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A BOUNDARY CONTROL PROBLEM FOR A STRING VIBRATIONS EQUATION. I

Abstract

In the paper we consider string vibrations equation. We assume that the state of a string is given at initial time and put a problem on finding on the ends of the string such boundary controls that reduce the state of the sting to the given state at the terminal time. Three mixed problems are considered and theorems on the uniqueness of the solution are proved for these problems.

In the paper we consider an one-dimensional wave equation $u_{tt}(x,t)-u_{xx}(x,t)=0$, describing a string vibrations process for time interval $[0,T]$, whose ends are the points $x=0$ and $x=l$. At each time t the vibrations process is characterized by the permutation $u(x,t)$ of the points of a string and velocity $u_t(x,t)$ of these points. For fixed t it is natural to call a pair of functions $\{u(x,t),u_t(x,t)\}$ given on the segment $0\le x\le l$ state of system at the time t. It is assumed that at initial time $t=0$ the permutation and velocity of the points of a string equal $u(x,0)=\varphi(x)$, $u_t(x,0)=\psi(x)$, at the time $t=T$ the permutation and velocity of the points of a string equal $u(x,T)=\varphi_1(x)$, $u_t(x,T)=\psi_1(x)$ where $\varphi(x)$, $\varphi_1(x)$ and $\psi(x)$, $\psi_1(x)$ are two arbitrary functions from the classes $W_2^2[0,l]$ and $W_2^1[0,l]$ respectively.

There arises a problem on the existence and obvious analytic representation of boundary controls on the ends of a string

$$u(0,t)=\mu(t), u_x(l,t)=\nu(t)$$

belonging to the classes $W_2^2[0,T]$ and $W_2^1[0,T]$, respectively, and providing passage of vibration process from the state $\{\varphi(x),\ \psi(x)\}$ for $t=0$ to the state $\{\varphi_1(x),\ \psi_1(x)\}$ for $t=T$.

The solution of this problem essentially depends on the ratio of the length of a string l and time T.

In the present paper we use the method of the paper [1] and establish necessary and sufficient conditions on four functions $\varphi(x)\in W_2^2[0,l]$, $\psi(x)\in$ $\in W_2^1[0,l]$, $\varphi_1(x)\in W_2^2[0,l]$ and $\psi_1(x)\in W_2^1[0,l]$ by fulfilling of which there exist desired boundary controls $\mu(t)$ and $\nu(t)$ from the classes $W_2^2[0,l]$ and $W_2^1[0,l]$ respectively, and obvious analytic form of these controls are found.

Besides the paper [1] investigations of many authors were devoted to the problems of boundary control of string vibrations process. We indicate the paper [2] wherein a problem on the existence of boundary controls in terms of generalized solution from L_2 of the solution of a wave equation was studied by the methods of the theory of Hilbert spaces, and the paper [3] wherein a boundary control problem is considered when $u(0,t)=\mu(t)$,$u(l,t)=\nu(t)$ and this problem in special cases is solved by means of the moments method the papers [4]-[12].

§1. Some notion and statement of mixed problems

By the symbol Q_T we'll denote a rectangle $Q_T = [0 \le x \le l] \times [0 \le t \le T]$.

Definition 1. *We'll say that the one-variable function $\nu(t)$ belongs to the class $\underline{W}_2^1[0,T]$ (respectively to the class $\overline{W}_2^1[0,T]$) if this function belongs to the class $W_2^1[0,T]$ and moreover, it satisfies the conditions $\nu(0) = 0$,$\nu(t) \equiv 0$ for $t \le 0$ (respectively satisfies the conditions $\nu(T) = 0$,$\nu(t) \equiv 0$ for $t \ge T$).*

Definition 2. *We'll say that the one-variable function $\mu(t)$ belongs to the class $\underline{W}_2^2[0,T]$ (respectively to the class $\overline{W}_2^2[0,T]$) if this function belongs to the class $W_2^2[0,T]$ and moreover, it satisfies the conditions $\mu(0) = 0$, $\mu'(0) = 0$, $\mu(t) \equiv 0$ for $t \le 0$ (respectively satisfies the conditions $\mu(T) = 0$, $\mu'(T) = 0$, $\mu(t) \equiv 0$ for $t \ge T$).*

It follows from definitions 1.1 and 1.2 that the function $\nu(t)(\mu(t))$, belonging to the class $\underline{W}_2^1[0,T]$ $\left(\underline{W}_2^2[0,T]\right)$ belongs to the class $W_2^1[-A,T]$ $\left(W_2^2[-A,T]\right)$ for any $A > 0$, and the function $\nu(t)(\mu(t))$ belonging to the class $\overline{W}_2^1[0,T]$ $\left(\overline{W}_2^2[0,T]\right)$ belongs to the class $W_2^1[0,A]$ $\left(W_2^2[0,A]\right)$ for any $A > T$.

Let's consider the following three problems for a wave equation in a rectangle Q_T.

Problem I. Find the solution of the equation

$$u_{tt} - u_{xx} = 0 \ in \ Q_T = [0 \le x \le l] \times [0 \le x \le T], \tag{1}$$

satisfying the boundary conditions

$$u(0,t) = \mu(t), u_x(l,t) = \nu(t) \ \ for \ \ 0 \le x \le T, \tag{2}$$

and initial conditions

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \ \ for \ \ 0 \le x \le l, \tag{3}$$

where $\mu(t) \in W_2^2[0,T]$,$\nu(t) \in W_2^1[0,T]$,$\varphi(x) \in W_2^2[0,l]$,$\psi(x) \in W_2^1[0,l]$, and the agreement conditions:

$$\mu(0) = \varphi(0), \nu(0) = \varphi'(l), \mu'(0) = \psi(0). \tag{4}$$

are satisfied.

Problem II. Find the solution of the equation

$$u_{tt} - u_{xx} = 0 \ \ in \ \ Q_T = [0 \le x \le l] \times [0 \le x \le T], \tag{5}$$

satisfying the boundary conditions

$$u(0,t) = \mu(t), u_x(l,t) = \nu(t) \ \ for \ \ 0 \le t \le T, \tag{6}$$

and terminal conditions

$$u(x,T) = \varphi_1(x), u_t(x,T) = \psi_1(x) \ \ for \ \ 0 \le x \le l, \tag{7}$$

where $\mu(t) \in W_2^2[0,T]$,$\nu(t) \in W_2^1[0,T]$,$\varphi_1(x) \in W_2^2[0,l]$,$\psi_1(x) \in W_2^1[0,l]$, and the agreement conditions

$$\mu(T) = \varphi_1(0), \nu(T) = \varphi_1'(l),\mu'(T) = \psi_1(0). \tag{8}$$

are satisfied.

Problem III. Find the solution of the equation

$$u_{tt} - u_{xx} = 0 \ \ in \ \ Q_T = [0 \le x \le l] \times [0 \le x \le T], \tag{9}$$

satisfying the boundary conditions

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \ \ for \ \ 0 \le x \le l, \tag{10}$$

and terminal conditions

$$u(x,T) = \varphi_1(x), u_t(x,T) = \psi_1(x) \ \ for \ \ 0 \le x \le l, \tag{11}$$

where $\varphi(x) \in W_2^2[0,l]$,$\psi(x) \in W_2^1[0,l]$,$\varphi_1(x) \in W_2^2[0,l]$,$\psi_1(x) \in W_2^1[0,l]$,

Definition 3. *We'll say that the function of two variables $u(x,t)$ belongs to the class $\widehat{W_2^2}[Q_T]$, if the function $u(x,t)$ itself and its partial derivatives of first order are continuous in a closed rectangle Q_T, and if all the generalized second order partial derivatives of this function exist, each of which belongs to the class $L_2[0,l]$ for any t from $[0,T]$ and belongs to $L_2[0,T]$ for any x from $[0,l]$.*

Notice at once that if the function $f(t)$ of one variable t is determined on the segment $[a,b]$ and belongs to the class $W_2^2[a,b]$ then any wave form function of two variables $f(x+t+c_1)$ (respectively $f(x-t+c_2)$) belongs to the class $\widehat{W_2^2}[Q_T]$ each time when for all x from $[0,l]$ and all t from $[0,T]$ the argument $(x+t+c_1)$ (respectively the argument $(x-t+c_2)$) belongs to the segment $[a,b]$.

Definition 4. *The function $u(x,t)$ from the class $\widehat{W_2^2}[Q_T]$ satisfying the equation $u_{tt} - u_{xx} = 0$ for any t from $[0,T]$ and almost for all x from $[0,l]$ and also for any x from $[0,l]$ and almost for all t from $[0,T]$ and satisfying in the classic since the boundary conditions (2) and initial conditions (3) (respectively the boundary conditions (6) and conditions (7)) is said to be a solution from $\widehat{W_2^2}[Q_T]$ of mixed problem I (respectively of mixed problem II).*

Definition 5. *The function $u(x,t)$ from the class $\widehat{W_2^2}[Q_T]$ satisfying the equation $u_{tt} - u_{xx} = 0$ for any t from $[0,T]$ and for almost all x from $[0,l]$ and also for any x from $[0,l]$ and for almost all t from $[0,T]$ and satisfying the conditions (10) and (11) in the classic since is said to be a solution from $\widehat{W_2^2}[Q_T]$ of problem III.*

Remark 1. The function $u(x,t)$ being a solution from $\widehat{W_2^2}[Q_T]$ of problem III by definition of the class $\widehat{W_2^2}[Q_T]$ for $x=0$ and $x=l$ has boundary values $u(0,t) = \mu(t)$ and $u_x(l,t) = \nu(t)$, moreover $u_{tt}(0,t) = \mu''(t)$ and $u_{xt}(l,t) = \nu'(t)$ and they belong to the class $L_2[0,T]$, i.e. $\mu(t) \in W_2^2[0,T]$,$\nu(t) \in W_2^1[0,T]$.

These boundary values $\mu(t)$ and $\nu(t)$ by definition of the class $\widehat{W_2^2}[Q_T]$ should be agreed with the functions $\varphi(x)$,$\psi(x)$,$\varphi_1(x)$,$\psi_1(x)$, in the conditions (10) and (11), i.e. $\mu(t)$ and $\nu(t)$ they should be fulfilled both for the argument conditions (4) and argeement conditions (8).

§2. Uniqueness of the solution of problems I, II, III

Let's prove the following two statements.

Theorem 1. *Both problem I and II may have at most one solution from $\widehat{W_2^2}[Q_T]$.*

Proof: Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions either of mixed problem I or mixed problem II and $u(x,t) = u_1(x,t) - u_2(x,t)$. Then $u(x,t)$ is a solution from $\widehat{W}_2^2[Q_T]$ either of mixed problem

$$u_{tt} - u_{xx} = 0 \quad in \quad Q_T, \tag{1*}$$

$$u(0,t) \equiv 0, \ u_x(l,t) \equiv 0 \quad for \quad 0 \le t \le T, \tag{2*}$$

$$u(x,0) \equiv 0, \ u_t(x,0) \equiv 0 \quad for \quad 0 \le x \le l, \tag{3*}$$

or the mixed problem

$$u_{tt} - u_{xx} = 0 \quad in \quad Q_T, \tag{5*}$$

$$u(0,t) \equiv 0, \ u_x(l,t) \equiv 0 \quad for \quad 0 \le t \le T, \tag{6*}$$

$$u(x,T) \equiv 0, \ u_t(x,T) \equiv 0 \quad for \quad 0 \le x \le l. \tag{7*}$$

For an arbitrary function $u(x,t)$ from the class $\widehat{W}_2^2[Q_T]$ the identity

$$\left(u_x^2 + u_t^2\right)_t = 2u_t(u_{tt} - u_{xx}) + 2(u_x \cdot u_t)_x \tag{12}$$

is valid for any t from $[0,T]$ and for almost all x from $[0,l]$.

In the case when the function $u(x,t)$ is a solution from $\widehat{W}_2^2[Q_T]$ either of mixed problem $(1^*) - (3^*)$, or mixed problem $(5^*) - (7^*)$, the identity (12) takes the form:

$$\left(u_x^2 + u_t^2\right)_t = 2(u_x \cdot u_t)_x \tag{13}$$

Integration of the identity (3) with respect to x in the ranges from zero to t gives

$$\frac{\partial}{\partial t}\int_0^l \left[u_x^2(x,t) + u_t^2(x,t)\right] dx = 2\left[u_t(l,t)u_x(l,t) - u_t(0,t)u_x(0,t)\right]. \tag{14}$$

The right hand side if the equality (14) equals zero (by the conditions (2^*) or (6^*), hence it follows for all t from the segment $0 \le t \le T$

$$\int_0^l \left[u_x^2(x,t) + u_t^2(x,t)\right] dt = c = const. \tag{15}$$

Assuming in (15) $t = 0$ in the case of problem $(1^*) - (3^*)$and $t = T$ in the case of problem $(5^*) - (7^*)$ we get $c = 0$. But for $c = 0$ from the equality (15) valid for any t from the segment $0 \le t \le T$ and continuity of the first derivatives it follows that $u_x(x,t) \equiv 0$ and $u_t(x,t) \equiv 0$ everywhere on a rectangle Q_T. Thus, $u(x,t) \equiv c_1$ everywhere in Q_T, and by one of the conditions (2^*) or (6^*) $c_1 = 0$, i.e. $u(x,t) \equiv 0$ in Q_t. Theorem 1 is proved.

Theorem 2. *For any $T \le l$ problem III may have at most one solution from $\widehat{W}_2^2[Q_T]$.*

Proof: Assume that for some T satisfying the condition $0 < T \le l$ there exist two solutions from $\widehat{W}_2^2[Q_T]$ of problem III: $u_1(x,t)$ and $u_2(x,t)$. The boundary values $u_1(0,t) = \mu_1(t)$, $u_{1x}(l,t) = \nu_1(t)$, $u_2(0,t) = \mu_2(t)$, $u_{2x}(l,t) = \nu_2(t)$, where $\mu_1, \mu_2 \in W_2^2[0,T]$, ν_1,$\nu_2 \in W_2^1[0,T]$ respond to these two solutions.

By the belong ness of the functions $u_1(x,t)$ and $u_2(x,t)$ to the class $\widehat{W}_2^2[Q_T]$ the agreement conditions:

$$\mu_1(0) = \varphi(0), \nu_1(0) = \varphi'(l), \mu_1'(0) = \psi(0),$$

$$\mu_1(T) = \varphi(0), \nu_1(T) = \varphi'(l), \mu_1'(T) = \psi_1(0), \tag{16}$$

$$\mu_2(0) = \varphi(0), \nu_2(0) = \varphi'(l), \mu_2'(0) = \psi(0),$$

$$\mu_2(T) = \varphi_1(0), \nu_2(T) = \varphi_1'(l), \mu_2'(T) = \psi_1(0), \tag{17}$$

should be fulfilled.

Assume $u(x,t) = u_1(x,t) - u_2(x,t)$ and consider that the function $u(x,t)$ is a solution from $\widehat{W}_2^2[Q_T]$ of the following problem of type III

$$u_{tt} - u_{xx} = 0 \quad in \quad Q_T, \tag{18}$$

$$u(x,0) \equiv 0, \;\; u_t(x,0) \equiv 0 \quad for \quad 0 \le x \le l, \tag{19}$$

$$u(x,T) \equiv 0, \;\; u_t(x,T) \equiv 0 \quad for \quad 0 \le x \le l, \tag{20}$$

and has boundary value $u(0,t) = \mu(t) = \mu_1(t) - \mu_2(t)$ belonging to the class $W_2^2[0,T]$ and boundary value $u_x(l,t) = \nu(t) = \nu_1(t) - \nu_2(t)$ belonging to the class $W_2^1[0,T]$. By (16) and (17) these boundary values satisfy the following agreement conditions:

$$\mu(0) = 0, \;\; \nu(0) = 0, \;\; \mu'(0) = 0, \tag{21}$$

$$\mu(T) = 0, \;\; \nu(T) = 0, \;\; \mu'(T) = 0. \tag{22}$$

Now, notice that the functions $u(x,t)$ being a solution from $\widehat{W}_2^2[Q_T]$ of the problem (18)-(20) may be considered as a solution from $\widehat{W}_2^2[Q_T]$ of a mixed problem of type II

$$u_{tt} - u_{xx} = 0 \quad in \quad Q_T, \tag{23}$$

$$u(0,t) = \mu(t), \;\; u_t(l,t) = \nu(t) \quad for \quad 0 \le t \le T, \tag{24}$$

$$u(x,T) \equiv 0, \;\; u_t(x,T) \equiv 0 \quad for \quad 0 \le x \le l. \tag{25}$$

Since the functions $\mu(t)$ and $\nu(t)$ satisfy the conditions (22), then continuing these functions by an identity zero on the values of $t \ge T$ we get the continued functions (denote them by the symbols $\overline{\mu}(t)$ and$\overline{\nu}(t)$) will belong to the classes $\overline{W}_2^2[0,T]$ and $\overline{W}_2^1[0,T]$, respectively. We can verify that provided $0 < T \le l$ a unique (by theorem 1) solution from $\widehat{W}_2^2[Q_T]$ of the problem (23)-(25) is of the form:

$$u(x,t) = \overline{\mu}(t+x) + \widetilde{\nu}_1(t+l-x), \tag{26}$$

where $\widetilde{\nu}_1(t) = \int\limits_t^l \nu(s)ds$. Since the function $u(x,t)$ for $t = 0$ satisfies the conditions (19), then from (26) and (19) we get that for all x from the segment $[0,l]$

$$u(x,0) = \overline{\mu}(x) + \widetilde{\nu}_1(l-x) \equiv 0, \tag{27}$$

$$u_t(x,0) = \overline{\mu}'(x) - \widetilde{\nu}_1'(l-x) \equiv 0. \tag{28}$$

Differentiating (27) with respect to x for all $x \in [0, l]$ we get

$$\overline{\mu}'(x) - \widetilde{\nu}_1'(l - x) \equiv 0. \quad (29)$$

It follows from (28) and (29) that for all x from $[0, l]$ $\overline{\mu}'(x) \equiv 0$ and $\widetilde{\nu}_1'(l - x) \equiv$ $\equiv 0$. Hence it follows that for all x from $[0, l]$ $\overline{\mu}(x) \equiv c_1 = const$, $\widetilde{\nu}_1(x) = c_2 =$ $= const$. It remains to notice that by the agreement conditions (21) $c_1 = 0$, therefore $\mu(x) \equiv 0$, then $u(x, t) = \widetilde{\nu}_1(t + l - x)$. By the condition (25) $u(x, T) =$ $= \widetilde{\nu}_1(T + l - x) \equiv 0$. In particular, here we can take $x = l$, i.e. $u(l, T) = \widetilde{\nu}_1(T) =$ $= 0$. Hence and from the conditions $\widetilde{\nu}_1(x) \equiv c_2$ it follows that $c_2 = 0$. So, $\widetilde{\nu}_1(x) \equiv$ $\equiv 0$, $x \in [0, l]$. Thus $\mu(x) \equiv 0$ and $\nu(x) \equiv 0$ on the segment $[0, l]$, therefore on the segment $[0, T]$ as well. Thereby the function $u(x, t)$ is a solution from $\widehat{W}_2^2 [Q_T]$ of the problem (23)-(25) wherein $\mu(t) \equiv 0$ and $\nu(t) \equiv 0$ on the segment $[0, T]$. By theorem 1 solution $u(x, t)$ of such a problem is an identity zero in Q_T. Theorem 2 is proved.

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