Rahila A. SULEYMANOVA

ESTIMATION OF DIFFUSION PROCESS PARAMETERS BY DISCRETE OBSERVATIONS

Abstract

Unknown parameters of the diffusion process are estimated by the results of discrete observations.

We apply martingale methods for estimating the parameters based on the approximation of likelihood function and obtain consistent and asymptotically normal estimates of parameters.

In the paper we consider the methods for estimating unknown parameters in discretely obseved diffusion type model. It is assumed that unknown parameters enter into the model linearly.

In majority of cases for discrete observations there are no exact expressions for the maximal likelihood function and the use of approximate likelihood function reduces to inconsistent estimates. For discretely observed diffusive processes described by stochastic diffusive equarion we apple martingale methods for estimating the parameters based on the approximation of likelihood function. As a result we obtain consistent and asymptotically normal estimates of parameters.

On a probability space (Ω, F, P) we consider a diffusive process determined by the stochastic differential equation

$$dX_t = f(X_t;\theta)\,dt + g(X_t)\,dW_t\ , \tag{1}$$

where $X_0 = x_0,\ t \geq 0,\ \theta = (\theta_1, \theta_2, ..., \theta_k)^T$ is a vector of parameters. Remind that in the case of continuous observation instead of the equation (1) we consider a diffusive type process with stochastic differential

$$dX_t = (a + bX_t)\,dt + X_t dW_t,\ t \geq 0,\quad X_0 = x_0 > 0, \tag{*}$$

where a and b are the unknown parameters and $\theta = (a, b)^T$, and W_t is a standard Wiener process. Index T indecates transposition. It is assumed that the coefficients of the drifts $f(X_t;\theta)$ and diffusion $g(X_t)$ are such that the equation (1) has a unique solution for all θ in the domain $\theta \subset R^k$, moreover, the functions $f(X_t;\theta)$ and $g(X_t)$ are continuously differentiable with respect θ and $g(X_t)$ is a positive function and doesn't depend on θ.

Our goal is to estimate an unknown vector parameter θ, by discrete observations $X_0,\ X_\Delta,\ X_{2\Delta}, ..., X_{n\Delta}$.

The likelihood function for the process X_t with continuous time given on the segment $[0, t]$ is expressed as

$$L_t(\theta) = \exp\left\{ \int_0^t \frac{f(X_s;\theta)}{g^2(X_s)} dX_s - \frac{1}{2}\int_0^t \frac{f^2(X_s;\theta)}{g^2(X_s)} ds \right\}. \tag{2}$$

This function is F-martingale that admits to get optimal estimates (see [9]).

If the process X_t is observed at the discrete points $X_0, X_\Delta, X_{2\Delta}, ..., X_{n\Delta}$, we'll approximate the function (2). To this end we'll substitute the Lebesgue and Ito integrals in (2) for the Riemann-Ito sum, take the upper integral sum and get the approxomate likelihood function:

$$\widetilde{L}_n(\theta) = \exp\left\{ \sum_{i=1}^{n} \frac{f(X_i;\theta)}{g^2(X_i)} \Delta X_i - \frac{1}{2}\sum_{i=1}^{n} \frac{f^2(X_i;\theta)}{g^2(X_i)} \right\}.$$

Differentiating the logarithm of the approximate likelihood function $\widetilde{L}_n(\theta)$ with respect to a vector parameter θ we get an estimator of a vector variable $\theta = (\theta_1, \theta_2, ..., \theta_k)^T$:

$$\frac{\partial \ln \widetilde{L}_n(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{f'_\theta\left(X_{(i-1)\Delta};\theta\right)}{g^2\left(X_{(i-1)\Delta}\right)} \left(X_{i\Delta} - X_{(i-1)\Delta}\right) -$$

$$-\Delta \sum_{i=1}^{n} \frac{f\left(X_{(i-1)\Delta};\theta\right)}{g^2\left(X_{(i-1)\Delta}\right)} \cdot f'_\theta\left(X_{(i-1)\Delta};\theta\right). \tag{3}$$

Thus, the estimator is a vector of dimension k.

It a diffusion coefficient entering in (1) depends on θ, i.e. $g(X_t;\theta)$, as is shown in the paper [2] (pp. 33-37), when fulfilling some regularity conditions for a likelihood function we can use the same estimator of a vector variable:

$$\frac{\partial \ln \widetilde{L}_n(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{f'_\theta\left(X_{(i-1)\Delta};\theta\right)}{g^2\left(X_{(i-1)\Delta};\theta\right)} \left(X_{i\Delta} - X_{(i-1)\Delta}\right) -$$

$$-\Delta \sum_{i=1}^{n} \frac{f\left(X_{(i-1)\Delta};\theta\right) f'_\theta\left(X_{(i-1)\Delta};\theta\right)}{g^2\left(X_{(i-1)\Delta};\theta\right)}. \tag{4}$$

In the same paper [2] it is established that the use of this expression for estimating a vector parameter θ allows to get strictly consistent estimations if and only if $\max\limits_{1\le i\le n} |t_i - t_{i-1}|$ is small. In the case of equidistant observations the time interval between the observations is determined by a fixed value of Δ and is bounded below with some quantity. The estimations of the vector parameter $\widehat{\theta}_n$ obtained by means of the estimator (4) become inconsistent [2] .

For solving this problem it was suggested [1] , [2] to use martingale estimation finctions.

Let for a stochastic sequence $A = (A_n, F_n)$ the elements A_n for each $n \geq 0$ be F_{n-1}-measurable. Assuming $F_{-1} = F_0$ we'll write $A = (A_n, F_{n-1})$ and call A a predictable sequence [6] . A stochastic sequence is said to be increasing if $A_0 = a_0 > 0$ with probability 1 and $A_n \geq A_{n-1}$ with probability 1 [6].

Now let's consider the D. Doub's expansion. It is as follows [6] , [7] . Let $V = (V_n, F_n)$ be a sub-martingale. Then there will be found a martingale $m = (m_n, F_n)$ and predictable increasing sequence $A = (A_n, F_{n-1})$ such that for each $n \geq 0$ it holds the Doub expansion

$$V_n = m_n + A_n \text{ with probability 1.} \tag{5}$$

Such type expansion is unique ([6] , [7]) .

It follows from the Doub's expansion (5) that the sequence $A = (A_n, F_{n-1})$ compensates the sub-matingale $V = (V_n, F_n)$ to the martingale and is said to be a compensator [6] , [7] .

As is shown in [2] , each coordinate of the estimator $\frac{\partial \ln \widetilde{L}_n(\theta_i)}{\partial \theta_i}$, $i = 1, ..., k$, determined in (3) is a sub-martingale.

The first method of construction of martingale estimation function is to compensate $\frac{\partial \ln \widetilde{L}_n(\theta)}{\partial \theta}$ and obtain the martingale $\widetilde{G}_n$. Subtracting a compensator from $\frac{\partial \ln \widetilde{L}_n(\theta)}{\partial \theta}$ we get a P_θ-martingale with zero mean with respect to sigma-algebra flow $F_i = \sigma(X_\Delta, ..., X_{i\Delta})$, $i \geq 1$.

In order to construct a compensator for $\frac{\partial \ln \widetilde{L}_n(\theta)}{\partial \theta}$ we prove the following theorem.

Theorem 1. *Let for the stochastic differential equation*

$$dX_t = f(X_t; \theta)\, dt + g(X_t)\, dW_t,$$

($X_0 = x_0 > 0$, $t \geq 0$, $\theta = (\theta_1, \theta_2, ..., \theta_k)^T$ *is a vector of parameters) observed at the discrete points* $X_\Delta, X_{2\Delta}, ..., X_{n\Delta}$ *from* $(X_t)_{t \geq 0}$ *, the estimation function of a vector parameter* θ *be determined by the relation* (3) *.*

Then a compensator for (3) *is expressed as follows:*

$$\sum_{i=1}^{n} E_\theta \left(\frac{\partial \ln \widetilde{L}_i(\theta)}{\partial \theta} - \frac{\partial \ln \widetilde{L}_{i-1}(\theta)}{\partial \theta} \middle| F_{i-1} \right) =$$

$$= \sum_{i=1}^{n} \frac{f'_\theta\left(X_{(i-1)\Delta}; \theta\right)}{g^2\left(X_{(i-1)\Delta}\right)} \left(F\left(X_{(i-1)\Delta}; \theta\right) - X_{(i-1)\Delta} \right) -$$

$$-\Delta \sum_{i=1}^{n} \frac{f\left(X_{(i-1)\Delta}; \theta\right) f'_\theta\left(X_{(i-1)\Delta}; \theta\right)}{g^2\left(X_{(i-1)\Delta}\right)}, \tag{6}$$

where

$$F\left(X_{(i-1)\Delta}; \theta\right) = E_\theta\left(X_{i\Delta} \middle| X_{(i-1)\Delta} \right). \tag{7}$$

Proof. In order to find an expression for the compensator, we consider the difference for $i = 2, ..., n$,

$$\frac{\partial \ln \widetilde{L}_i(\theta)}{\partial \theta} - \frac{\partial \ln \widetilde{L}_{i-1}(\theta)}{\partial \theta} = \frac{f'_\theta\left(X_{(i-1)\Delta}; \theta\right)}{g^2\left(X_{(i-1)\Delta}\right)} \left(X_{i\Delta} - X_{(i-1)\Delta} \right) -$$

$$-\Delta \frac{f\left(X_{(i-1)\Delta}; \theta\right) f'_\theta\left(X_{(i-1)\Delta}; \theta\right)}{g^2\left(X_{(i-1)\Delta}\right)}.$$

Let's find conditional mathematical expectation of the previous expression:

$$E_0 \left(\frac{\partial \ln \widetilde{L}_i(\theta)}{\partial \theta} - \frac{\partial \ln \widetilde{L}_{i-1}(\theta)}{\partial \theta} \middle| X_{(i-1)\Delta} \right) = \frac{f'_\theta\left(X_{(i-1)\Delta}; \theta\right)}{g^2\left(X_{(i-1)\Delta}\right)} \times$$

$$\times \left(E_0\left(X_{i\Delta}|\, X_{(i-1)\Delta}\right)-X_{(i-1)\Delta}\right)-\Delta\frac{f\left(X_{(i-1)\Delta};\theta\right)f'_\theta\left(X_{(i-1)\Delta};\theta\right)}{g^2\left(X_{(i-1)\Delta}\right)}.$$

Using the denotation (7) we transform the last expression as follows

$$E_0\left(\frac{\partial\ln\widetilde{L}_i(\theta)}{\partial\theta}-\frac{\partial\ln\widetilde{L}_{i-1}(\theta)}{\partial\theta}\right|X_{(i-1)\Delta}\Bigg)=$$

$$=\frac{f'_\theta\left(X_{(i-1)\Delta};\theta\right)}{g^2\left(X_{(i-1)\Delta}\right)}\left(F\left(X_{(i-1)\Delta};\theta\right)-X_{(i-1)\Delta}\right)-\Delta\frac{f\left(X_{(i-1)\Delta};\theta\right)f'_\theta\left(X_{(i-1)\Delta};\theta\right)}{g^2\left(X_{(i-1)\Delta}\right)}.$$

Summing the right hand sides of the last expression with respect to $i=1,...,n$, the compensator:

$$A_n=\sum_{i=1}^{n}\frac{f'_\theta\left(X_{(i-1)\Delta};\theta\right)}{g^2\left(X_{(i-1)\Delta}\right)}\left(F\left(X_{(i-1)\Delta};\theta\right)-X_{(i-1)\Delta}\right)-$$

$$-\Delta\frac{f\left(X_{(i-1)\Delta};\theta\right)f'_\theta\left(X_{(i-1)\Delta};\theta\right)}{g^2\left(X_{(i-1)\Delta}\right)}.$$

Statement (6) of the theorem follows from the last expression.

The theorem is proved.

Thus, subtracting from (3) the expression (6) and assuming that in (1) $g(X_t;\theta)=g(X_t)$ (does't depend on the parameter θ) we obtain a martingale zero mean estimation function of the form

$$\widetilde{G}_n(\theta)=\frac{\partial\ln\widetilde{L}_i(\theta)}{\partial\theta}-A_n=\sum_{i=1}^{n}\frac{f'_\theta\left(X_{(i-1)\Delta};\theta\right)}{g^2\left(X_{(i-1)\Delta}\right)}\left(X_{i\Delta}-F\left(X_{(i-1)\Delta};\theta\right)\right). \quad (8)$$

General class of zero mean P_θ martingale estimation functions of the form [2]

$$G_n(\theta)=\sum_{i=1}^{n}g_{i-1}\left(X_{(i-1)\Delta};\theta\right)\left(X_{i\Delta}-F\left(X_{(i-1)\Delta};\theta\right)\right), \quad (9)$$

is an alternative variant, where for $i=1,...,n$ the function $g_{i-1}\left(X_{(i-1)\Delta};\theta\right)$ is F_{i-1} -measurable and continuously-differentiable with respect to θ, and the function $F\left(X_{(i-1)\Delta};\theta\right)$ is determined in (7).

The function allowing to get optimal in asymptotic sense estimation of parameters (1) interior to the class (9) $[1],[2]$, is determined in the following way:

$$G^*_n(\theta)=\sum_{i=1}^{n}\frac{F'_\theta\left(X_{(i-1)\Delta};\theta\right)}{\phi\left(X_{(i-1)\Delta};\theta\right)}\left(X_{i\Delta}-F\left(X_{(i-1)\Delta};\theta\right)\right) \quad (10)$$

where

$$\phi\left(X_{(i-1)\Delta};\theta\right)=E_\theta\left[\left(X_{i\Delta}-F\left(X_{(i-1)\Delta};\theta\right)\right)^2\Big|X_{(i-1)\Delta}\right],\quad i=1,..,n. \quad (11)$$

Optimality in asymptotic sense for estimation of parameters obtained on the basis of (10) means [2] that as $\Delta\to0$ with probability 1 the estimation of parameters converge to the maximal likelihood estimations.

As it was established in [10] the function $G_n^*(\theta)$ interior the class (9) in some sense is the nearest to the function based on ordinary unknown exact likelihood function.

For small Δ martigale estimation function $\widetilde{G}_n(\theta)$ determined in (8) is a first order approximation with respect to Δ of the function $G_n^*(\theta)$ determined in (10) [2]. This means that the function $\widetilde{G}_n(\theta)$ allows to get as $\Delta \to 0$ with probability 1 estimations of parameters of the stochastic equation (1), approximately being the of maximal likelihood estimations.

Asymptotic properties of the estimation $\widehat{\theta}_n$ obtained from the martingale estimation functions (8) and (10), or by a general way-from the class of martingale estimation functions G_n in the form of (9), are considered in [10].

Let's consider a diffusion process determined by a stochastic differential equation (*). To estimate the unknown parameters a and b by discrete observations $X_0, X_\Delta, X_{2\Delta}, ..., X_{n\Delta}$ from $(X_n)_{t\geq 0}$ we use the martingale estimation function $\widetilde{G}_n$.

Comparing the equations (1) and (*) we get the correspondence:

$$g(X_t,\theta) = X_{(i-1)\Delta}, \quad f(X_t,\theta) = \left(a + bX_{(i-1)\Delta}\right), \quad \theta = (a,b)^T, \quad (i-1)\Delta = t_{i-1},$$

where the index T is transposition

Lemma 1. *Let the diffusion process X_t be given by the equation (*).*
Then the function

$$f(t) = E_{a,b}(X_t|X_0)$$

is a solution of the differential equation

$$f'(t) = a + bf(t). \tag{12}$$

Proof. Let's write the (*) in an integral form.

$$X_t = X_0 + \int_0^t (a + bX_s)\,ds + \int_0^t X_s dW_s\ .$$

By the condition of X_0 we have

$$E_{a,b}(X_t|X_0) = E_{a,b}(X_0|X_0) +$$
$$+E_{a,b}\left(\left(\int_0^t (a+bX_s)\,ds\right)\Bigg| X_0\right) + E_{a,b}\left(\left(\int_0^t X_s dW_s\right)\Bigg| X_0\right),$$

that is equivalent to

$$E_{a,b}(X_t|X_0) = X_0 + at + b\int_0^t E_{a,b}(X_s|X_0)\,ds,$$

since $E_{a,b}\left(\left(\int_0^t X_s dW_s\right)\Big| X_0\right) = 0$. Consequently,

$$\frac{dE_{a,b}(X_t|X_0)}{dt} = a + bE_{a,b}(X_t|X_0).$$

Notice that the function $f_\tau(t) = E_{a,b}(X_t | X_T)$, $0 \le \tau \le t$ satisfies the equation (12) as well. The proof remains the same, as the expression

$$E_{a,b}\left(\left(\int_0^t X_s dW_s\right)\middle| X_T\right) = \int_0^T X_s dW_s,$$

is independent of t and that is why it doesn't influence on the derivative.

The lemma is proved.

Corollary. *The function*

$$F\left(X_{(i-1)\Delta}; a, b\right) = E_{a,b}\left(X_{i\Delta} | X_{(i-1)\Delta}\right),$$

determined by the expression (7) , *for the stochastic equation* (*) *has the form:*

$$F\left(X_{(i-1)\Delta}; a, b\right) = X_{(i-1)\Delta} \cdot e^{b\Delta} + \frac{a}{b}\left(e^{b\Delta} - 1\right). \tag{13}$$

Proof. The solution of the linear differential equation

$$f'(t) = a + bf(t), \qquad f(t_0) = f_0, \quad t \ge t_0,$$

under the given initial conditions has the form

$$f(t) = f_0 e^{b(t-t_0)} + \frac{a}{b}\left(e^{b(t-t_0)} - 1\right).$$

Consequently, for $E_{a,b}(X_{t_i} | X_{t_i-1}) = f(t_i)$ when observations are performed by the interval $\Delta = t_i - t_{i-1}$ for all $1 \le i \le n$ we get

$$E_{a,b}(X_{t_i} | X_{t_i-1}) = E_{a,b}(X_{t_i-1} | X_{t_i-1}) e^{b\Delta} + \frac{a}{b}\left(e^{b\Delta} - 1\right) = X_{t_i-1} e^{b\Delta} + \frac{a}{b}\left(e^{b\Delta} - 1\right).$$

Hence the statement of the corollary (13) follows:

The corollary is proved.

Substituting (13) into (8) we get that for the equation (*) the function $\widetilde{G}_n(a,b)$ is given by the following formula:

$$\widetilde{G}_n(a,b) = \left[\sum_{i=1}^n \frac{1}{X^2_{(i-1)\Delta}}\left(X_{i\Delta} - X_{(i-1)\Delta} e^{b\Delta} + \frac{a}{b}\left(1 - e^{b\Delta}\right)\right),\right.$$

$$\left.\sum_{i=1}^n \frac{1}{X_{(i-1)\Delta}}\left(X_{i\Delta} - X_{(i-1)\Delta} e^{b\Delta} + \frac{a}{b}\left(1 - e^{b\Delta}\right)\right)\right]^T.$$

Obviously, any estimation function is a vector, whose number of coordinates equals the number of estimated parameters.

Substituting into (10) from (11) and (13) we get the estimation, function $G^*_n(a,b)$ for the differential stochastic equation $(*)$

$$G^*_n(a,b) = \left[\sum_{i=1}^n \frac{e^{b\Delta} - 1}{b\phi\left(X_{(i-1)\Delta}; a, b\right)}\left(X_{i\Delta} - X_{(i-1)\Delta} e^{b\Delta} + \frac{a}{b}\left(1 - e^{b\Delta}\right)\right),\right.$$

$$\sum_{i=1}^{n}\frac{\Delta e^{b\Delta}\left(X_{(i-1)\Delta}+\frac{a}{b}\right)+\frac{a}{b^2}\left(1-e^{b\Delta}\right)}{\phi\left(X_{(i-1)\Delta};a,b\right)}\left(X_{i\Delta}-X_{(i-1)\Delta}e^{b\Delta}+\frac{a}{b}\left(1-e^{b\Delta}\right)\right)\Bigg]^{T},$$

where $\phi\left(X_{(i-1)\Delta};a,b\right)$ is given in (11).

Let's find estimates of parameters $\widetilde{a}_n$ and $\widetilde{b}_n$ that are the solutions of a system of equations $\widetilde{G}_n(a,b)=0$.

Theorem 2. *Let* $X_t,\ 0\le t\le T$ *be a random process given by the equation* $(^{*})$.

Then the estimates of the parameters $\widetilde{a}_n$ *and* $\widetilde{b}_n$ *obtained on the basis of the solution of the equation* $\widetilde{G}_n(a,b)=0$*, where the function* $\widetilde{G}_n(a,b)$ *is given in* (8) *are of the form:*

$$e^{\widetilde{b}_n\Delta}=\frac{\left(\sum\limits_{i=1}^{n}\frac{X_{i\Delta}}{X_{(i-1)\Delta}}\right)\left(\sum\limits_{i=1}^{n}\frac{1}{X^2_{(i-1)\Delta}}\right)-\left(\sum\limits_{i=1}^{n}\frac{1}{X_{(i-1)\Delta}}\right)\left(\sum\limits_{i=1}^{n}\frac{X_{i\Delta}}{X^2_{(i-1)\Delta}}\right)}{n\sum\limits_{i=1}^{n}\frac{1}{X^2_{(i-1)\Delta}}-\left(\sum\limits_{i=1}^{n}\frac{1}{X_{(i-1)\Delta}}\right)^2},\tag{14}$$

$$\widetilde{a}_n=\frac{\widetilde{b}_n}{1-e^{\widetilde{b}_n\Delta}}\frac{e^{\widetilde{b}_n\Delta}\sum\limits_{i=1}^{n}\frac{1}{X_{(i-1)\Delta}}-\sum\limits_{i=1}^{n}\frac{X_{i\Delta}}{X^2_{(i-1)\Delta}}}{\sum\limits_{i=1}^{n}\frac{1}{X^2_{(i-1)\Delta}}}.\tag{15}$$

Proof. To get the equations for estimating the parameters $\widetilde{a}_n,\widetilde{b}_n$ we equate each coordinate of the function $\widetilde{G}_n(a,b)$ to zero, and get

$$\begin{cases}\sum\limits_{i=1}^{n}\frac{X_{i\Delta}}{X^2_{(i-1)\Delta}}-e^{b\Delta}\sum\limits_{i=1}^{n}\frac{1}{X_{(i-1)\Delta}}+\frac{a}{b}\sum\limits_{i=1}^{n}\frac{1}{X^2_{(i-1)\Delta}}=0,\\[2ex]\sum\limits_{i=1}^{n}\frac{X_{i\Delta}}{X_{(i-1)\Delta}}-ne^{b\Delta}+\frac{a}{b}\left(1-e^{b\Delta}\right)\sum\limits_{i=1}^{n}\frac{1}{X_{(i-1)\Delta}}=0.\end{cases}\tag{16}$$

We transform the obtained system in the following way

$$\begin{cases}\sum\limits_{i=1}^{n}\frac{X_{i\Delta}}{X^2_{(i-1)\Delta}}-e^{b\Delta}\sum\limits_{i=1}^{n}\frac{1}{X_{(i-1)\Delta}}=-\frac{a}{b}\sum\limits_{i=1}^{n}\frac{1}{X^2_{(i-1)\Delta}},\\[2ex]\sum\limits_{i=1}^{n}\frac{X_{i\Delta}}{X_{(i-1)\Delta}}-ne^{b\Delta}=-\frac{a}{b}\left(1-e^{b\Delta}\right)\sum\limits_{i=1}^{n}\frac{1}{X_{(i-1)\Delta}}=0.\end{cases}$$

Having divided the first equation into the second one in the obtained system, we get an equation for the parameter b.

$$\frac{\sum\limits_{i=1}^{n}\frac{X_{i\Delta}}{X^2_{(i-1)\Delta}}-e^{b\Delta}\sum\limits_{i=1}^{n}\frac{1}{X_{(i-1)\Delta}}}{\sum\limits_{i=1}^{n}\frac{X_{i\Delta}}{X_{(i-1)\Delta}}-ne^{b\Delta}}=\frac{\sum\limits_{i=1}^{n}\frac{1}{X^2_{(i-1)\Delta}}}{\sum\limits_{i=1}^{n}\frac{1}{X_{(i-1)\Delta}}}.$$

From the last equation we get (14) .

In order to get the estimate $\widetilde{a}_n$ for the parameter a, we substitute $\widetilde{b}_n$, for example, into the first equation of the system (16) . After obvious transformations we get (15) .

The theorem is proved.

Asymptotic properties of estimates $\widetilde{a}_n$ and $\widetilde{b}_n$ obtained from the class of martingale estimation functions $\widetilde{G}_n(a,b)$ are defined in theorem 3 that is reduced without proof.

Theorem 3. *Let* X_t, $0 \le t \le T$ *be random process given by the equation* (1) *.*

Let $\widetilde{a}_n$ *and* $\widetilde{b}_n$ *be estimates obtained by the values of the process* $X_{i\Delta}$ *,* $i = 0, 1, ..., n = [T/\Delta]$ *on the basis of the martingale estimation function* (8) *. Then, as* $n \to \infty$, $\Delta \to 0$ *with probability 1 it holds:*

1) $\widetilde{a}_n \to \widehat{a}_T$ *,*

2) $\widetilde{b}_n \to \widehat{b}_T$ *,*

where $\widehat{a}_T$ *,* $\widehat{b}_T$ *are the maximal likelihood estimates obtained by continuous observations.*

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Rahila A. Suleymanova

Institute of Mathematics and Mechanics of NAS of Azerbaijan.

9, F. Agayev str., AZ1141, Baku, Azerbaijan.

Tel.: (99412) 439 47 20 (off.)

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