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NECESSARY OPTIMALITY CONDITIONS IN A VARIABLE STRUCTURE CONTROL PROBLEM DESCRIBED BY A SYSTEM OF TWO-DIMENSIONAL VOLTERRA TYPE INTEGRAL EQUATIONS

Abstract

We consider an optimal control problem described by a system of Volterra type two-dimensional integral equations. Necessary optimality conditions of first and second order are derived under assumption that the control domain is open.

We consider a variable structure control problem described by a system of two-dimensional Volterra type integral equations.

Necessary and sufficient optimality conditions are derived provided that the control domain is open.

Introduction. The optimal control problems described by integral equations have been studied relatively little to day. In this direction we can notice the papers [1-4].

Recently, the questions related with investigations of variable structure control problems that in applications simulate multi-stage or so-called multi-step processes [5-7] are of great interest.

The papers [8-12] and others are devoted to variable structure optimal control problems described by different differential and difference equations.

The suggested paper is devoted to deriving necessary optimality conditions of first and second order in a variable structure control problem described by a system of nonlinear two-dimensional Volterra type equations.

1. Problem Statement. Assume that in the given domain

$$D = D_1 \cup D_2 \left(D_i = [t_{i-1}, t_i] \times [x_0, X]\,,\ i = \overline{1,2}\right)$$

a controlled process is described by a system of non-linear two-dimensional integral equations

$$z(t,x) = \int_{t_0}^{t}\int_{x_0}^{x} f(t,x,\tau,s,z(\tau,s),u(\tau,s))\,dsd\tau, \qquad (t,x)\in D_1\ ,$$

$$y(t,x) = \int_{t_1}^{t}\int_{x_0}^{x} g(t,x,\tau,s,y(\tau,s),v(\tau,s))\,dsd\tau + G(z(t_1,x))\,,\ (t,x)\in D_2\ . \quad (1)$$

Here $f(t,x,\tau,s,z,u)$ $(g(t,x,\tau,s,y,v))$ is a given n (m)-dimensional vector function continuous in $D_1\times D_1\times R^n\times R^r$ $(D_2\times D_2\times R^m\times R^q)$ together with partial derivatives with respect to (z,u) $((y,v))$ up to the second order inclusively,

$t_0 < t_1 < t_2$, $x_0 < X$, $G(z)$ is a given twice differentiable in R^n $m-$dimensional vector function, $u(t,x)$ $(v(t,x))$ is r (q) dimensional piecewise-continuous (in the sense of [13]) vector of controlling effects with values from the given non-empty bounded and open set $U(V)$, i.e.

$$u(t,x) \in U \subset R^r, \quad (t,x) \in D_1 \;\; v(t,x) \in V \subset R^q, \quad (t,x) \in D_2\ . \tag{2}$$

Such controlling functions are said to be admissible.

We estimate the quality of the process by the functional

$$S(u,v) = \varphi_1(z(t_1,X)) + \varphi_2(y(t_2,X))\,. \tag{3}$$

Here $\varphi_1(z)$, $\varphi_2(y)$ are the given twice continuously-differentiable scalar functions.

The optimal control problem is in finding of such an admissible control $\left(u^0(t,x), v^0(t,x)\right)$ at which the values of the functional (3) is least in comparison with other values for admissible controls.

In this case the admissible control $\left(u^0(t,x),\ v^0(t,x)\right)$ is said to be an optimal control, its appropriate state $\left(z^0(t,x),\ y^0(t,x)\right)$-an optimal state, the process $\left(u^0(t,x),\ v^0(t,x), z^0(t,x),\ y^0(t,x)\right)$-an optimal process.

2. The first and second variations of the quality functional. Let in the system (1) the state $\left(z^0(t,x),\ y^0(t,x)\right)$ correspond to the admissible control $\left(u^0(t,x),\ v^0(t,x)\right)$, the state

$$\left(\overline{u}(t,x) = u^0(t,x) + \Delta u(t,x),\ \overline{v}(t,x) = v^0(t,x) + \Delta v(t,x)\right)$$

to the admissible control

$$\left(\overline{z}(t,x) = z^0(t,x) + \Delta z(t,x),\ \overline{y}(t,x) = y^0(t,x) + \Delta y(t,x)\right).$$

Then, it is clear that $\left(z^0(t,x),\ y^0(t,x)\right)$ will satisfy the system of integral equations

$$\Delta z(t,x) = \int_{t_0}^{t}\int_{x_0}^{x} \left[f(t,x,\tau,s,\overline{z}(\tau,s),\overline{u}(\tau,s)) - \right.$$

$$\left. -f\left(t,x,\tau,s,z^0(\tau,s),u^0(\tau,s)\right)\right] ds d\tau,$$

$$\Delta y(t,x) = \int_{t_1}^{t}\int_{x_0}^{x} \left[g(t,x,\tau,s,\overline{y}(\tau,s),\overline{v}(\tau,s)) - \right.$$

$$\left. -g\left(t,x,\tau,s,y^0(\tau,s),v^0(\tau,s)\right)\right] ds d\tau + \left[G(\overline{z}(t_1,x)) - G\left(z^0(t_1,x)\right)\right]. \tag{4}$$

By $\psi_i^0(t,x)$, $i = 1,2$ we denote until unknown n and $m-$dimensional vector-functions, respectively.

Then, the identities are valid

$$\int_{t_0}^{t_1}\int_{x_0}^{X} \psi_1^{0'}(t,x)\,\Delta z(t,x)\,dxdt = \int_{t_0}^{t_1}\int_{x_0}^{X} \left[\int_{t}^{t_1}\int_{x}^{X} \left[\psi_1^{0'}(\tau,s)\left[f(\tau,s,t,x,\overline{z}(t,x),\overline{u}(t,x)) - \right.\right.\right.$$

$$-f\left(\tau,s,t,x,z^0(t,x),u^0(t,x)\right)\right]dsd\tau\Big]dtdx,$$

$$\int_{t_1}^{t_2}\int_{x_0}^{X}\psi_2^{0\prime}(t,x)\,\Delta y(t,x)\,dxdt=\int_{t_1}^{t_2}\int_{x_0}^{X}\psi_2^{0\prime}(t,x)\left[G(\overline{z}(t_1,x))-G\left(z^0(t_1,x)\right)\right]dxdt+$$

$$+\int_{t_1}^{t_2}\int_{x_0}^{X}\left[\int_{t}^{t_2}\int_{x}^{X}\Big[\psi_2^{0\prime}(\tau,s)\left[g(\tau,s,t,x,\overline{y}(t,x),\overline{v}(t,x))-\right.\right.$$

$$-g\left(\tau,s,t,x,y^0(t,x),v^0(t,x)\right)\right]dsd\tau\Big]dtdx. \qquad (5)$$

Assuming

$$M\left(\psi_2^0(t,x),\ z\right)=\psi_2^{0\prime}(t,x)\,G(z),$$

and using the identities (4), (5) we write an increment of the quality functional (3) in the form

$$\Delta S\left(u^0,v^0\right)=\varphi_1(\overline{z}(t_1,X))-\varphi_1\left(z^0(t_1,X)\right)+\varphi_2(\overline{y}(t_2,X))-\varphi_2\left(y^0(t_2,X)\right)+$$

$$+\int_{t_0}^{t_1}\int_{x_0}^{X}\psi_1^{0\prime}(t,x)\,\Delta z(t,x)\,dxdt-\int_{t_0}^{t_1}\int_{x_0}^{X}\left[\int_{t}^{t_1}\int_{x}^{X}\psi_1^{0}(\tau,s)\left[f(\tau,s,t,x,\overline{z}(t,x),\overline{u}(t,x))-\right.\right.$$

$$-f\left(\tau,s,t,x,z^0(t,x),u^0(t,x)\right)\right]d\tau ds\Big]dxdt-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{X}\left[\int_{t}^{t_2}\int_{x}^{X}\psi_2^{0}(\tau,s)\left[g(\tau,s,t,x,\overline{y}(t,x),\overline{v}(t,x))-\right.\right.$$

$$-g\left(\tau,s,t,x,y^0(t,x),v^0(t,x)\right)\right]dsd\tau\Big]dtdx-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{X}\left[M\left(\psi_2^0(t,x),\overline{z}(t_1,x)\right)-M\left(\psi_2^0(t,x),z^0(t_1,x)\right)\right]dxdt+$$

$$+\int_{t_1}^{t_2}\int_{x_0}^{X}\psi_2^{0\prime}(t,x)\,\Delta y(t,x)\,dxdt. \qquad (6)$$

Using the Taylor formula from (6) we get

$$\Delta S\left(u^0,v^0\right)=\frac{\partial\varphi_1'\left(z^0(t_1,X)\right)}{\partial z}\Delta z(t_1,X)+\frac{\partial\varphi_2'\left(y^0(t_2,X)\right)}{\partial y}\Delta y(t_2,X)+$$

$$+\frac{1}{2}\Delta z'(t_1,X)\,\frac{\partial^2\varphi_1\left(z^0(t_1,X)\right)}{\partial z^2}\Delta z(t_1,X)+$$

$$+\frac{1}{2}\Delta y'(t_2,X)\,\frac{\partial^2\varphi_2\left(y^0(t_2,X)\right)}{\partial y^2}\Delta y(t_2,X)+$$

$$+o_1\left(\|\Delta z(t_1,X)\|^2\right)+o_2\left(\|\Delta y(t_2,X)\|^2\right)+\int_{t_0}^{t_1}\int_{x_0}^{X}\psi_1^{0\prime}(t,x)\,\Delta z(t,x)\,dxdt-$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{X}\left[\int_{t}^{t_1}\int_{x}^{X}\psi_1^{0'}(\tau,s)\left[f(\tau,s,t,x,\overline{z}(t,x),\overline{u}(t,x))-\right.\right.$$

$$\left.\left.-f\left(\tau,s,t,x,z^0(t,x),u^0(t,x)\right)\right]ds d\tau\right]dxdt+$$

$$+\int_{t_1}^{t_2}\int_{x_0}^{X}\psi_2^{0'}(t,x)\Delta y(t,x)\,dxdt-\int_{t_1}^{t_2}\int_{x_0}^{X}\left[\int_{t}^{t_2}\int_{x}^{X}\psi_1^{0'}(\tau,s)\left[g(\tau,s,t,x,\overline{y}(t,x),\overline{v}(t,x))-\right.\right.$$

$$\left.\left.-g\left(\tau,s,t,x,y^0(t,x),v^0(t,x)\right)\right]ds d\tau\right]dtdx-$$

$$-\int_{t}^{t_2}\int_{x_0}^{X}M_z'\left(\psi_2^0(t,x),z^0(t_1,x)\right)\Delta z(t_1,x)\,dxdt-$$

$$-\frac{1}{2}\int_{t_1}^{t_2}\int_{x_0}^{X}\Delta z'(t_1,x)\,M_{zz}\left(\psi_2^0(t,x),z^0(t_1,x)\right)\Delta z(t_1,x)\,dxdt-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{X}o_3\left(\left\|\Delta z(t_1,X)\right\|^2\right)dxdt. \qquad (7)$$

It is clear that

$$\Delta z(t_1,X)=\int_{t_0}^{t_1}\int_{x_0}^{X}\left[f(t_1,X,\tau,s,\overline{z}(\tau,s),\overline{u}(\tau,s))-\right.$$

$$\left.-f\left(t_1,X,\tau,s,z^0(\tau,s),u^0(\tau,s)\right)\right]ds d\tau,$$

$$\Delta y(t_2,X)=\int_{t_1}^{t_2}\int_{x_0}^{X}\left[g(t_2,X,t,x,\overline{y}(t,x),\overline{v}(t,x))-\right.$$

$$\left.-g\left(t_2,X,t,x,y^0(t,x),v^0(t,x)\right)\right]dxdt+\left[G(\overline{z}(t_1,X))-G\left(z^0(t_1,X)\right)\right],$$

$$\Delta z(t_1,x)=\int_{t_0}^{t_1}\int_{x_0}^{x}\left[f(t_1,x,\tau,s,\overline{z}(\tau,s),\overline{u}(\tau,s))-\right.$$

$$\left.-f\left(t_1,x,\tau,s,z^0(\tau,s),u^0(\tau,s)\right)\right]ds d\tau.$$

Allowing for these identities the increment formula (7) takes the form

$$\Delta S\left(u^0,v^0\right)=\int_{t_0}^{t_1}\int_{x_0}^{X}\frac{\partial\varphi_1'\left(z^0(t_1,X)\right)}{\partial z}\left[f(t_1,X,t,x,\overline{z}(t,x),\overline{u}(t,x))-\right.$$

$$\left.-f\left(t_1,X,t,x,z^0(t,x),u^0(t,x)\right)\right]dxdt+$$

$$+\int_{t_1}^{t_2}\int_{x_0}^{X}\frac{\partial\varphi_2'\left(y^0\left(t_2,X\right)\right)}{\partial y}\left[g\left(t_2,X,t,x,\overline{y}\left(t,x\right),\overline{v}\left(t,x\right)\right)-\right.$$

$$\left.-g\left(t_2,X,t,x,y^0\left(t,x\right),v^0\left(t,x\right)\right)\right]dxdt+$$

$$+\frac{\partial\varphi_2'\left(y^0\left(t_2,X\right)\right)}{\partial y}\left[G\left(\overline{z}\left(t_1,X\right)\right)-G\left(z^0\left(t_1,X\right)\right)\right]+$$

$$+\frac{1}{2}\Delta z'\left(t_1,X\right)\frac{\partial^2\varphi_1\left(z^0\left(t_1,X\right)\right)}{\partial z^2}\Delta z\left(t_1,X\right)+$$

$$+\frac{1}{2}\Delta y'\left(t_2,X\right)\frac{\partial^2\varphi_2\left(y^0\left(t_2,X\right)\right)}{\partial y^2}\Delta y\left(t_2,X\right)+$$

$$+o_1\left(\left\|\Delta z\left(t_1,X\right)\right\|^2\right)+o_2\left(\left\|\Delta y\left(t_2,X\right)\right\|^2\right)-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{X}o_3\left(\left\|\Delta z\left(t_1,x\right)\right\|^2\right)dxdt+\int_{t_0}^{t_1}\int_{x_0}^{x}\psi_1^{0'}\left(t,x\right)\Delta z\left(t,x\right)dxdt-$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{X}\left[\int_{t}^{t_1}\int_{x}^{X}\psi_1^{0'}\left(\tau,s\right)\left[f\left(\tau,s,t,x,\overline{z}\left(t,x\right),\overline{u}\left(t,x\right)\right)-\right.\right.$$

$$\left.\left.-f\left(\tau,s,t,x,z^0\left(t,x\right),u^0\left(t,x\right)\right)\right]dsd\tau\right]dxdt+$$

$$+\int_{t_1}^{t_2}\int_{x_0}^{X}\psi_2^{0'}\left(t,x\right)\Delta y\left(t,x\right)dxdt-\int_{t_1}^{t_2}\int_{x_0}^{X}\left[\int_{t}^{t_2}\int_{x}^{X}\psi_2^{0'}\left(\tau,s\right)\left[g\left(\tau,s,t,x,\overline{y}\left(t,x\right),\overline{v}\left(t,x\right)\right)-\right.\right.$$

$$\left.\left.-g\left(\tau,s,t,x,y^0\left(t,x\right),v^0\left(t,x\right)\right)\right]dsd\tau\right]dtdx-$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{X}\left[\int_{t_1}^{t_2}\int_{x}^{X}M_z'\left(\psi_2^0\left(\tau,s\right),z^0\left(t_1,s\right)\right)\left[f\left(t_1,s,t,x,\overline{z}\left(t,x\right),\overline{v}\left(t,x\right)\right)-\right.\right.$$

$$\left.\left.-f\left(t_1,s,t,x,z^0\left(t,x\right),v^0\left(t,x\right)\right)\right]d\tau ds\right]dtdx-$$

$$-\frac{1}{2}\int_{t_1}^{t_2}\int_{x_0}^{X}\Delta z'\left(t_1,x\right)M_{zz}\left(\psi_2^0\left(t,x\right),z^0\left(t_1,x\right)\right)\Delta z\left(t_1,x\right)dxdt.$$

Hence, assuming $N\left(z\right)=\dfrac{\partial\varphi_2'\left(y^0\left(t_2,X\right)\right)}{\partial y}G\left(z\right)$, we'll have

$$\Delta S\left(u^0,v^0\right)=\int_{t_0}^{t_1}\int_{x_0}^{X}\frac{\partial\varphi_1'\left(z^0\left(t_1,X\right)\right)}{\partial z}\left[f\left(t_1,X,t,x,\overline{z}\left(t,x\right),\overline{u}\left(t,x\right)\right)-\right.$$

$$\left.-f\left(t_1,X,t,x,z^0\left(t,x\right),u^0\left(t,x\right)\right)\right]dxdt+$$

$$+\int_{t_1}^{t_2}\int_{x_0}^{X}\frac{\partial\varphi_2'\left(y^0\left(t_2,X\right)\right)}{\partial y}\left[g\left(t_2,X,t,x,\overline{y}\left(t,x\right),\overline{v}\left(t,x\right)\right)-\right.$$

$$\left.-g\left(t_2,X,t,x,y^0\left(t,x\right),v^0\left(t,x\right)\right)\right]dxdt+$$

$$+\int_{t_0}^{t_1}\int_{x_0}^{X}N_z'\left(z^0\left(t_1,X\right)\right)\left[f\left(t_1,X,t,x,\overline{z}\left(t,x\right),\overline{u}\left(t,x\right)\right)-\right.$$

$$\left.-f\left(t_1,X,t,x,z^0\left(t,x\right),u^0\left(t,x\right)\right)\right]dxdt+$$

$$\left.+\frac{1}{2}\Delta z'\left(t_1,X\right)\frac{\partial^2\varphi_1\left(z^0\left(t_1,X\right)\right)}{\partial z^2}\Delta z\left(t_1,X\right)\right]+o_1\left(\left\|\Delta z\left(t_1,X\right)\right\|^2\right)+$$

$$+\frac{1}{2}\Delta y'\left(t_2,X\right)\frac{\partial^2\varphi_2\left(y^0\left(t_2,X\right)\right)}{\partial y^2}\Delta y\left(t_2,X\right)+o_2\left(\left\|\Delta y\left(t_2,X\right)\right\|^2\right)+$$

$$+\Delta z'\left(t_1,X\right)N_{zz}\left(z^0\left(t_1,X\right)\right)\Delta z\left(t_1,X\right)-\int_{t_1}^{t_2}\int_{x_0}^{X}o_3\left(\left\|\Delta z\left(t_1,x\right)\right\|^2\right)dxdt+$$

$$+\int_{t_0}^{t_1}\int_{x_0}^{X}\psi_1^{0'}\left(t,x\right)\Delta z\left(t,x\right)dxdt-\int_{t_0}^{t_1}\int_{x_0}^{X}\left[\int_{t}^{t_1}\int_{x}^{X}\psi_1^{0'}\left(\tau,s\right)\left[f\left(\tau,s,t,x,\overline{z}\left(t,x\right),\overline{u}\left(t,x\right)\right)-\right.\right.$$

$$\left.\left.-f\left(\tau,s,t,x,z^0\left(t,x\right),u^0\left(t,x\right)\right)\right]dsd\tau\right]dxdt+$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{X}\left[\int_{t}^{t_2}\int_{x}^{X}\psi_2^{0'}\left(\tau,s\right)\left[g\left(\tau,s,t,x,\overline{y}\left(t,x\right),\overline{v}\left(t,x\right)\right)-\right.\right.$$

$$\left.\left.-g\left(\tau,s,t,x,y^0\left(t,x\right),v^0\left(t,x\right)\right)\right]dsd\tau\right]dxdt-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{X}\psi_2^{0'}\left(t,x\right)\Delta y\left(t,x\right)dxdt-$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{X}\left[\int_{t_1}^{t_2}\int_{x}^{X}M_z'\left(\psi_2^0\left(\tau,s\right),z^0\left(t_1,s\right)\right)\left[f\left(t_1,s,t,x,\overline{z}\left(t,x\right),\overline{u}\left(t,x\right)\right)-\right.\right.$$

$$\left.\left.-f\left(t_1,s,t,x,z^0\left(t,x\right),u^0\left(t,x\right)\right)\right]dsd\tau\right]dtdx-$$

$$-\frac{1}{2}\int_{t_1}^{t_2}\int_{x_0}^{X}\Delta z'\left(t_1,x\right)M_{zz}\left(\psi_2^0\left(t,x\right),z^0\left(t_1,x\right)\right)\Delta z\left(t_1,x\right)dxdt+$$

$$+o_4\left(\left\|\Delta z\left(t_1,X\right)\right\|^2\right). \tag{8}$$

Assuming

$$H_1\left(t,x,z,u,\psi_1^0\right)=\frac{\partial\varphi_1'\left(z^0\left(t_1,X\right)\right)}{\partial z}f\left(t_1,X,t,x,z\left(t,x\right),u\left(t,x\right)\right)-$$

$$-N_z'\left(z^0(t_1,X)\right)f(t_1,X,t,x,z(t,x),u(t,x))+$$

$$+\int_t^{t_1}\int_x^X \psi_1^{0'}(\tau,s)\,f(\tau,s,t,x,z(t,x),u(t,x))\,dsd\tau+$$

$$+\int_{t_1}^{t_2}\int_x^X M_z'\left(\psi_2^0(\tau,s),z^0(t_1,s)\right)f(t_1,s,t,x,z(t,x),u(t,x))\,dsd\tau,$$

$$H_2\left(t,x,y,v,\psi_2^0\right)=-\frac{\partial\varphi_2'\left(y^0(t_2,X)\right)}{\partial y}g(t_2,X,t,x,y(t,x),v(t,x))+$$

$$+\int_t^{t_2}\int_x^X \psi_2^{0'}(\tau,s)\,g(\tau,s,t,x,z(t,x),u(t,x))\,dsd\tau,$$

we write increment (8) in the form

$$\Delta S\left(u^0,v^0\right)=-\int_{t_0}^{t_1}\int_{x_0}^X\left[H_1\left(t,x,\overline{z}(t,x),\overline{u}(t,x),\psi_1^0(t,x)\right)-\right.$$

$$\left.-H_1\left(t,x,z^0(t,x),u^0(t,x),\psi_1^0(t,x)\right)\right]dxdt-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^X\left[H_2\left(t,x,\overline{y}(t,x),\overline{v}(t,x),\psi_2^0(t,x)\right)-\right.$$

$$\left.-H_2\left(t,x,y^0(t,x),v^0(t,x),\psi_2^0(t,x)\right)\right]dxdt+$$

$$+\frac{1}{2}\Delta z'(t_1,X)\,\frac{\partial^2\varphi_1\left(z^0(t_1,X)\right)}{\partial z^2}\Delta z(t_1,X)+$$

$$+\frac{1}{2}\Delta y'(t_2,X)\,\frac{\partial^2\varphi_2\left(y^0(t_2,X)\right)}{\partial y^2}\Delta y(t_2,X)+$$

$$+o_1\left(\|\Delta z(t_1,X)\|^2\right)+o_2\left(\|\Delta y(t_2,X)\|^2\right)-\int_{t_1}^{t_2}\int_{x_0}^X o_3\left(\|\Delta z(t_1,x)\|^2\right)dxdt+$$

$$+\int_{t_0}^{t_1}\int_{x_0}^X \psi_1^{0'}(t,x)\,\Delta z(t,x)\,dxdt+\int_{t_1}^{t_2}\int_{x_0}^X \psi_2^{0'}(t,x)\,\Delta y(t,x)\,dxdt+$$

$$+\frac{1}{2}\Delta z'(t_1,X)\,N_{zz}\left(z^0(t_1,X)\right)\Delta z(t_1,X)+o_4\left(\|\Delta z(t_1,x)\|^2\right)-$$

$$-\frac{1}{2}\int_{t_1}^{t_2}\int_{x_0}^X \Delta z'(t_1,x)\,M_{zz}\left(\psi_2^0(t,x),\Delta z^0(t_1,x)\right)\Delta z(t_1,x)\,dxdt.$$

Hence, using the denotation of type

$$\frac{\partial H_1[t,x]}{\partial z}\equiv\frac{\partial H_1\left(t,x,z^0(t,x),u^0(t,x),\psi_1^0(t,x)\right)}{\partial z};$$

$$\frac{\partial H_2[t,x]}{\partial y}\equiv\frac{\partial H_2\left(t,x,y^0(t,x),v^0(t,x),\psi_2^0(t,x)\right)}{\partial y},$$

$$\frac{\partial^2 H_1[t,x]}{\partial z^2}\equiv\frac{\partial^2 H_1\left(t,x,z^0(t,x),u^0(t,x),\psi_1^0(t,x)\right)}{\partial z^2};$$

$$\frac{\partial^2 H_2[t,x]}{\partial y^2}\equiv\frac{\partial^2 H_2\left(t,x,y^0(t,x),v^0(t,x),\psi_2^0(t,x)\right)}{\partial y^2}$$

and assuming that $\psi_i^0(t,x)$, $i=1,2$ are the solutions of the following system of two-dimensional Volterra equations

$$\psi_1^0(t,x)=\frac{\partial H_1[t,x]}{\partial z},\ (t,x)\in D_1\ ,\quad \psi_2^0(t,x)=\frac{\partial H_2[t,x]}{\partial y},\ (t,x)\in D_2\ ,$$

and using the Taylor formula, we'll have

$$\Delta S\left(u^0,v^0\right)=\frac{1}{2}\Delta z'(t_1,X)\frac{\partial^2\varphi_1(z_1(t_1,X))}{\partial z^2}\Delta z(t_1,X)+$$
$$+\frac{1}{2}\Delta y(t_2,X)\frac{\partial^2\varphi_2(y(t_2,X))}{\partial y^2}\Delta y(t_2,X)+$$
$$+\frac{1}{2}\Delta z'(t_1,X)\frac{\partial^2 N\left(z^0(t_1,X)\right)}{\partial z^2}\Delta z(t_1,X)-$$
$$-\frac{1}{2}\int_{t_1}^{t_2}\int_{x_0}^{X}\Delta z'(t_1,x)M_{zz}\left(\psi_2^0(t,x),z^0(t_1,x)\right)\Delta z(t_1,x)\,dxdt-$$
$$-\int_{t_0}^{t_1}\int_{x_0}^{X}\frac{\partial H_1'[t,x]}{\partial u}\Delta u(t,x)\,dxdt-\frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{X}\left[\Delta z'(t,x)\frac{\partial^2 H_1[t,x]}{\partial z^2}\Delta z(t,x)+\right. \tag{9}$$
$$\left.+2\Delta u'(t,x)\frac{\partial^2 H_1[t,x]}{\partial u\partial z}\Delta z(t,x)+\Delta u'(t,x)\frac{\partial^2 H_1[t,x]}{\partial u^2}\Delta u(t,x)\right]dxdt-$$
$$-\int_{t_1}^{t_2}\int_{x_0}^{X}\frac{\partial H_2'[t,x]}{\partial v}\Delta v(t,x)\,dxdt-\frac{1}{2}\int_{t_1}^{t_2}\int_{x_0}^{X}\left[\Delta y'(t,x)\frac{\partial^2 H_2[t,x]}{\partial y^2}\Delta y(t,x)+\right.$$
$$\left.+2\Delta v'(t,x)\frac{\partial^2 H_2[t,x]}{\partial v\partial y}\Delta y(t,x)+\Delta v'(t,x)\frac{\partial^2 H_2[t,x]}{\partial v^2}\Delta y(t,x)\right]dxdt+$$
$$+o_1\left(\|\Delta z(t_1,X)\|^2\right)+o_2\left(\|\Delta y(t_2,X)\|^2\right)-\int_{t_1}^{t_2}\int_{x_0}^{X}o_3\left(\|\Delta z(t_1,x)\|^2\right)dxdt-$$
$$-\int_{t_0}^{t_1}\int_{x_0}^{X}o_5\left(\|\Delta z(t,x)\|^2\right)dxdt+o_4\left(\|\Delta z(t_1,X)\|^2\right)-\int_{t_1}^{t_2}\int_{x_0}^{X}o_6\left(\|\Delta y(t,x)\|^2\right)dxdt.$$

We define a special increment of the admissible control $\left(u^0(t,x),\ v^0(t,x)\right)$ by the formula

$$\begin{array}{l}\Delta u_\varepsilon(t,x)=\varepsilon\delta u(t,x),\quad (t,x)\in D_1\ ,\\ \Delta v_\varepsilon(t,x)=\varepsilon\delta v(t,x),\quad (t,x)\in D_2\ .\end{array} \tag{10}$$

Here $\delta u(t,x) \in R^r$, $(t,x) \in D_1$, $(\delta v(t,x) \in R^q$, $(t,x) \in D_2)$ is an arbitrary piecewise-continuous $r(q)$ −dimensional vector-function.

By $(\Delta z_\varepsilon(t,x), \ \Delta y_\varepsilon(t,x))$ we denote a special increment of the state $(z^0(t,x)$, $y^0(t,x))$. It is clear from (4) that $(\Delta z_\varepsilon(t,x), \ \Delta y_\varepsilon(t,x))$ is a solution of the integral equation

$$\Delta z_\varepsilon(t,x) = \int_{t_0}^{t}\int_{x_0}^{x} \left[f\left(t,x,\tau,s,z^0(\tau,s)+\Delta z_\varepsilon(\tau,s), u^0(\tau,s)+\Delta z_\varepsilon(\tau,s)\right)-\right.$$

$$\left.-f\left(t,x,\tau,s,z^0(\tau,s), u^0(\tau,s)\right)\right] dsd\tau,$$

$$\Delta y_\varepsilon(t,x) = G\left(z^0(t_1,x)+\Delta z_\varepsilon(t_1,x)\right) - G\left(z^0(t_1,x)\right)+$$

$$+\int_{t_1}^{t}\int_{x_0}^{x} \left[g\left(t,x,\tau,s,y^0(\tau,s)+\Delta y(\tau,s), v^0(\tau,s)+\Delta v_\varepsilon(\tau,s)\right)-\right. \tag{11}$$

$$\left.-g\left(t,x,\tau,s,y^0(\tau,s), v^0(\tau,s)\right)\right] dsd\tau.$$

It follows from (11) that $(\Delta z_\varepsilon(t,x), \ \Delta y_\varepsilon(t,x))$ is a solution of the linearzed equation

$$\Delta z_\varepsilon(t,x) = \int_{t_0}^{t}\int_{x_0}^{x} \left[f_z[t,x,\tau,s]\Delta z_\varepsilon(\tau,s) + f_u[t,x,\tau,s]\Delta u_\varepsilon(\tau,s)\right] dsd\tau+$$

$$+\int_{t_0}^{t}\int_{x_0}^{x} o_7\left(\|\Delta z_\varepsilon(\tau,s)\| + \|\Delta u_\varepsilon(\tau,s)\|\right) dsd\tau, \tag{12}$$

$$\Delta y_\varepsilon(t,x) = \int_{t_1}^{t}\int_{x_0}^{x} \left[g_y[t,x,\tau,s]\Delta y_\varepsilon(\tau,s) + g_v[t,x,\tau,s]\Delta v_\varepsilon(\tau,s)\right] dsd\tau+$$

$$+G_z\left(z^0(t_1,x)\right)\Delta z_\varepsilon(t_1,x) + \int_{t_0}^{t}\int_{x_0}^{x} o_8\left(\|\Delta z_\varepsilon(\tau,s)\| +\right.$$

$$\left.+\|\Delta u_\varepsilon(\tau,s)\|\right) dsd\tau + o_9\left(\|\Delta z_\varepsilon(t_1,x)\|\right),$$

where here and in the sequel, by definition

$$f_z[t,x;\tau,s] \equiv f_z\left(t,x,\tau,s,z^0(\tau,s), u^0(\tau,s)\right),$$

$$f_u[t,x;\tau,s] \equiv f_u\left(t,x,\tau,s,z^0(\tau,s), u^0(\tau,s)\right),$$

$$g_y[t,x;\tau,s] \equiv g_y\left(t,x,\tau,s,z^0(\tau,s), v^0(\tau,s)\right), .$$

$$g_v[t,x;\tau,s] \equiv g_v\left(t,x,\tau,s,z^0(\tau,s), v^0(\tau,s)\right)$$

Applying the Vendroff-Bellman lemma (see e.g. [14, 15]) we prove the validity of the estimates.

$$\|\Delta z_\varepsilon(t,x)\| \le L_1\varepsilon, \quad (t,x)\in D_1 \ ; \quad \|\Delta y_\varepsilon(t,x)\| \le L_2\varepsilon, \quad (t,x)\in D_2 \ . \tag{13}$$

Here $L_i > 0, \quad i = 1,2$ are some constants.
Taking into account (10), (13) by means of (12) we prove the following statement.
Lemma. *Under the made assumptions the expansions are valid.*

$$\begin{array}{l} \Delta z_{\varepsilon}(t,x) = \varepsilon \delta z(t,x) + 0(\varepsilon;t,x), \quad (t,x) \in D_1 , \\ \Delta y_{\varepsilon}(t,x) = \varepsilon \delta y(t,x) + 0(\varepsilon;t,x), \quad (t,x) \in D_2 . \end{array} \tag{14}$$

Here $(\delta z(t,x), \ \delta y(t,x))$ *is a solution of the system of equations*

$$\delta z(t,x) = \int_{t_0}^{t}\int_{x_0}^{x} [f_z[t,x,\tau,s]\,\delta z(\tau,s) + f_u[t,x,\tau,s]\,\delta u(\tau,s)]\,dsd\tau, \tag{15}$$

$$\delta z(t,x) = \int_{t_0}^{t}\int_{x_0}^{x} [f_z[t,x,\tau,s]\,\delta z(\tau,s) + f_u[t,x,\tau,s]\,\delta u(\tau,s)]\,dsd\tau +$$
$$+ G_z\left(z^0(t_1,x)\right)\delta z(t_1,x).$$

The system of linear integral equations (15) *is said to be an equation in variations in the problem* (1)-(3)

Allowing for (10), (15) *from* (9) *we conclude that the first and second variations (in the classic sense* [16, 17, 18] *in the functional* (3) *have the following form, respectively*

$$\delta^1 S\left(u^0, v^0, \delta u, \delta v\right) =$$
$$= -\int_{t_0}^{t_1}\int_{x_0}^{X} \frac{\partial H_1'[t,x]}{\partial u}\delta u(t,x)\,dxdt - \int_{t_1}^{t_2}\int_{x_0}^{X} \frac{\partial H_2'[t,x]}{\partial v}\delta v(t,x)\,dxdt, \tag{16}$$

$$\delta^2 S\left(u^0, v^0, \delta u, \delta v\right) =$$
$$= \delta z'(t_1,X)\left[\frac{\partial^2 \varphi_1\left(z^0(t_1,X)\right)}{\partial z^2} + \frac{\partial^2 N\left(z^0(t_1,X)\right)}{\partial z^2}\right]\delta z(t_1,X) -$$
$$-\delta y'(t_2,X)\frac{\partial^2 \varphi_2\left(y^0(t_2,X)\right)}{\partial y^2}\delta y(t_2,X) - \int_{t_0}^{t_1}\int_{x_0}^{X}\left[\delta z'(t,x)\frac{\partial^2 H_1[t,x]}{\partial z^2}\delta z(t,x) -\right.$$
$$\left. -2\delta u'(t,x)\frac{\partial^2 H_1[t,x]}{\partial u \partial z}\delta z(t,x) + \delta u'(t,x)\frac{\partial^2 H_1[t,x]}{\partial u^2}\delta u(t,x)\right] dxdt - \tag{17}$$
$$-\int_{t_1}^{t_2}\int_{x_0}^{X} \delta z'(t_1,x)\frac{\partial^2 M[t,x]}{\partial z^2}\delta z(t_1,x)\,dxdt - \frac{1}{2}\int_{t_1}^{t_2}\int_{x_0}^{X}\left[\delta y'(t,x)\frac{\partial^2 H_2[t,x]}{\partial y^2}\delta y(t,x) +\right.$$
$$\left. +2\delta v'(t,x)\frac{\partial^2 H_2[t,x]}{\partial v \partial y}\delta y(t,x) + \delta v'(t,x)\frac{\partial^2 H_2[t,x]}{\partial v^2}\delta v(t,x)\right] dxdt.$$

4. Necessary optimality conditions. It is known that (see e.g. [14, 16]) if control domains U and V are open for optimality of admissible control $(u^0(t,x),$

$v^0(t,x))$ it is necessary that for all admissible variations $(\delta u(t,x),\ \delta v(t,x))$ of control $\left(u^0(t,x),\ v^0(t,x)\right)$ the first variation of the functional $S(u,v)$ be equal to zero, and the second one be non negative, i.e.

$$\delta^1 S\left(u^0, v^0, \delta u, \delta v\right) = 0 \tag{18}$$

$$\delta^2 S\left(u^0, v^0, \delta u, \delta v\right) \geq 0 \tag{19}$$

for all $\delta u(t,x) \in R^r$, $(t,x) \in D_1$, $\delta v(t,x) \in R^q$, $(t,x) \in D_2$.

Relations (18) and (19) are implicit necessary optimality conditions of first and second order.

The following theorem follows from identity (18) by virtue of independence of $\delta u(t,x)$ and $\delta v(t,x)$ by the scheme, for example, of the paper [14].

Theorem 1. *It the sets U and V are open, then for the optimality of the admissible control $\left(u^0(t,x),\ v^0(t,x)\right)$ in the paper (1)-(3) the relation*

$$\begin{aligned} \frac{\partial H_1[\theta,\xi]}{\partial u} &= 0, \quad (\theta,\xi) \in [t_0,t_1) \times [x_0,X), \\ \frac{\partial H_2[\theta,\xi]}{\partial v} &= 0, \quad (\theta,\xi) \in [t_1,t_2) \times [x_0,X). \end{aligned} \tag{20}$$

should be fulfilled.

Here $(\theta,\xi) \in [t_0,t_1) \times [x_0,X)$ $\left((\theta,\xi) \in [t_1,t_2) \times [x_0,X)\right)$ is an arbitrary continuity point of the controlling function $u^0(t,x)$ $\left(v^0(t,x)\right)$.

The relation (20) is said to be an analogy of the Euler equation [16, 17, 18] for the considered problem.

Analogy of the Euler equation is a necessary optimality condition of first order.

Any admissible control $\left(u^0(t,x),\ v^0(t,x)\right)$ satisfying the Euler equation is said to be classic extremal.

Now, let's derive necessary second order optimality conditions. We use the Dirichlet formula and prove that the solutions of the system of integral equations (15) admit the representations

$$\begin{gathered} \delta z(t,x) = \int_{t_0}^{t}\int_{x_0}^{x} f_u[t,x;\tau,s]\,\delta u(\tau,s)\,ds d\tau + \\ + \int_{t_0}^{t}\int_{x_0}^{x}\left[\int_{\tau}^{t}\int_{s}^{x} R_1[t,x,\alpha,\beta]\, f_u[\alpha,\beta;\tau,s]\,d\alpha d\beta\right]\delta u(\tau,s)\,ds d\tau, \end{gathered} \tag{21}$$

$$\begin{gathered} \delta y(t,x) = \int_{t_1}^{t}\int_{x_0}^{x} g_v[t,x;\tau,s]\,\delta v(\tau,s)\,ds d\tau + \\ + \int_{t_1}^{t}\int_{x_0}^{x}\left[\int_{\tau}^{t}\int_{s}^{x} R_2[t,x,\alpha,\beta]\, g_v[\alpha,\beta;\tau,s]\,d\alpha d\beta\right]\delta v(\tau,s)\,ds d\tau + \\ + G_z(z(t_1,x))\,\delta z(t_1,x) + \int_{t_1}^{t}\int_{x_0}^{x} R_2(t,x;\tau,s)\,G_z(z(t_1,s))\,\delta z(t_1,s)\,ds d\tau. \end{gathered} \tag{22}$$

Here $R_i(t,x;\tau,s)$, $i=1,2$ are the solutions of the following matrix integral equations of Volterra type, respectively.

$$R_1(t,x;\tau,s)=\int_\tau^t\int_s^x R_1(t,x;\alpha,\beta)\,f_z[\alpha,\beta;\tau,s]\,d\alpha d\beta+f_z[t,x;\tau,s],$$

$$R_2(t,x;\tau,s)=\int_\tau^t\int_s^x R_2(t,x;\alpha,\beta)\,g_y[\alpha,\beta;\tau,s]\,d\alpha d\beta+g_y[t,x;\tau,s].$$

Assuming

$$Q_1(t,x;\tau,s)=\int_\tau^t\int_s^x R_1(t,x,\alpha,\beta)\,f_u[\alpha,\beta,\tau,s]\,d\alpha d\beta+f_u[t,x;\tau,s] \tag{23}$$

we write representation (21) in the form

$$\delta z(t,x)=\int_{t_0}^t\int_{x_0}^x Q_1(t,x;\tau,s)\,\delta u(\tau,s)\,dsd\tau. \tag{24}$$

Allowing for this in (22) and assuming

$$Q_2(t,x;\tau,s)=g_v[t,x;\tau,s]+\int_\tau^t\int_s^x R_2(t,x;\alpha,\beta)\,g_v[\alpha,\beta;\tau,s]\,d\alpha d\beta,$$

$$Q_3(t,x;\tau,s)=\int_{t_1}^t\int_s^x R_2(t,x;\alpha,\beta)\,G_z(z(t_1,\beta))\,Q_1(t_1,\beta;\tau,s)\,d\alpha d\beta+$$
$$+G_z(z(t_1,x))\,Q_1(t_1,x;\tau,s),$$

we'll have

$$\delta y(t,x)=\int_{t_1}^t\int_{x_0}^x Q_2(t,x;\tau,s)\,\delta v(\tau,s)\,dsd\tau+\int_{t_0}^{t_1}\int_{x_0}^x Q_3(t,x;\tau,s)\,\delta u(\tau,s)\,dsd\tau. \tag{25}$$

The obtained representations play a great role in deriving necessary second order optimality conditions.

We use the representations (24), (25) and transform the inequality (19).

If we assume that $\delta u(t,x)\neq 0$, $(t,x)\in D_1$ and $\delta v(t,x)\equiv 0$, $(t,x)\in D_2$ then representations (24),(25) and inequality (19) will take the following form, respectively:

$$\delta z(t,x)=\int_{t_0}^t\int_{x_0}^x Q_1(t,x;\tau,s)\,\delta u(\tau,s)\,dsd\tau,\quad (t,x)\in D_1\ , \tag{26}$$

$$\delta y(t,x)=\int_{t_0}^{t_1}\int_{x_0}^{x} Q_3(t,x;\tau,s)\,\delta u(\tau,s)\,dsd\tau, \quad (t,x)\in D_2\ . \tag{27}$$

$$\delta z'(t_1,X)\left[\frac{\partial^2\varphi_1\left(z^0(t_1,X)\right)}{\partial z^2}+\frac{\partial^2 N\left(z^0(t_1,X)\right)}{\partial z^2}\right]\delta z(t_1,X)-$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{X}\left[\delta z'(t,x)\,\frac{\partial^2 H_1[t,x]}{\partial z^2}\delta z(t,x)-\right.$$

$$\left.-2\delta u'(t,x)\,\frac{\partial^2 H_1[t,x]}{\partial u\partial z}\delta z(t,x)+\delta u'(t,x)\,\frac{\partial^2 H_1[t,x]}{\partial u^2}\delta u(t,x)\right]dxdt-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{X}\delta z'(t_1,x)\,\frac{\partial^2 M[t,x]}{\partial z^2}\delta z(t_1,x)\,dxdt-$$

$$-\int_{t_1}^{t_2}\int_{x_0}^{X}\delta y'(t,x)\,\frac{\partial^2 H_2[t,x]}{\partial y^2}\delta y(t,x)\,dxdt\geq 0. \tag{28}$$

Allowing for $(26),(27)$ we get

$$\delta z'(t_1,X)\left[\frac{\partial^2\varphi_1(z(t_1,X))}{\partial z^2}+\frac{\partial^2 N\left(z^0(t_1,X)\right)}{\partial z^2}\right]\delta z(t_1,X)=$$

$$=\int_{t_0}^{t_1}\int_{x_0}^{X}\int_{t_0}^{t_1}\int_{x_0}^{X}\delta u'(\tau,s)\,Q_1'(t_1,X;\tau,s)\left[\frac{\partial^2\varphi_1(z(t_1,X))}{\partial z^2}+\right.$$

$$\left.+\frac{\partial^2 N\left(z^0(t_1,X)\right)}{\partial z^2}\right]Q_1(t_1,X;\alpha,\beta)\,\delta u(\alpha,\beta)\,dsd\tau d\alpha d\beta, \tag{29}$$

$$\delta y'(t_2,X)\,\frac{\partial^2\varphi_2\left(y^0(t_2,X)\right)}{\partial y^2}\delta y(t_2,X)=$$

$$=\int_{t_0}^{t_1}\int_{x_0}^{X}\int_{t_0}^{t_1}\int_{x_0}^{X}\delta u'(\tau,s)\,Q_3'(t_2,X;\tau,s)\,\frac{\partial^2\varphi_2\left(y^0(t_2,X)\right)}{\partial y^2}\times$$

$$\times Q_3(t_2,X;\alpha,\beta)\,\delta u(\alpha,\beta)\,d\tau dsd\alpha d\beta. \tag{30}$$

Further, it is clear that

$$\int_{t_0}^{t_1}\int_{x_0}^{X}\delta u'(t,x)\,\frac{\partial^2 H_1[t,x]}{\partial u\partial z}\delta z(t,x)\,dxdt=$$

$$\int_{t_0}^{t_1}\int_{x_0}^{X}\left[\int_{t}^{t_1}\int_{x}^{X}\delta u'(\tau,s)\,\frac{\partial^2 H_1[\tau,s]}{\partial u\partial z}Q_1(\tau,s;t,x)\,d\tau ds\right]\delta u(t,x)\,dxdt. \tag{31}$$

Finally, following the schemes of the papers [14, 19] we prove the identities

$$\int_{t_1}^{t_2}\int_{x_0}^{X}\delta z'(t_1,x)\frac{\partial^2 M[t,x]}{\partial z^2}\delta z(t_1,x)\,dxdt=$$

$$=\int_{t_1}^{t_2}\int_{x_0}^{X}\left(\int_{t_0}^{t_1}\int_{x_0}^{x}\delta u'(\tau,s)\,Q_1'(t_1,x;\tau,s)\,dsd\tau\right)\times$$

$$\times\frac{\partial^2 M[t,x]}{\partial z^2}\left(\int_{t_0}^{t_1}\int_{x_0}^{x}Q_1(t_1,x;\alpha,\beta)\,\delta u(\alpha,\beta)\,d\alpha d\beta\right)dxdt=$$

$$=\int_{t_0}^{t_1}\int_{x_0}^{X}\int_{t_0}^{t_1}\int_{x_0}^{X}\delta u'(\tau,s)\left[\int_{t_1}^{t_2}\int_{\max(s,\beta)}^{X}Q_1'(t_1,x;\tau,s)\frac{\partial^2 M[t,x]}{\partial z^2}\times\right.$$

$$\left.\times Q_1(t_1,x;\alpha,\beta)\,dxdt\right]\delta u(\alpha,\beta)\,d\alpha d\beta d\tau ds, \tag{32}$$

$$\int_{t_1}^{t}\int_{x_0}^{X}\delta y'(t,x)\frac{\partial^2 H_2[t,x]}{\partial y^2}\delta y(t,x)\,dxdt=$$

$$=\int_{t_1}^{t_1}\int_{x_0}^{X}\int_{t_0}^{t_1}\int_{x_0}^{X}\delta u'(\tau,s)\left[\int_{t_1}^{t_1}\int_{x_0}^{X}Q_3'(t,x;\tau,s)\frac{\partial^2 H_2[t,x]}{\partial y^2}\times\right.$$

$$\left.\times Q_3(t,x;\alpha,\beta)\,dxdt\right]\delta u(\alpha,\beta)\,dsd\tau d\alpha d\beta, \tag{33}$$

$$\int_{t_1}^{t_2}\int_{x_0}^{X}\delta z'(t,x)\frac{\partial^2 H_1[t,x]}{\partial z^2}\delta z(t,x)\,dxdt=$$

$$=\int_{t_1}^{t_1}\int_{x_0}^{X}\int_{t_0}^{t_1}\int_{x_0}^{X}\delta u'(\tau,s)\left[\int_{\max(\tau,\beta)}^{t_1}\int_{\max(s,\beta)}^{X}Q_1'(t,x;\tau,s)\frac{\partial^2 H_2[t,x]}{\partial y^2}\times\right.$$

$$\left.\times Q_1(t,x;\alpha,\beta)\,dxdt\right]\delta u(\alpha,\beta)\,d\alpha d\beta dsd\tau. \tag{34}$$

Let by definition

$$K_1(\tau,s,\alpha,\beta)=-Q_3'(t_2,X;\tau,s)\frac{\partial^2\varphi_2\left(y^0(t_2,X)\right)}{\partial y^2}Q_3(t_2,X;\alpha,\beta)-$$

$$-Q_1'(t_1,X;\alpha,\beta)\left(\frac{\partial^2\varphi_1\left(z^0(t_1,X)\right)}{\partial z^2}+\frac{\partial^2 N\left(z^0(t_1,X)\right)}{\partial z^2}\right)Q_1(t_1,X;\alpha,\beta)-$$

$$-\int_{t_1}^{t_2}\int_{\max(s,\beta)}^{X}\left[Q_1'(t_1,x;\tau,s)\frac{\partial^2 M[t,x]}{\partial z^2}Q_1(t_1,x;\alpha,\beta)+\right.$$

$$+Q_3'(t,x,\tau,s,)\frac{\partial^2 H_2[t,x]}{\partial y^2}Q_3(t,x;\alpha,\beta)\Bigg]dxdt-$$

$$-\int\limits_{\max(\tau,\alpha)}^{t_1}\int\limits_{\max(s,\beta)}^{X}Q_1'(t,x;\tau,s)\frac{\partial^2 H_2[t,x]}{\partial y^2}Q_1(t,x;\alpha,\beta)\,dxdt. \qquad (35)$$

Then allowing for identities (29)-(34), inequality (28) will take the form

$$\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{X}\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{X}\delta u'(\tau,s)K_1(\tau,s,\alpha,\beta)\delta u(\alpha,\beta)\,dsd\tau d\alpha d\beta+$$

$$+2\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{X}\left[\int\limits_{t}^{t_1}\int\limits_{x}^{X}\delta u'(\tau,s)\frac{\partial^2 H_1[t,x]}{\partial u\partial z}Q_1(\tau,s;t,x)\,d\tau ds\right]\delta u(t,x)\,dxdt+$$

$$+\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{X}\delta u'(t,x)\frac{\partial^2 H_1[t,x]}{\partial u^2}\delta u(t,x)\,dxdt\leq 0 \qquad (36)$$

If we assume $\delta u(t,x)\equiv 0$, $(t,x)\in D_1$ and $\delta v(t,x)\neq 0$, $(t,x)\in D_2$ the representations (24),(35) and inequality (19) will take the following form, respectively:

$$\delta z(t,x)\equiv 0,\quad (t,x)\in D_1\ ,$$

$$\delta y(t,x)=\int\limits_{t_1}^{t}\int\limits_{x_0}^{x}Q_2(t,x;\tau,s)\delta v(\tau,s)\,dsd\tau,\quad (t,x)\in D_2 \qquad (37)$$

$$\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{X}\Bigg[\delta y'(t,x)\frac{\partial^2 H_2[t,x]}{\partial y^2}\delta y(t,x)+\delta v'(t,x)\frac{\partial^2 H_2[t,x]}{\partial v^2}\delta v(t,x)+$$

$$+2\delta v'(t,x)\frac{\partial^2 H_2[t,x]}{\partial v\partial y}\delta y(t,x)\Bigg]dxdt-$$

$$-\delta y'(t_2,X)\frac{\partial^2\varphi_2\left(y^0(t_2,X)\right)}{\partial y^2}\delta y(t_2,X)\leq 0. \qquad (38)$$

By means of representation (37) we prove the validity of identities

$$\delta y'(t_2,X)\frac{\partial^2\varphi_2\left(y^0(t_2,X)\right)}{\partial y^2}\delta y(t_2,X)=$$

$$=\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{X}\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{X}\delta u'(\tau,s)Q_2'(t_2,X;\tau,s)\frac{\partial^2\varphi_2\left(y^0(t_2,X)\right)}{\partial y^2}\times$$

$$\times Q_2(t_2,X;\alpha,\beta)\delta u(\alpha,\beta)\,d\tau dsd\alpha d\beta, \qquad (39)$$

$$\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{X}\delta u'(t,x)\frac{\partial^2 H_2[t,x]}{\partial v\partial y}\delta y(t,x)\,dxdt=$$

$$=\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{X}\left[\int\limits_{t}^{t_2}\int\limits_{x}^{X}\delta v'(\tau,s)\frac{\partial^2 H_2[\tau,s]}{\partial v\partial y}Q_2(\tau,s;t,x)\,d\tau ds\right]\delta v(t,x)\,dxdt, \tag{40}$$

$$\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{X}\delta y'(t,x)\frac{\partial^2 H_2[t,x]}{\partial y^2}\delta y(t,x)\,dxdt=$$

$$=\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{X}\int\limits_{t}^{t_2}\int\limits_{x}^{X}\delta v'(\tau,s)\left[\int\limits_{\max(\tau,\alpha)}^{t_2}\int\limits_{\max(s,\beta)}^{X}Q_2(t,x,\tau,s)\frac{\partial^2 H_2[t,x]}{\partial y^2}\times\right.$$

$$\left.\times Q_2(t,x;\alpha,\beta)\,d\alpha d\beta\right]d\tau ds dx dt. \tag{41}$$

Assume

$$K_2(t,s;\alpha,\beta)=-Q_2'(t_2,X;\tau,s)\frac{\partial^2\varphi_2\left(y^0(t_2,X)\right)}{\partial y^2}Q_2(t_2,X;\alpha,\beta)+$$

$$+\int\limits_{\max(\tau,\alpha)}^{t_2}\int\limits_{\max(s,\beta)}^{X}Q_2'(t,x;\tau,s)\frac{\partial^2 H_2[t,x]}{\partial y^2}Q_2(t,x;\alpha,\beta)\,dxdt. \tag{42}$$

Then allowing for identities (39)-(41) and denotation (42), the inequality (38) will take the form:

$$\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{X}\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{X}\delta v'(\tau,s)K_2(\tau,s;\alpha,\beta)\delta v(\alpha,\beta)\,d\alpha d\beta ds d\tau+$$

$$+2\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{X}\left[\int\limits_{t}^{t_2}\int\limits_{x}^{X}\delta v'(\tau,s)\frac{\partial^2 H_2[\tau,s]}{\partial v\partial y}Q_2(\tau,s,t,x)\,d\tau ds\right]\delta v(t,x)\,dxdt+$$

$$+\int\limits_{t_1}^{t_2}\int\limits_{x_0}^{X}\delta v'(t,x)\frac{\partial^2 H_2[t,x]}{\partial v^2}\delta v(t,x)\,dxdt\le 0. \tag{43}$$

Summarize the obtained result.

Theorem 2. *For optimality of the classic extremal* $\left(u^0(t,x),\ v^0(t,x)\right)$ *the relations* (36), (43) *should be fulfilled for all* $\delta u(t,x)\in R^r$, $(t,x)\in D_1$ *and* $\delta v(t,x)\in R^q$, $(t,x)\in D_2$ *, respectively.*

Inequalities (36), (43) *are necessary second order optimality conditions.*

Notice that the matrix functions of type (35), (42) *for investigating singular controls in control problems were first introduced in the paper* [20].

Remark. Relations (35) and (43) are the second order integral necessary optimality conditions. From them we can obtain different easily verifiable pointwise necessary optimality conditions.

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