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THE SOLUTION OF A NONLACAL BOUNDARY VALUE PROBLEM FOR A FOURTH ORDER PARTIAL DIFFERENTIAL EQUATION

Abstract

In the paper we prove existence of a classic solution of a non-local boundary value problem for a fourth order partial differential equation.

For the equation [1]

$$u_{tt}(x,t) - u_{ttxx}(x,t) - u_{xx}(x,t) = f(x,t) \tag{1}$$

in the domain $D_T = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le T\}$ we consider a problem under the conditions

$$u(x,0) + \delta u(x,T) = \varphi(x),$$

$$u_t(x,0) + \delta u_t(x,T) = \psi(x), \quad 0 \le x \le 1 \tag{2}$$

and boundary conditions

$$u(x,0) = 0,\ u_x(1,t) + du_{xx}(1,t) = 0, \quad 0 \le t \le T, \tag{3}$$

where $d > 0,\ \delta$ are the given numbers $f(x,t),\ \varphi(x),\ \psi(x)$ are the given functions, $u(x,t)$ is the desired function. Under the classic solution of the problem $(1)-(3)$ we understand the function $u(x,t)$ continuous in a closed domain D_T together with all derivatives entering in to the equation (1) and satisfying all the conditions $(1)-(3)$ in the ordinary sense.

Let's consider a spectral problem [2]

$$y''(x) + \lambda y(x) = 0,\ 0 \le x \le 1, \tag{4}$$

$$y(0) = 0,\ y'(1) = d\lambda y(1),\ d > 0 \tag{5}$$

that has only ligen functions

$$y_k(x) = \sqrt{2}\sin\left(\sqrt{\lambda_k}x\right), k = 0, 1, 2, ...,$$

with positive eigen numbers λ_k from the equation $ctg\sqrt{\lambda} = d\sqrt{\lambda}$. We give a zero index to any eigen function and enumerate the other ones in increase order of eigen numbers.

The following statements are formulated and justified in the paper [2].

Lemma 1. *Beginning with some number N it holds the estimate*

$$0 < \sqrt{\lambda_k} - k\pi < (dk\pi)^{-1}.$$

Lemma 2. *The system $\{z_k(x)\}$ conjugated to the system*

$$\{y_k(x)\},\ k = 1, 2, ...,$$

is determined by the formula

$$z_k(x)=\sqrt{2}\left(\sin\left(\sqrt{\lambda_k}x\right)-\sin\sqrt{\lambda_k}\sin\left(\sqrt{\lambda_0}x\right)/\sin\sqrt{\lambda_0}\right)/\left(1+d\sin^2\sqrt{\lambda_n}\right).$$

Theorem 1. *The system $\{y_k(x)\}$, $k=1,2,...$ forms a basis in the space $L_p(0,1)$, $p>1$, for $p=2$ even the Riesz basis.*

In the paper [3], *it is known that*

$$\sqrt{\lambda_k}=k\pi+O\left(\frac{1}{k}\right),$$

$$y_k(x)=\sqrt{2}\sin(k\pi x)+O\left(\frac{1}{k}\right),$$

$$\sqrt{2}\cos\sqrt{\lambda_k}x=\sqrt{2}\cos(k\pi x)+O\left(\frac{1}{k}\right).$$

Assume $g(x)\in L_p(0,1)$. Then

$$\left(\sum_{k=1}^{\infty}\left(\sqrt{2}\int_0^1 g(x)\sin\sqrt{\lambda_k}x\,dx\right)^2\right)^{1/2}\leq$$

$$\leq\left(\frac{\sqrt{2}}{2}+\left(\sum_{k=1}^{\infty}0\left(k^{-2}\right)\right)^{1/2}\right)\|g(x)\|_{L_2(0,1)}, \tag{6}$$

$$\left(\sum_{k=1}^{\infty}\left(\sqrt{2}\int_0^1 g(x)\cos\sqrt{\lambda_k}x\,dx\right)^2\right)^{1/2}\leq$$

$$\leq\left(\frac{\sqrt{2}}{2}+\left(\sum_{k=1}^{\infty}0\left(k^{-2}\right)\right)^{1/2}\right)\|g(x)\|_{L_2(0,1)}. \tag{7}$$

Theorem 2. [4]. *Let the function $g(x)$ satisfy the conditions*

$$g(x)\in W_2^{2n}(0,1),\ g(0)=g''(0)=...=g^{(2n-2)}=0,\ n\geq 1,$$

$$J\equiv g(1)+\left(1/d\sin\sqrt{\lambda_0}\right)\int_0^1 g(x)\sin\sqrt{\lambda_0}x\,dx=0,$$

$$g''(1)+g'(1)/d=0,...,g^{(2n-2)}(1)+g^{(2n-3)}(1)/d=0,\ n\geq 2.$$

then the Fourier series

$$\sum_{k=1}^{\infty}g_k y_k(x)$$

converges in the metric $W_2^{2n}(0,1)$.

But if the function $g(x)$ is subjected to reguirements:

$$g(x) \in W_2^{(2n+1)}(0,1), \; g(0) = g''(0) = ... = g^{(2n)} = (0), \; n \geq 0, \; J = 0,$$

$$g''(0) + g'(1)/d = 0, ..., g^{(2n)}(1) + g^{(2n-1)}(1)/d = 0, \; n \geq 1,$$

then the Fourier series

$$\sum_{k=1}^{\infty} g_k y_k(x)$$

converges in the metric $W_2^{(2n+1)}(0,1)$ where

$$g_k = \int_0^1 g(x) z_k(x) dx.$$

Let $g(x) \in L_2(0,1)$, then

$$g_k = \frac{\sqrt{2}}{1 + d\sin^2\sqrt{\lambda_k}} \left(\int_0^1 g(x) \sin\sqrt{\lambda_k} x dx - \frac{\sin\sqrt{\lambda_k}}{\sin\sqrt{\lambda_0}} \int_0^1 g(x) \sin\sqrt{\lambda_0} x dx \right) =$$

$$= \frac{\sqrt{2}}{1 + d\sin^2\sqrt{\lambda_k}} \left(\int_0^1 g(x) \sin\sqrt{\lambda_k} x dx - \frac{\cos\sqrt{\lambda_k}}{d\sqrt{\lambda_k}\sin\sqrt{\lambda_0}} \int_0^1 g(x) \sin\sqrt{\lambda_0} x dx \right).$$

Hence, allowing for (6) we have:

$$\left(\sum_{k=1}^{\infty} g_k^2 \right)^{1/2} \leq M_0 \left\| g(x) \right\|_{L_2(0,1)}, \tag{8}$$

where

$$M_0 = \left(\frac{\sqrt{2}}{2} + \left(\sum_{k=1}^{\infty} 0\left(k^{-2}\right) \right)^{1/2} + \frac{1}{d\sqrt{6}\sin\sqrt{\lambda_0}} \right) \sqrt{2}.$$

Assume that $g'(x) \in L_2(0,1)$ and

$$g(0) = 0, \; J = 0. \tag{9}$$

It is known that $[3]$,

$$g_k = \frac{\sqrt{2}}{1 + d\sin^2\sqrt{\lambda_k}} \frac{1}{\sqrt{\lambda_k}} \int_0^1 g'(x) \cos\sqrt{\lambda_k} x dx. \tag{10}$$

Hence, allowing for (7) we find:

$$\left(\sum_{k=1}^{\infty} \left(\sqrt{\lambda_k} \left| g_k \right| \right)^2 \right)^{1/2} \leq M_1 \left\| g'(x) \right\|_{L_2(0,1)}, \tag{11}$$

where

$$M_1 = \frac{\sqrt{2}}{2} + \left(\sum_{k=1}^{\infty} 0 \left(k^{-2} \right) \right)^{1/2}.$$

We assume that for $g(x) \in C^{-1}[0,1]$, $g''(x) \in L_2(0,1)$ the conditions (9) are filfilled and obtain the equality

$$g_k = \frac{-\sqrt{2}}{1 + d \sin^2 \sqrt{\lambda_k}} \frac{1}{\lambda_k} \int_0^1 g''(x) \sin \sqrt{\lambda_k} x dx + \frac{\cos \sqrt{\lambda_k}}{d \sqrt{\lambda_k} \lambda_k} g'(1).$$

Hence, allowing for (6) we have:

$$\left(\sum_{k=1}^{\infty} \left(\lambda_k |g_k| \right)^2 \right)^{1/2} \le M_2 \left\| g''(x) \right\|_{L_2(0,1)} + \frac{1}{d\sqrt{6}} \left| g'(1) \right|. \tag{12}$$

Now, assume that $g(x) \in C^2[0,1]$, $g'''(x) \in L_2(0,1)$ and

$$g(0) = g''(0) = 0, \ J = 0, \ g''(1) + g'(1)/d = 0.$$

Then

$$g_k = -\frac{\sqrt{2}}{1 + d \sin^2 \sqrt{\lambda_k}} \frac{1}{\lambda_k \sqrt{\lambda_k}} \int_0^1 g'''(x) \cos \sqrt{\lambda_k} x dx. \tag{13}$$

Hence, allowing for (7) we find:

$$\left(\sum_{k-1}^{\infty} \left(\lambda_k^{3/2} |g_k| \right)^2 \right)^{1/2} \le M_1 \left\| g'''(x) \right\|_{L_2(0,1)}. \tag{14}$$

We'll look for the classic solution of the problem $(1) - (2)$ in the form

$$u(t,x) = \sum_{k-1}^{\infty} u_k(t) y_k(x), \tag{15}$$

where

$$u_k(t) = \int_0^1 u(x,t) z_k(x) dx \tag{16}$$

is a solution of the following problem:

$$(1 + \lambda_k) u_k''(t) + \lambda_k u_k(t) = f_k(t) \tag{17}$$

$$u_k(0) + \delta u_k(T) = \varphi_k, \quad u_k'(0) + \delta u_k'(T) = \psi_k \qquad (k = 1, 2, ...), \tag{18}$$

where

$$f_k(t) = \int_0^1 f(x,t) z_k(x) dx, \ \varphi_k = \int_0^1 \varphi(x) z_k(x) dx, \ \psi_k = \int_0^1 \psi(x) z_k(x) dx.$$

Solving the problem $(17)-(18)$ we find:

$$u_k(t)=\frac{1}{\beta_k\rho_k(T)}\{\beta_k(\cos\beta_k t+\delta\cos\beta_k(T-t))\varphi_k+$$

$$(\sin\beta_k t-\delta\sin\beta_k(T-t))\psi_k-\frac{\delta}{1+\lambda_k}\int_0^T f_k(\tau)(\sin\beta_k(T+t-\tau)+$$

$$+\delta\sin\beta_k(t-\tau))d\tau\}+\frac{1}{\beta_k(1+\lambda_k)}\int_0^t f_k(\tau)\sin\beta_k(t-\tau)d\tau \quad (k=1,2), \qquad (19)$$

where

$$\beta_k=\sqrt{\frac{\lambda_k}{1+\lambda_k}},\quad \rho_k(T)=1+2\delta\cos\beta_k T+\delta^2 \quad (k=1,2,...).$$

Obviously

$$u_k'(t)=\frac{1}{\rho_k(T)}\{\beta_k(-\sin\beta_k t+\delta\sin\beta_k(T-t))\varphi_k+(\cos\beta_k t-$$

$$-\delta\cos\beta_k(T-t))\psi_k-\frac{\delta}{1+\lambda_k}\int_0^T f_k(\tau)(\cos\beta_k(T+t-\tau)+\delta\cos\beta_k(t-\tau))d\tau\}+$$

$$+\frac{1}{1+\lambda_k}\int_0^t f_k(\tau)\,cos\beta_k(t-\tau)d\tau \quad (k=1,2), \qquad (20)$$

$$u_k''(t)=\frac{1}{1+\lambda_k}f_k(\tau)-\frac{\beta_k}{\rho_k(T)}\{\beta_k(\cos\beta_k t+\delta\cos\beta_k(T-t))\varphi_k+$$

$$+(\sin\beta_k t-\delta\sin\beta_k(T-t))\psi_k-\frac{\delta}{1+\lambda_k}\int_0^T f_k(\tau)(\sin\beta_k(T+t-\tau)+$$

$$+\delta\sin\beta_k(t-\tau))d\tau\}-\frac{\beta_k}{1+\lambda_k}\int_0^t f_k(\tau)\sin\beta_k(t-\tau)d\tau \quad (k=1,2...). \qquad (21)$$

The following theorem is true.

Theorem 3. *Let*

1. $\varphi(x)\in C^2[0,1],\ \varphi'''(x)\in L_2(0,1),\ \varphi(0)=\varphi''(0)=0,$
$\varphi(1)+\left(1/\left(d\sin\sqrt{\lambda_0}\right)\right)\int_0^1\varphi(x)\sin\sqrt{\lambda_0}x dx=0,\ \varphi''(1)+\varphi'(1)/d=0;$
2. $\psi(x)\in C^2[0,1],\ \psi'''(x)\in L_2(0,1),\ \psi(0)=\psi''(0)=0,$
$\psi(1)+\left(1/\left(d\sin\sqrt{\lambda_0}\right)\right)\int_0^1\psi(x)\sin\sqrt{\lambda_0}x dx=0,\ \psi''(1)+\psi'(1)/d=0;$
3. $f(x,t)\in C(D_T),\ f_x(x,t)\in L_2(D_T),\ f(0,t)=0,$
$f(1,t)+\left(1/\left(d\sin\sqrt{\lambda_0}\right)\right)\int_0^1 f(x,t)\sin\sqrt{\lambda_0}x dx=0,\ (t\in[0,T]).$

4. $\delta \neq \pm 1, \quad 1+2\delta\cos\beta_k T+\delta^2 \neq 0 \quad (k=1,2,...)$.

Then the function

$$u(x,t)=\sum_{k=1}^{\infty}\left\{\frac{1}{\beta_k\rho_k(T)}\left[\beta_k\left(\cos\beta_k t+\delta\cos\beta_k(T-t)\right)\varphi_k\right]+\right.$$

$$+\left(\sin\beta_k t-\delta\sin\beta_k(T-t)\right)\psi_k-$$

$$\left.-\frac{\delta}{1+\lambda_k}\int_0^T f_k(\tau)\left(\sin\beta_k(T+t-\tau)+\delta\sin\beta_k(t-\tau)\right)d\tau\right]+$$

$$\left.+\frac{1}{\beta_k(1+\lambda_k)}\int_0^t f_k(\tau)\sin\beta_k(t-\tau)\,d\tau\right\}y_k(x) \qquad (22)$$

is a classic solution of the problem $(1)-(3)$.

Proof. Obviously

$$1/\sqrt{2}<\beta_k<1, \quad |\rho_k(T)|\geq 1+2\delta^2-|\delta|\equiv 1/\rho.$$

Taking these into account, from (19), (20), (21) we have:

$$|u_k(t)|\leq\sqrt{2}\,(1+|\delta|)\left(|\varphi_k|+|\psi_k|\right)+$$

$$+\sqrt{2T}\,(1+\rho|\delta|(1+|\delta|))\frac{1}{\lambda_k}\left(\int_0^T|f_k(\tau)|^2d\tau\right)^{1/2},$$

$$\left|u_k'(t)\right|\leq\frac{1}{\lambda_k}|f_k(t)|+\rho(1+|\delta|)\left(|\varphi_k|+|\psi_k|\right)+$$

$$+(1+\rho|\delta|(1+|\delta|))\frac{\sqrt{T}}{\lambda_k}\left(\int_0^T|f_k(\tau)|^2d\tau\right)^{1/2}.$$

Hence we get

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}\|u_k(t)\|_{C[0,T]}\right)^2\right)^{1/2}\leq\sqrt{6}\rho(1+|\delta|)\times$$

$$\times\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}|\varphi_k|\right)^2\right)^{1/2}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}|\psi_k|\right)^2\right)^{1/2}\right]+$$

$$+\sqrt{6}\,(1+\rho|\delta|(1+|\delta|))\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^{1/2}|f_k(\tau)|\right)^2d\tau\right)^{1/2}, \qquad (23)$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}\|u_k'(t)\|_{C[0,T]}\right)^2\right)^{1/2}\leq\sqrt{3}\rho(1+|\delta|)\times$$

$$\times\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}\left|\varphi_k\right|\right)^2\right)^{1/2}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}\left|\psi_k\right|\right)^2\right)^{1/2}\right]+$$

$$\sqrt{3}\left(1+\rho\left|\delta\right|\left(1+\left|\delta\right|\right)\right)\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^{1/2}\left|f_k\left(\tau\right)\right|\right)^2d\tau\right)^{1/2}, \quad (24)$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}\left\|u_k''\left(t\right)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq\sqrt{2}\left(\sum_{k=1}^{\infty}\left(\lambda_k^{1/2}\left|f_k\left(t\right)\right|\right)^2\right)^{1/2}+2\rho\left(1+\left|\delta\right|\right)\times$$

$$\times\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}\left|\varphi_k\right|\right)^2\right)^{1/2}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}\left|\psi_k\right|\right)^2\right)^{1/2}\right]+$$

$$+2\left(1+\rho\left|\delta\right|\left(1+\left|\delta\right|\right)\right)\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^{1/2}\left|f_k\left(\tau\right)\right|\right)^2d\tau\right)^{1/2}. \quad (25)$$

Then from $(23)-(25)$ allowing for (11), (14) we find:

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}\left\|u_k\left(t\right)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq$$

$$\leq\sqrt{6}\rho\left(1+\left|\delta\right|\right)M_1\left(\left\|\varphi'''\left(x\right)\right\|_{L_2(0,1)}+\left\|\psi'''\left(x\right)\right\|_{L_2(0,1)}\right)+$$

$$+\sqrt{6T}\left(1+\rho\left|\delta\right|\left(1+\left|\delta\right|\right)\right)M_1\left\|f_x\left(x,t\right)\right\|_{L_2(D_T)}, \quad (26)$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}\left\|u_k'\left(t\right)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq$$

$$\leq\sqrt{3}\rho\left(1+\left|\delta\right|\right)M_1\left(\left\|\varphi'''\left(x\right)\right\|_{L_2(0,1)}+\left\|\psi'''\left(x\right)\right\|_{L_2(0,1)}\right)+$$

$$+\sqrt{3T}\left(1+\rho\left|\delta\right|\left(1+\left|\delta\right|\right)\right)M_1\left\|f_x\left(x,t\right)\right\|_{L_2(D_T)}, \quad (27)$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}\left\|u_k''\left(t\right)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq 2M_1\left\|\left\|f_x\left(x,t\right)\right\|_{L_2(0,1)}\right\|_{C[0,T]}+$$

$$+2\rho\left(1+\left|\delta\right|\right)M_1\left(\left\|\varphi'''\left(x\right)\right\|_{L_2(0,1)}+\left\|\psi'''\left(x\right)\right\|_{L_2(0,1)}\right)+$$

$$+2\left(1+\rho\left|\delta\right|\left(1+\left|\delta\right|\right)\right)\sqrt{T}M_1\left\|f_x\left(x,t\right)\right\|_{L_2(D_T)}. \quad (28)$$

Obviously

$$\left|u\left(x,t\right)\right|\leq\left(\sum_{k=1}^{\infty}\lambda_k^{-3}\right)^{1/2}\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}\left\|u_k\left(t\right)\right\|_{C[0,T]}\right)^2\right)^{1/2}, \quad (29)$$

$$\left|u_t\left(x,t\right)\right|\leq\left(\sum_{k=1}^{\infty}\lambda_k^{-3}\right)^{1/2}\left(\sum_{k=1}^{\infty}\left(\lambda_k^{3/2}\left\|u_k'\left(t\right)\right\|_{C[0,T]}\right)^2\right)^{1/2}, \quad (30)$$

$$|u_{xx}(x,t)| \le \left(\sum_{k=1}^{\infty} \lambda_k^{-1}\right)^{1/2} \left(\sum_{k=1}^{\infty} \left(\lambda_k^{3/2} \|u_k(t)\|_{C[0,T]}\right)^2\right)^{1/2}, \quad (31)$$

$$|u_{tt}(x,t)| \le \left(\sum_{k=1}^{\infty} \lambda_k^{-3}\right)^{1/2} \left(\sum_{k=1}^{\infty} \left(\lambda_k^{3/2} \|u''_k(t)\|_{C[0,T]}\right)^2\right)^{1/2}, \quad (32)$$

$$|u_{ttxx}(x,t)| \le \left(\sum_{k=1}^{\infty} \lambda_k^{-1}\right)^{1/2} \left(\sum_{k=1}^{\infty} \left(\lambda_k^{3/2} \|u''_k(t)\|_{C[0,T]}\right)^2\right)^{1/2}. \quad (33)$$

From $(29)-(33)$ allowing for $(26)-(28)$ it follows that the functions $u(x,t)$, $u_t(x,t)$, $u_{xx}(x,t)$, $u_{tt}(x,t)$, $u_{ttxx}(x,t)$ are continuous in D_T. We can see by direct verification that the functions $u(x,t)$ satisfy the equation and conditions (2), (3) in the ordinary sense. The theorem is proved.

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