Esmira N. AKHMEDOVA, Hidayat M. HUSEYNOV

ON SOLUTION OF THE INVERSE STURM-LIOUVILLE PROBLEM WITH DISCONTINUOUS COEFFICIENTS

Abstract

In the paper we prove a theorem on necessary and sufficient conditions on solvability of the inverse problem of spectral analysis for Sturm-Liouville operator with discontinuous coefficients and give solution algorithm of the inverse problem.

Consider the boundary value problem

$$-y'' + q(x)\,y = \lambda^2 \rho(x)\,y, \quad 0 \le x \le \pi, \tag{1}$$

$$y(0) = 0, \qquad y(\pi) = 0, \tag{2}$$

where $q(x) \in L_2(0,\pi)$ is a real-valued function, λ is a complex parameter and $\rho(x)$ is a piecewise-constant function:

$$\rho(x) = \begin{cases} 1, & 0 \le x \le a, \\ \alpha^2, & a < x \le \pi, \end{cases} \qquad 0 < \alpha \ne 1. \tag{3}$$

We assume that $a(1+\alpha) > \pi\alpha$.

The following theorem gives necessary and sufficient conditions for solvability of the inverse problem of spectral analysis for boundary value problem $(1)-(3)$.

Theorem. *For the sequences* $\left\{\lambda_n^2\,,\ \alpha_n\right\}_{n\ge1}$ *, where* $\lambda_n \ne \lambda_m$ *for* $n \ne m$, $\alpha_n > 0$ *for all* n *to be the spectral data of a problem* $L(q(x))$ *of the form* $(1)-(3)$ *with* $q(x) \in L_2(0,\pi)$, *it is necessary and sufficient to satisfy conditions*

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}, \qquad \alpha_n = \alpha_n^0 + \frac{t_n}{n}. \tag{4}$$

Here λ_n^0 *are the zeros of the function*

$$\Delta_0(\lambda) \equiv \frac{1}{2}\left(1+\frac{1}{\alpha}\right)\frac{\sin\lambda(\alpha\pi - \alpha a + a)}{\lambda} + \frac{1}{2}\left(1-\frac{1}{\alpha}\right)\frac{\sin\lambda(-\alpha\pi + \alpha a + a)}{\lambda};$$

$$\alpha_n^0 = \int_0^\pi s_0^2\left(x,\lambda_n^0\right)\rho(x)\,dx;$$

$$s_0(x,\lambda) = \frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\frac{\sin\lambda\mu^+(x)}{\lambda} + \frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\frac{\sin\lambda\left(\mu^-(x)\right)}{\lambda};$$

$$\mu^\pm(x) = \pm x\sqrt{\rho(x)} + a\left(1 \mp \sqrt{\rho(x)}\right);$$

d_n *is a bounded sequence;* $\{k_n\} \in l_2$, $\{t_n\} \in l_2$.

Proof. Necessity of the theorem is proved in [1]. Let's prove sufficiency. Let the real numbers $\left\{\lambda_n^2\ ,\ \alpha_n\right\}_{n\geq 1}$ of the form (4) be given. Construct the functions $F_0(x,t)$ and $F(x,t)$ by the formulas

$$F_0(x,t)=\sum_{n=1}^{\infty}\left(\frac{s_0(t,\lambda_n)\sin\lambda_n x}{\alpha_n\lambda_n}-\frac{s_0\left(t,\lambda_n^0\right)\sin\lambda_n^0 x}{\alpha_n^0\lambda_n^0}\right) \tag{5}$$

and

$$F(x,t)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)F_0\left(\mu^+(x),t\right)+\frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)F_0\left(\mu^-(x),t\right). \tag{6}$$

Let $A(x,t)$ be a solution of the main equation of the inverse problem for boundary value problem (1) – (3) (see [2]):

$$\frac{2}{1+\sqrt{\rho(t)}}A\left(x,\mu^+(t)\right)+\frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A(x,2a-t)+F(x,t)+$$

$$+\int_0^{\mu^+(x)}A(x,\xi)F_0(\xi,t)\,d\xi=0,\qquad 0<t<x. \tag{7}$$

Construct the function $s(x,\lambda)$ by the formula

$$s(x,\lambda)=s_0(x,\lambda)+\int_0^{\mu^+(x)}A(x,t)\frac{\sin\lambda t}{\lambda}dt, \tag{8}$$

and the function $q(x)$ by the formula

$$q(x)=\frac{4\rho(x)}{\sqrt{\rho(x)}+1}\frac{d}{dx}A\left(x,\mu^+(x)\right). \tag{9}$$

Denote

$$b(x)=\sum_{n=1}^{\infty}\left(\frac{\cos\lambda_n x}{\alpha_n\lambda_n^2}-\frac{\cos\lambda_n^0 x}{\alpha_n\lambda_n^{02}}\right).$$

We can show that $b(x)\in W_2^1(0,\pi)$,

$$F_0(x,t)=\frac{1}{4}\left(1+\frac{1}{\sqrt{\rho(t)}}\right)\left[b\left(x-\mu^+(t)\right)-b\left(x+\mu^+(t)\right)\right]+$$

$$+\frac{1}{4}\left(1-\frac{1}{\sqrt{\rho(t)}}\right)\left[b\left(x-\mu^-(t)\right)-b\left(x+\mu^-(t)\right)\right],$$

and the functions $F_0(x,t)$, $F(x,t)$, $A(x,t)$ have the same smoothness.

According to (5) and (6) we have

$$F''_{0tt}(x,t)=\rho(t)F''_{0xx}(x,t),\qquad \rho(t)F''_{xx}(x,t)=\rho(x)F''_{tt}(x,t), \tag{10}$$

$$F_0(x,t)|_{x=0}=0,\qquad F_0(x,t)|_{t=0}=0, \tag{11}$$

$$\frac{\partial}{\partial x}F_0\left(\mu^{\pm}(x),t\right)=\pm\sqrt{\rho(x)}\left.\frac{\partial}{\partial\xi}F_0(\xi,t)\right|_{\xi=\mu^{\pm}(x)}. \tag{12}$$

Using the main equation (7) it is easy to prove that

$$A(x,0)=0, \tag{13}$$

$$\frac{\sqrt{\rho(x)}-1}{\sqrt{\rho(x)}+1}\frac{d}{dx}A\left(x,\mu^{+}(x)\right)=\frac{d}{dx}\left\{A\left(x,\mu^{-}(x)+0\right)-A\left(x,\mu^{-}(x)-0\right)\right\}. \tag{14}$$

Lemma 1. *The relations*

$$-s''(x,\lambda)+q(x)s(x,\lambda)=\lambda^2\rho(x)s(x,\lambda), \tag{15}$$

$$s(0,\lambda)=0,\qquad s'(0,\lambda)=1 \tag{16}$$

hold.

Proof. Assume that $b(x)\in W_2^2(0,\pi)$. Differentiating twice on x the identity

$$J(x,t):=\frac{2}{1+\sqrt{\rho(t)}}A\left(x,\mu^{+}(t)\right)+\frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A(x,2a-t)+$$

$$+F(x,t)+\int_0^{\mu^{+}(x)}A(x,\xi)F_0(\xi,t)\,d\xi=0, \tag{17}$$

using (11) we have

$$J'_x(x,t)=\frac{2}{1+\sqrt{\rho(t)}}A'_x\left(x,\mu^{+}(t)\right)+\frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A'_x(x,2a-t)+$$

$$+F'_x(x,t)+\int_0^{\mu^{+}(x)}A'_x(x,\xi)F_0(\xi,t)\,d\xi+\sqrt{\rho(x)}A\left(x,\mu^{+}(x)\right)F_0\left(\mu^{+}(x),t\right)+$$

$$+\sqrt{\rho(x)}F_0\left(\mu^{-}(x),t\right)\left(A\left(x,\mu^{-}(x)+0\right)-A\left(x,\mu^{-}(x)-0\right)\right)=0,$$

$$J''_{xx}(x,t)=\frac{2}{1+\sqrt{\rho(t)}}A''_{xx}\left(x,\mu^{+}(t)\right)+\frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A''_{xx}(x,2a-t)+$$

$$+F''_{xx}(x,t)+\int_0^{\mu^{+}(x)}A''_{xx}(x,\xi)F_0(\xi,t)\,d\xi+\sqrt{\rho(x)}F_0\left(\mu^{+}(x),t\right)\left.\frac{\partial A(x,\xi)}{\partial x}\right|_{\xi=\mu^{+}(x)}+$$

$$+\sqrt{\rho(x)}F_0\left(\mu^{-}(x),t\right)\left(\left.\frac{\partial A(x,\xi)}{\partial x}\right|_{\xi=\mu^{-}(x)+0}-\left.\frac{\partial A(x,\xi)}{\partial x}\right|_{\xi=\mu^{-}(x)-0}\right)+$$

$$+\sqrt{\rho(x)}F_0\left(\mu^{+}(x),t\right)\frac{d}{dx}A\left(x,\mu^{+}(x)\right)+\sqrt{\rho(x)}A\left(x,\mu^{+}(x)\right)\frac{\partial}{\partial x}F_0\left(\mu^{+}(x),t\right)+$$

$$+\sqrt{\rho(x)}\left(A\left(x,\mu^{-}(x)+0\right)-A\left(x,\mu^{-}(x)-0\right)\right)\frac{\partial}{\partial x}F_0\left(\mu^{-}(x),t\right)+$$

$$+\sqrt{\rho(x)}F_0\left(\mu^{-}(x),t\right)\frac{d}{dx}\left\{A\left(x,\mu^{-}(x)+0\right)-A\left(x,\mu^{-}(x)-0\right)\right\}=0. \tag{18}$$

Differentiating the identity (17) on t we have

$$J'_t(x,t) = \frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}} A'_t\left(x,\mu^+(t)\right) - \frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}} A'_t(x,2a-t) +$$

$$+F'_t(x,t) + \int_0^{\mu^+(x)} A(x,\xi)\, F'_{0_t}(\xi,t)\, d\xi = 0,$$

$$J''_{tt}(x,t) = \frac{2\rho(t)}{1+\sqrt{\rho(t)}} A''_{tt}\left(x,\mu^+(t)\right) + \frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}} A''_{tt}(x,2a-t) +$$

$$+F''_{tt}(x,t) + \int_0^{\mu^+(x)} A(x,\xi)\, F''_{0_{tt}}(\xi,t)\, d\xi = 0.$$

According to (10)

$$J''_{tt}(x,t) = \frac{2}{1+\sqrt{\rho(t)}} A''_{tt}\left(x,\mu^+(t)\right) + \frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}} A''_{tt}(x,2a-t) +$$

$$+\frac{1}{\rho(t)} F''_{tt}(x,t) + \int_0^{\mu^+(x)} A(x,\xi)\, F''_{0_{\xi\xi}}(\xi,t)\, d\xi = 0.$$

Integrating twice by parts, using (11) and (13) we get

$$\int_0^{\mu^+(x)} A(x,\xi)\, F''_{0_{\xi\xi}}(\xi,t)\, d\xi =$$

$$= \left(A\left(x,\mu^-(x)-0\right) - A\left(x,\mu^-(x)+0\right)\right) \frac{\partial}{\partial\xi} F_0(\xi,t)\Big|_{\xi=\mu^-(x)} +$$

$$+A\left(x,\mu^+(x)\right) \frac{\partial}{\partial\xi} F_0(\xi,t)\Big|_{\xi=\mu^+(x)} - F_0\left(x,\mu^-(x)\right) \frac{\partial A(x,\xi)}{\partial\xi}\Big|_{\xi=\mu^-(x)-0} +$$

$$+F_0(x,0) \frac{\partial A(x,\xi)}{\partial\xi}\Big|_{\xi=0} - F_0\left(x,\mu^+(x)\right) \frac{\partial A(x,\xi)}{\partial\xi}\Big|_{\xi=\mu^+(x)} +$$

$$+F_0\left(x,\mu^-(x)\right) \frac{\partial A(x,\xi)}{\partial\xi}\Big|_{\xi=\mu^-(x)+0} + \int_0^{\mu^+(x)} A''_{\xi\xi}(x,\xi)\, F_0(\xi,t)\, d\xi.$$

Thus we obtain:

$$J''_{tt}(x,t) = \frac{2}{1+\sqrt{\rho(t)}} A''_{tt}\left(x,\mu^+(t)\right) + \frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}} A''_{tt}(x,2a-t) +$$

$$+\frac{1}{\rho(t)} F''_{tt}(x,t) + \left(A\left(x,\mu^-(x)-0\right) - A\left(x,\mu^-(x)+0\right)\right) \frac{\partial}{\partial\xi} F_0(\xi,t)\Big|_{\xi=\mu^-(x)} +$$

$$+A\left(x,\mu^+(x)\right) \frac{\partial}{\partial\xi} F_0(\xi,t)\Big|_{\xi=\mu^+(x)} - F_0\left(x,\mu^-(x)\right) \frac{\partial A(x,\xi)}{\partial\xi}\Big|_{\xi=\mu^-(x)-0} +$$

$$+F_0(x,0) \frac{\partial A(x,\xi)}{\partial\xi}\Big|_{\xi=0} - F_0\left(x,\mu^+(x)\right) \frac{\partial A(x,\xi)}{\partial\xi}\Big|_{\xi=\mu^+(x)} +$$

$$+F_0\left(x,\mu^-(x)\right)\left.\frac{\partial A(x,\xi)}{\partial\xi}\right|_{\xi=\mu^-(x)+0}+\int\limits_0^{\mu^+(x)}A''_{\xi\xi}(x,\xi)\,F_0(\xi,t)\,d\xi. \qquad (19)$$

Using (17) – (19) in the identity

$$J''_{xx}(x,t)-\rho(x)\,J''_{tt}(x,t)-q(x)\,J(x,t)\equiv 0$$

we have

$$\frac{2}{1+\sqrt{\rho(t)}}A''_{xx}\left(x,\mu^+(t)\right)+\frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A''_{xx}(x,2a-t)+F''_{xx}(x,t)+$$

$$+\int\limits_0^{\mu^+(x)}A''_{xx}(x,\xi)\,F_0(\xi,t)\,d\xi+\sqrt{\rho(x)}F_0\left(\mu^+(x),t\right)\left.\frac{\partial A(x,\xi)}{\partial x}\right|_{\xi=\mu^+(x)}+$$

$$+\sqrt{\rho(x)}F_0\left(\mu^-(x),t\right)\left(\left.\frac{\partial A(x,\xi)}{\partial x}\right|_{\xi=\mu^-(x)+0}-\left.\frac{\partial A(x,\xi)}{\partial x}\right|_{\xi=\mu^-(x)-0}\right)+$$

$$+\sqrt{\rho(x)}F_0\left(\mu^+(x),t\right)\frac{d}{dx}A\left(x,\mu^+(x)\right)+\sqrt{\rho(x)}A\left(x,\mu^+(x)\right)\frac{\partial}{\partial x}F_0\left(\mu^+(x),t\right)+$$

$$+\sqrt{\rho(x)}\left(A\left(x,\mu^-(x)+0\right)-A\left(x,\mu^-(x)-0\right)\right)\frac{\partial}{\partial x}F_0\left(\mu^-(x),t\right)+$$

$$+\sqrt{\rho(x)}F_0\left(\mu^-(x),t\right)\frac{d}{dx}\left\{A\left(x,\mu^-(x)+0\right)-A\left(x,\mu^-(x)-0\right)\right\}-$$

$$-\rho(x)\left(\frac{2}{1+\sqrt{\rho(t)}}A''_{tt}\left(x,\mu^+(t)\right)+\frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A''_{tt}(x,2a-t)+\right.$$

$$+\frac{1}{\rho(t)}F''_{tt}(x,t)+\left(A\left(x,\mu^-(x)-0\right)-A\left(x,\mu^-(x)+0\right)\right)\left.\frac{\partial}{\partial\xi}F_0(\xi,t)\right|_{\xi=\mu^-(x)}+$$

$$+A\left(x,\mu^+(x)\right)\left.\frac{\partial}{\partial\xi}F_0(\xi,t)\right|_{\xi=\mu^+(x)}-F_0\left(x,\mu^-(x)\right)\left.\frac{\partial A(x,\xi)}{\partial\xi}\right|_{\xi=\mu^-(x)-0}+$$

$$+F_0(x,0)\left.\frac{\partial A(x,\xi)}{\partial\xi}\right|_{\xi=0}-F_0\left(x,\mu^+(x)\right)\left.\frac{\partial A(x,\xi)}{\partial\xi}\right|_{\xi=\mu^+(x)}+$$

$$\left.+F_0\left(x,\mu^-(x)\right)\left.\frac{\partial A(x,\xi)}{\partial\xi}\right|_{\xi=\mu^-(x)+0}+\int\limits_0^{\mu^+(x)}A''_{\xi\xi}(x,\xi)\,F_0(\xi,t)\,d\xi\right)-$$

$$-q(x)\left(\frac{2}{1+\sqrt{\rho(t)}}A\left(x,\mu^+(t)\right)+\frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A(x,2a-t)+\right.$$

$$\left.+F(x,t)+\int\limits_0^{\mu^+(x)}A(x,\xi)\,F_0(\xi,t)\,d\xi\right)=0,$$

from which using the formulas (6) , (9) − (12) and (14) we get

$$\left[\frac{2}{1+\sqrt{\rho(t)}}A''_{xx}\left(x,\mu^{+}(t)\right)+\frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A''_{xx}(x,2a-t)-\right.$$

$$-\rho(x)\left(\frac{2}{1+\sqrt{\rho(t)}}A''_{tt}\left(x,\mu^{+}(t)\right)+\frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A''_{tt}(x,2a-t)\right)-$$

$$\left.-q(x)\left(\frac{2}{1+\sqrt{\rho(t)}}A\left(x,\mu^{+}(t)\right)+\frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A(x,2a-t)\right)\right]+$$

$$+\int_{0}^{\mu^{+}(x)}\left[A''_{xx}(x,\xi)-\rho(x)A''_{\xi\xi}(x,\xi)-q(x)A(x,\xi)\right]F_0(\xi,t)\,d\xi=0. \tag{20}$$

Homogeneous equation (20) has only trivial solution (see [2]), i.e.

$$A''_{xx}(x,t)-\rho(x)A''_{tt}(x,t)-q(x)A(x,t)=0,\quad 0<t<x. \tag{21}$$

Differentiating (8) twice we calculate

$$s'(x,\lambda)=s'_0(x,\lambda)+\int_{0}^{\mu^{+}(x)}A'_x(x,t)\frac{\sin\lambda t}{\lambda}dt+\sqrt{\rho(x)}A\left(x,\mu^{+}(x)\right)\frac{\sin\lambda\mu^{+}(x)}{\lambda}+$$

$$+\frac{\sqrt{\rho(x)}\sin\lambda\mu^{-}(x)}{\lambda}\left\{A\left(x,\mu^{-}(x)+0\right)-A\left(x,\mu^{-}(x)-0\right)\right\}, \tag{22}$$

$$s''(x,\lambda)=s''_0(x,\lambda)+\int_{0}^{\mu^{+}(x)}A''_{xx}(x,t)\frac{\sin\lambda t}{\lambda}dt+\sqrt{\rho(x)}\left.\frac{\partial}{\partial x}A(x,t)\right|_{t=\mu^{+}(x)}\frac{\sin\lambda\mu^{+}(x)}{\lambda}+$$

$$+\sqrt{\rho(x)}\left(\left.\frac{\partial}{\partial x}A(x,t)\right|_{t=\mu^{-}(x)+0}-\left.\frac{\partial}{\partial x}A(x,t)\right|_{t=\mu^{-}(x)-0}\right)\frac{\sin\lambda\mu^{-}(x)}{\lambda}+$$

$$+\frac{\sqrt{\rho(x)}\sin\lambda\mu^{+}(x)}{\lambda}\frac{d}{dx}A\left(x,\mu^{+}(x)\right)+\rho(x)A\left(x,\mu^{+}(x)\right)\cos\lambda\mu^{+}(x)+$$

$$+\frac{\sqrt{\rho(x)}\sin\lambda\mu^{-}(x)}{\lambda}\frac{d}{dx}\left\{A\left(x,\mu^{-}(x)+0\right)-A\left(x,\mu^{-}(x)-0\right)\right\}-$$

$$-\rho(x)\left\{A\left(x,\mu^{-}(x)+0\right)-A\left(x,\mu^{-}(x)-0\right)\right\}\cos\lambda\mu^{-}(x). \tag{23}$$

On the other hand, integrating twice by parts, using (13) we have

$$\lambda^2\rho(x)s(x,\lambda)=\lambda^2\rho(x)s_0(x,\lambda)+\lambda^2\rho(x)\int_{0}^{\mu^{+}(x)}A(x,t)\frac{\sin\lambda t}{\lambda}dt=$$

$$=-s''_0(x,\lambda)+\rho(x)\cos\lambda\mu^{-}(x)\left[A\left(x,\mu^{-}(x)+0\right)-A\left(x,\mu^{-}(x)-0\right)\right]-$$

$$-\rho(x)\cos\lambda\mu^+(x)\,A\left(x,\mu^+(x)\right)+\rho(x)\frac{\sin\lambda\mu^-(x)}{\lambda}\left[\left.\frac{\partial}{\partial t}A(x,t)\right|_{t=\mu^-(x)-0}-\right.$$

$$\left.-\left.\frac{\partial}{\partial t}A(x,t)\right|_{t=\mu^-(x)+0}\right]+\rho(x)\frac{\sin\lambda\mu^+(x)}{\lambda}\left.\frac{\partial}{\partial t}A(x,t)\right|_{t=\mu^+(x)}-$$

$$-\rho(x)\int\limits_0^{\mu^+(x)}A''_{tt}(x,t)\frac{\sin\lambda t}{\lambda}dt. \qquad (24)$$

According to (8), $(23)-(24)$ we get

$$s''(x,\lambda)+\lambda^2\rho(x)s(x,\lambda)-q(x)s(x,\lambda)=s''_0(x,\lambda)+\int\limits_0^{\mu^+(x)}A''_{xx}(x,t)\frac{\sin\lambda t}{\lambda}dt+$$

$$+\sqrt{\rho(x)}\left.\frac{\partial}{\partial x}A(x,t)\right|_{t=\mu^+(x)}\frac{\sin\lambda\mu^+(x)}{\lambda}+$$

$$+\sqrt{\rho(x)}\left(\left.\frac{\partial}{\partial x}A(x,t)\right|_{t=\mu^-(x)+0}-\left.\frac{\partial}{\partial x}A(x,t)\right|_{t=\mu^-(x)-0}\right)\frac{\sin\lambda\mu^-(x)}{\lambda}+$$

$$+\frac{\sqrt{\rho(x)}\sin\lambda\mu^+(x)}{\lambda}\frac{d}{dx}A\left(x,\mu^+(x)\right)+\rho(x)A\left(x,\mu^+(x)\right)\cos\lambda\mu^+(x)+$$

$$+\frac{\sqrt{\rho(x)}\sin\lambda\mu^-(x)}{\lambda}\frac{d}{dx}\left\{A\left(x,\mu^-(x)+0\right)-A\left(x,\mu^-(x)-0\right)\right\}-$$

$$-\rho(x)\left\{A\left(x,\mu^-(x)+0\right)-A\left(x,\mu^-(x)-0\right)\right\}\cos\lambda\mu^-(x)-s''_0(x,\lambda)+\rho(x)\cos\lambda\mu^-(x)\times$$

$$\times\left[A\left(x,\mu^-(x)+0\right)-A\left(x,\mu^-(x)-0\right)\right]-\rho(x)\cos\lambda\mu^+(x)A\left(x,\mu^+(x)\right)+$$

$$+\rho(x)\frac{\sin\lambda\mu^-(x)}{\lambda}\left[\left.\frac{\partial}{\partial t}A(x,t)\right|_{t=\mu^-(x)-0}-\left.\frac{\partial}{\partial t}A(x,t)\right|_{t=\mu^-(x)+0}\right]+$$

$$+\rho(x)\frac{\sin\lambda\mu^+(x)}{\lambda}\left.\frac{\partial}{\partial t}A(x,t)\right|_{t=\mu^+(x)}-\rho(x)\int\limits_0^{\mu^+(x)}A''_{tt}(x,t)\frac{\sin\lambda t}{\lambda}dt-$$

$$-q(x)\left[\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\frac{\sin\lambda\mu^+(x)}{\lambda}+\right.$$

$$\left.+\frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\frac{\sin\lambda\mu^-(x)}{\lambda}+\int\limits_0^{\mu^+(x)}A(x,t)\frac{\sin\lambda t}{\lambda}dt\right].$$

Hence using (9), (14) and (21) we obtain (15). For $x=0$ formulas (8) and (22) imply (16). Lemma 1 is proved in the case $b(x)\in W_2^2(0,\pi)$.

The proof of Lemma 1 in the case $b(x)\in W_2^1(0,\pi)$ is carried out by a standard method (see e.g. $[3]$, p. 56).

Lemma 2. *For each function $g(x) \in L_2(0,\pi;\rho)$ the relation*

$$\int_0^\pi \rho(x)\, g^2(x)\, dx = \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \left(\int_0^\pi \rho(t)\, g(t)\, s(t,\lambda_n)\, dt \right)^2 \tag{25}$$

holds.

Proof. Using formula (5),

$$F(x,t) = \sum_{n=1}^{\infty} \left(\frac{s_0(t,\lambda_n)\, s_0(x,\lambda_n)}{\alpha_n} - \frac{s_0\left(t,\lambda_n^0\right) s_0\left(x,\lambda_n^0\right)}{\alpha_n^0} \right), \tag{26}$$

and

$$\frac{\sin \lambda\xi}{\lambda} = \begin{cases} s_0(\xi,\lambda), & \xi < a, \\ \dfrac{2\alpha}{1+\alpha} s_0\left(\dfrac{\xi}{\alpha} + a - \dfrac{a}{\alpha}, \lambda\right) + \dfrac{1-\alpha}{1+\alpha} s_0(2a-\xi,\lambda), & \xi > a \end{cases}$$

it is easy to transform formula (8) to the form

$$s(x,\lambda) = s_0(x,\lambda) + \int_0^x \rho(t)\, \Phi(x,t)\, s_0(t,\lambda)\, dt, \tag{27}$$

and the main equation (7) to the form

$$\Phi(x,t) + F(x,t) + \int_0^x \rho(\xi)\, \Phi(x,\xi)\, F(\xi,t)\, d\xi = 0, \tag{28}$$

where

$$\Phi(x,t) = \frac{2}{1+\sqrt{\rho(t)}} A\left(x,\mu^+(t)\right) + \frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}} A(x,2a-t).$$

Solving relation (27) with respect to $s_0(x,\lambda)$ we get

$$s_0(x,\lambda) = s(x,\lambda) + \int_0^x \rho(t)\, H(x,t)\, s(t,\lambda)\, dt. \tag{29}$$

By the standard method we can prove that

$$H(x,t) = F(x,t) + \int_0^t \rho(\xi)\, \Phi(t,\xi)\, F(x,\xi)\, d\xi, \quad 0 \le t \le x, \tag{30}$$

where $H(x,t)$ is defined by (29).

Denote $Q(\lambda) = \int_0^\pi \rho(t)\, g(t)\, s(t,\lambda)\, dt$ and using (27) transform it to the form

$$Q(\lambda) = \int_0^\pi \rho(t)\, h(t)\, s_0(t,\lambda)\, dt,$$

where

$$h(t) = g(t) + \int_t^\pi \rho(\xi)\, g(\xi)\, \Phi(\xi, t)\, d\xi. \tag{31}$$

In a similar way, using (29) we get

$$g(t) = h(t) + \int_t^\pi \rho(\xi)\, h(\xi)\, H(\xi, t)\, d\xi. \tag{32}$$

According to (31) we have

$$\int_0^\pi \rho(t)\, h(t)\, F(x,t)\, dt = \int_0^\pi \left[g(t) + \int_t^\pi \rho(\xi)\, g(\xi)\, \Phi(\xi,t)\, d\xi \right] \rho(t)\, F(x,t)\, dt =$$

$$= \int_0^\pi \rho(t)\, g(t) \left[F(x,t) + \int_0^t \rho(\xi)\, \Phi(t,\xi)\, F(x,\xi)\, d\xi \right] dt =$$

$$= \int_0^x \rho(t)\, g(t) \left[F(x,t) + \int_0^t \rho(\xi)\, \Phi(t,\xi)\, F(x,\xi)\, d\xi \right] dt +$$

$$+ \int_x^\pi \rho(t)\, g(t) \left[F(x,t) + \int_0^t \rho(\xi)\, \Phi(t,\xi)\, F(x,\xi)\, d\xi \right] dt.$$

Using (28) and (30) we get

$$\int_0^\pi \rho(t)\, h(t)\, F(x,t)\, dt = \int_0^x \rho(t)\, g(t)\, H(x,t)\, dt - \int_x^\pi \rho(t)\, g(t)\, \Phi(t,x)\, dt. \tag{33}$$

From (26) and Parseval's equality we obtain

$$\int_0^\pi \rho(t)\, h^2(t)\, dt + \int_0^\pi \int_0^\pi \rho(t)\, \rho(x)\, h(t)\, h(x)\, F(x,t)\, dx dt =$$

$$= \int_0^\pi \rho(t)\, h^2(t)\, dt + \sum_{n=1}^\infty \left[\frac{1}{\alpha_n} \left(\int_0^\pi \rho(t)\, h(t)\, s_0(t,\lambda_n)\, dt \right)^2 - \right.$$

$$\left. - \frac{1}{\alpha_n^0} \left(\int_0^\pi \rho(t)\, h(t)\, s_0\left(t,\lambda_n^0\right) dt \right)^2 \right] =$$

$$= \sum_{n=1}^\infty \frac{1}{\alpha_n} \left(\int_0^\pi \rho(t)\, h(t)\, s_0(t,\lambda_n)\, dt \right)^2 = \sum_{n=1}^\infty \frac{Q^2(\lambda_n)}{\alpha_n}.$$

According to (33)

$$\sum_{n=1}^{\infty}\frac{Q^2(\lambda_n)}{\alpha_n}=\int_0^{\pi}\rho(t)h^2(t)\,dt+\int_0^{\pi}\rho(x)h(x)\left[\int_0^{x}\rho(t)g(t)H(x,t)\,dt\right]dx-$$

$$-\int_0^{\pi}\rho(x)h(x)\left[\int_x^{\pi}\rho(t)g(t)\Phi(t,x)\,dt\right]dx=\int_0^{\pi}\rho(t)h^2(t)\,dt+$$

$$+\int_0^{\pi}\rho(t)g(t)\left[\int_t^{\pi}\rho(x)h(x)H(x,t)\,dx\right]dt-$$

$$-\int_0^{\pi}\rho(x)h(x)\left[\int_x^{\pi}\rho(t)g(t)\Phi(t,x)\,dt\right]dx,$$

whence by the formulas $(31),(32)$

$$\sum_{n=1}^{\infty}\frac{Q^2(\lambda_n)}{\alpha_n}=\int_0^{\pi}\rho(t)h^2(t)\,dt+\int_0^{\pi}\rho(t)g(t)(g(t)-h(t))\,dt-$$

$$-\int_0^{\pi}\rho(x)h(x)(h(x)-g(x))\,dx=\int_0^{\pi}\rho(t)g^2(t)\,dt,$$

i.e. relation (25) holds.

Lemma 2 is proved.

Corollary. *For any functions $f(x),\ g(x)\in L_2(0,\pi;\rho)$ the relation*

$$\int_0^{\pi}\rho(x)f(x)g(x)\,dx=\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\int_0^{\pi}\rho(t)f(t)s(t,\lambda_n)\,dt\int_0^{\pi}\rho(t)g(t)s(t,\lambda_n)\,dt \quad (34)$$

holds.

Lemma 3. *The relation*

$$\int_0^{\pi}\rho(t)s(t,\lambda_n)s(t,\lambda_k)\,dt=\begin{cases}0, & n\neq k,\\ \alpha_n\ , & n=k\end{cases} \quad (35)$$

holds.

Proof. 1) Let $f(x)\in W_2^2[0,\pi]$, $f(0)=f(\pi)=0$. Consider the series

$$f^*(x)=\sum_{n=1}^{\infty}c_n s(x,\lambda_n), \quad (36)$$

where

$$c_n=\frac{1}{\alpha_n}\int_0^{\pi}\rho(x)f(x)s(x,\lambda_n)\,dx. \quad (37)$$

Using Lemma 1 and integrating twice by parts, we get

$$c_n = \frac{1}{\alpha_n \lambda_n^2} \int_0^\pi f(x) \left(-s''(x, \lambda_n) + q(x) s(x, \lambda_n)\right) dx =$$

$$= \frac{1}{\alpha_n \lambda_n^2} \left[f'(\pi) s(\pi, \lambda_n) + \int_0^\pi s(x, \lambda_n) \left(-f''(x) + f(x) q(x)\right) dx \right].$$

Applying asymptotic formulas (4) and

$$s(x, \lambda) = O\left(\frac{1}{|\lambda|} e^{|\operatorname{Im} \lambda| \mu^+(x)}\right), \quad s'(x, \lambda) = O\left(e^{|\operatorname{Im} \lambda| \mu^+(x)}\right)$$

we can see that as $n \to \infty$

$$c_n = O\left(\frac{1}{n^3}\right), \quad s(x, \lambda_n) = O\left(\frac{1}{n}\right)$$

uniformly on $[0, \pi]$. Hence, series (36) converges absolutely and uniformly on $[0, \pi]$. By (34) and (37) we have

$$\int_0^\pi \rho(x) f(x) g(x) dx = \sum_{n=1}^\infty c_n \int_0^\pi \rho(t) g(t) s(t, \lambda_n) dt =$$

$$= \int_0^\pi \rho(t) g(t) \sum_{n=1}^\infty c_n s(t, \lambda_n) dt = \int_0^\pi \rho(t) g(t) f^*(t) dt.$$

By arbitrariness of $g(x)$ we conclude that $f^*(x) = f(t)$, i.e.

$$f(x) = \sum_{n=1}^\infty c_n s(x, \lambda_n). \tag{38}$$

2) Fix $k \geq 1$ and assume $f(x) = s(x, \lambda_k)$. Then, by (38)

$$s(x, \lambda_k) = \sum_{n=1}^\infty c_{nk} s(x, \lambda_n),$$

where

$$c_{nk} = \frac{1}{\alpha_n} \int_0^\pi \rho(x) s(x, \lambda_n) s(x, \lambda_k) dx.$$

We can show that the system of functions $\{s_0(x, \lambda_n)\}_{n \geq 1}$ is minimal in $L_2(0, \pi; \rho)$. Then by (8) the system of functions $\{s(x, \lambda_n)\}_{n \geq 1}$ is minimal in $L_2(0, \pi; \rho)$ as well. Therefore $c_{nk} = \delta_{nk}$ (δ_{nk} is a Kronecker symbol) and we arrive at (35).

Lemma 3 is proved.

Lemma 4. *For all* $n \geq 1$ *the equality*

$$s(\pi, \lambda_n) = 0$$

holds.

Proof. It is easy to show that

$$\left(\lambda_n^2-\lambda_m^2\right)\int_0^{\pi}\rho(x)\,s(x,\lambda_n)\,s(x,\lambda_m)\,dx=s(\pi,\lambda_n)\,s'(\pi,\lambda_m)-s'(\pi,\lambda_n)\,s(\pi,\lambda_m).$$

By (35) we get

$$s(\pi,\lambda_n)\,s'(\pi,\lambda_m)-s'(\pi,\lambda_n)\,s(\pi,\lambda_m)=0. \tag{39}$$

We shall prove that for any n, $s'(\pi,\lambda_n)\neq 0$. Assume the contrary, i.e. there exists such m that $s'(\pi,\lambda_m)=0$. Then from (39) we have $s'(\pi,\lambda_n)=0$ for $n\neq m$.

On the other hand

$$s'(\pi,\lambda_n)=s_0'(\pi,\lambda_n)+\int_0^{\mu^+(\pi)}A_x'(\pi,t)\frac{\sin\lambda_n t}{\lambda_n}dt+\frac{\alpha\sin\lambda_n\mu^+(\pi)}{\lambda_n}A\left(\pi,\mu^+(\pi)\right)+$$
$$+\frac{\alpha\sin\lambda_n\mu^-(\pi)}{\lambda_n}\{A\left(\pi,\mu^-(\pi)+0\right)-A\left(\pi,\mu^-(\pi)-0\right)\},$$

i.e. $s'(\pi,\lambda_n)\sim s_0'\left(\pi,\lambda_n^0\right)\neq 0$ as $n\to\infty$, that contradicts the condition $s'(\pi,\lambda_n)=0$, $n\neq m$. Thus $s'(\pi,\lambda_n)\neq 0$ for any n, and from (39) we have

$$\frac{s(\pi,\lambda_n)}{s'(\pi,\lambda_n)}=\frac{s(\pi,\lambda_m)}{s'(\pi,\lambda_m)}\overset{def}{=}H,$$

i.e. for any n, $s'(\pi,\lambda_n)\,H=s(\pi,\lambda_n)$. Since $s(\pi,\lambda_n)=o(1)$ as $n\to\infty$, we have $H=0$, i.e. $s(\pi,\lambda_n)=0$.

Lemma 4 is proved.

It follows from Lemmas 1, 3 and 4 that the numbers $\left\{\lambda_n^2\ ,\alpha_n\right\}_{n\geq 1}$ are spectral data of the constructed boundary value problem $L(q(x))$.

The theorem is proved.

Algorithm of the consruction of the function $q(x)$ by spectral data $\left\{\lambda_n^2\ ,\alpha_n\right\}$ follows from the proof of the theorem:

1) By the given numbers $\left\{\lambda_n^2\ ,\alpha_n\right\}_{n\geq 1}$ the functions $F_0(x,t)$ and $F(x,t)$ are constructed by the formulas (5) and (6), respectively;

2) The function $A(x,t)$ is found from equation (7);

3) $q(x)$ is calculated by the formula (9).

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Esmira N. Akhmedova, Hidayat M. Huseynov
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F. Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 47 20 (off.)

Received June 29, 2007; Revised October 16, 2007.