|Q|}-1\right)\|f\|^{|P|+|Q|-1}} \le E_f.$$

The lower bound is established.

Upper bound. It is easy to see that the function

$$g_0(x) = \frac{\prod\limits_N f}{\prod\limits_{Ç\setminus(x_1,\ldots,x_n)} f}$$

has the form $\prod\limits_{i=1}^{m}\varphi_i(u_i)$. Considering this we have

$$E_f = \inf_g\|f-g\| \le \|f-g_0\| = \left\| f(x_1,\ldots,x_n) - \frac{\prod\limits_N f}{\prod\limits_{Ç\setminus(x_1,\ldots,x_n)} f}\right\| = \frac{\nabla f}{\prod\limits_{Ç\setminus(x_1,\ldots,x_n)} f}.$$

Theorem 2 is proved.

Remark. Indeed, it was proved more, namely, bilateral estimates with concrete coefficients were established.

We can improve the lower bound from theorem 2. The following theorem will help us.

Theorem 3. *The following lower bound is true*

$$\left\|\frac{\nabla f}{K(f)}\right\| \le E_f, \tag{9}$$

where

$$K(f) = \left|\prod_N f\right| \cdot \sum_{i=1}^{|M|} 2^{i-1}\left|\prod_{M\setminus M_i} f\right| + \left|\prod_Q f\right| \cdot \sum_{i=1}^{|P|} 2^{i-1}\left|\prod_{P\setminus P_i} f\right| + 2^{|M|}\cdot$$

$$\cdot\left|\prod_M f\right| \cdot \sum_{i=1}^{|N|} 2^{i-1}\left|\prod_{N\setminus N_i} f\right| + 2^{|P|}\left|\prod_P f\right| \cdot \sum_{i=1}^{|Q|} 2^{i-1}\left|\prod_{Q\setminus Q_i} f\right|,$$

where we mean

$$R = \left\{R_1,...,R_{|R|}\right\} \quad R = M, N, P, Q.$$

Proof. Let's use the equality $(3^{'})$ from theorem 2.

$$\nabla f = I - II + III - IV\ . \tag{$3^{'}$}$$

Estimate each addend somewhat otherwise. For estimate I we use the earlier established equality

$$\prod_M f - \prod_M g = [f(M_1) - g(M_1)]) \prod_{M\backslash M_1} f + g(M_1)\left[\prod_{M\backslash M_1} f - \prod_{M\backslash M_1} g\right]. \tag{10}$$

Applying this method to the last summand we get

$$\prod_M f - \prod_M g = [f(M_1) - g(M_1)] \prod_{M\backslash M_1} f + g(M_1)\times$$

$$\left\{[f(M_2) - g(M_2)] \prod_{M\backslash M_1 M_2} f + g(M_2)\left[\prod_{M\backslash M_1 M_2} f - \prod_{M\backslash M_1 M_2} g\right]\right\}.$$

Then from (10) we get

$$\left|\prod_M f - \prod_M g\right| \leq \|f-g\| \cdot \left|\prod_{M\backslash M_1} f\right| + |g(M_1)| \left\{\|f-g\| \cdot \left|\prod_{M\backslash M_1 M_2} f\right| + \right.$$

$$\left. + |g(M_2)| \left|\prod_{M\backslash M_1 M_2} f - \prod_{M\backslash M_1 M_2} g\right|\right\} = \|f-g\| \cdot \left|\prod_{M\backslash M_1} f\right| + |g(M_1)| \times$$

$$\times \|f-g\| \cdot \left|\prod_{M\backslash M_1 M_2} f\right| + |g(M_1)|\,|g(M_2)| \left|\prod_{M\backslash M_1 M_2} f - \prod_{M\backslash M_1 M_2} g\right|.$$

Continuing this process we get

$$\left|\prod_M f - \prod_M g\right| \leq \|f-g\| \left\{\left|\prod_{M\backslash M_1} f\right| + |g(M_1)| \cdot \left|\prod_{M\backslash M_1 M_2} f\right| + \right.$$

$$\left. + |g(M_1)|\,|g(M_2)| \left|\prod_{M\backslash M_1 M_2 M_3} f\right| + ... + \left|g(M_1)...g(M_{|M|-1}|)\right.\right\}. \tag{11}$$

Considering that we can conduct the approximation with g satisfying $|g| \leq 2|f|$ from (11) we get

$$\left|\prod_M f - \prod_M g\right| \leq \|f-g\| \cdot \left\{\left|\prod_{M\backslash M_1 M_2} f\right| + 2|f(M_1)| \cdot \left|\prod_{M\backslash M_1 M_2} f\right| + \right.$$

$$+2^2 |f(M_1) f(M_2)| \cdot \left|\prod_{M\backslash M_1 M_2 M_3} f\right| + ... + 2^{|M|-1} f(M_1) f(M_2) ...$$

$$...\left|f(M_{|M|-1})\right|\} = \|f-g\|\left\{\left|\prod_{M\setminus M_1} f\right| + 2\left|\prod_{M\setminus M_2} f\right| +\right.$$

$$\left.+2^2\left|\prod_{M\setminus M_3} f\right| + ... + 2^{|M|-1}\left|\prod_{M\setminus M_{|M|}} f\right|\right\} = \|f-g\|\cdot\sum_{i=1}^{|M|} 2^{i-1}\left|\prod_{M\setminus M_i} f\right|. \quad (12)$$

Relation (12) allows to write

$$I = \prod_N f\left(\prod_M f - \prod_M g\right) \le \left|\prod_N f\right|\cdot\sum_{i=1}^{|M|} 2^{i-1}\left|\prod_{M\setminus M_1} f\right|\cdot\|f-g\|. \quad (13)$$

Acting in a similar way we get

$$II = \prod_Q f\left(\prod_P f - \prod_P g\right) \le \left|\prod_Q f\right|\cdot\sum_{i=1}^{|P|} 2^{i-1}\left|\prod_{P\setminus P_i} f\right|\cdot\|f-g\| \quad (14)$$

$$III = \prod_M f\left(\prod_N f - \prod_N g\right) \le \left|\prod_M g\right|\cdot\sum_{i=1}^{|N|} 2^{i-1}\left|\prod_{N\setminus N_i} f\right|\cdot\|f-g\| \quad (15)$$

$$IV = \prod_P g\left(\prod_Q f - \prod_Q g\right) \le \left|\prod_P g\right|\cdot\sum_{i=1}^{|Q|} 2^{i-1}\left|\prod_{Q\setminus Q_i} f\right|\cdot\|f-g\|. \quad (16)$$

Allowing for (13)-(16) we have

$$\nabla f = I - II + III - IV \le$$

$$\le \|f-g\|\left\{\prod_N f\left|\sum_{i=1}^{|M|}\right|\left|\prod_{M\setminus M_i} 2^{i-1} f\right| + \left|\prod_Q f\right|\cdot\sum_{i=1}^{|P|} 2^{i-1}\left|\prod_{P\setminus P_i} f\right| + 2^{|M|}\left|\prod_M f\right|\cdot\right.$$

$$\left.\cdot\sum_{i=1}^{|N|} 2^{i-1}\left|\prod_{N\setminus N_i} f\right| + 2^{|P|}\left|\prod_P f\right|\sum_{i=1}^{|Q|} 2^{i-1}\left|\prod_{Q\setminus Q_i} f\right|\right\} \Longrightarrow \left\|\frac{\nabla f}{K(f)}\right\| \le E_f,$$

where

$$K(f) = \left|\prod_N f\right|\cdot\sum_{i=1}^{|M|} 2^{i-1}\left|\prod_{M\setminus M_i} f\right| + \left|\prod_Q f\right|\cdot\sum_{i=1}^{|P|} 2^{i-1}\left|\prod_{P\setminus P_i} f\right| + 2^{|M|}\cdot$$

$$\cdot\left|\prod_M f\right|\cdot\sum_{i=1}^{|N|} 2^{i-1}\left|\prod_{N\setminus N_i} f\right| + 2^{|P|}\left|\prod_P f\right|\cdot\sum_{i=1}^{|Q|} 2^{i-1}\left|\prod_{Q\setminus Q_i} f\right|.$$

Theorem 3 is proved.

Now, let's show that the estimate (9) is better than the lower bound E_f in theorem 2.

We have

$$\left|\prod_N f\right| \le \|f\|^{|N|};\quad \sum_{i=1}^{|M|} 2^{i-1}\left|\prod_{M\setminus M_i} f\right| \le \|f\|^{|M-1|}\sum_{i=1}^{|M|} 2^{i-1} = \left(2^{(M)}-1\right)\|f\|^{|M-1|}.$$

Therefore

$$\left|\prod_N f\right| \cdot \sum_{i=1}^{|M|} 2^{i-1} \left|\prod_{M\backslash M_i} f\right| \le \left(2^{(M)}-1\right) \|f\|^{|M|+|N|-1}.$$

Similarly

$$\left|\prod_Q f\right| \cdot \sum_{i=1}^{|P|} 2^{i-1} \left|\prod_{P\backslash P_i} f\right| \le \left(2^{(P)}-1\right) \|f\|^{|P|+|Q|-1}$$

$$\left|\prod_M f\right| \cdot \sum_{i=1}^{|N|} 2^{i-1} \left|\prod_{N\backslash N_i} f\right| \le \left(2^{(N)}-1\right) \|f\|^{|M|+|N|-1}$$

$$\left|\prod_P f\right| \cdot \sum_{i=1}^{|Q|} 2^{i-1} \left|\prod_{Q\backslash Q_i} f\right| \le \left(2^{(Q)}-1\right) \|f\|^{|P|+|Q|-1}.$$

Now, let's estimate the denominator of the left hand side of the inequality (9). Summing the four inequalities obtained above we get.

$$K(f) = \left|\prod_N f\right| \sum_{i=1}^{|M|} 2^{i-1} \left|\prod_{M\backslash M_i} f\right| + \left|\prod_Q f\right| \sum_{i=1}^{|P|} 2^{i-1} \left|\prod_{P\backslash P_i} f\right| + 2^{|M|} \left|\prod_M f\right| \times$$

$$\times \sum_{i=1}^{|N|} 2^{i-1} \left|\prod_{N\backslash N_i} f\right| + 2^{|P|} \left|\prod_P f\right| \sum_{i=1}^{|Q|} 2^{i-1} \left|\prod_{Q\backslash Q_i} f\right| \le \left(2^{|M|}-1\right) \times$$

$$\times \|f\|^{|M|+|N|-1} + \left(2^{|P|}-1\right) \|f\|^{|P|+|Q|-1} + 2^{|M|} \left(2^{|N|}-1\right) \times$$

$$\times \|f\|^{|M|+|N|-1} + 2^{|P|} \left(2^{|Q|}-1\right) \|f\|^{|P|+|Q|-1} = \left[\left(2^{|M|}-1\right) + \right.$$

$$\left. + 2^{|M|} \left(2^{|N|}-1\right)\right] \|f\|^{|M|+|N|-1} + \left[\left(2^{|P|}-1\right) + 2^{|P|} \left(2^{|Q|}-1\right)\right] \times$$

$$\times \|f\|^{|P|+|Q|-1} = \left(2^{|M|+|N|}-1\right) \|f\|^{|M|+|N|-1} + \left(2^{|P|+|Q|}-1\right) \|f\|^{|P|+|Q|-1}.$$

This means that the estimate in theorem 3 is better that the lower bound from theorem 2.

Remark. We can simplify lower bound E_f in theorem 2.

By definition of the sets M, N, P, Q, Ç and H,

$$|M| + |N| = |\text{Ç}|, \quad |P| + |Q| = |H|.$$

It is easy to see that the number of elements (subsets) of the set $\overline{n} = \{1,...,n\}$ containing even number elements (and zero) equals the number of its elements containing odd number elements: $|\text{Ç}| + |H| = 2^n$, and it is also known that $|\text{Ç}| = |H| = 2^{n-1}$.

Taking this into account we can simplify the denominator of the fraction E_f in the lower bound as follows

$$\left[\left(2^{|M|+|N|}-1\right) \|f\|^{|M|+|N|-1} + \left(2^{|P|+|Q|-1}\right) \|f\|^{|P|+|Q|-1}\right] =$$

$$\left(2^{|\text{Ç}|}-1\right) \|f\|^{|M|+|N|-1} + \left(2^{|H|}-1\right) \|f\|^{|P|+|Q|-1} =$$

$$\|f\|^{2^{n-1}-1}\left(2^{2^{n-1}}-1\right)+\|f\|^{2^{n-1}-1}\left(2^{2^{n-1}}-1\right)=2\left(2^{2^{n-1}-1}-1\right)\|f\|^{2^{n-1}-1}.$$

Thus we can write the bilateral estimate E_f in theorem 2 in the following form

$$\frac{\|\nabla f\|}{2\left(2^{2^{n-1}}-1\right)\|f\|^{2^{n-1}-1}}\leq E_f\leq\left\|\frac{\nabla f}{\prod\limits_{Ç\setminus(x_1,\ldots,x_n)} f}\right\|.$$

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