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EXISTENCE AND UNIQUENESS OF THE SOLUTION OF AN OPTIMAL CONTROL PROBLEM FOR A SCHRODINGER EQUATION WITH PURE IMAGINARY COEFFICIENT IN THE NON-LINEAR PART OF THIS EQUATION

Abstract

In the present paper we consider an optimal control problem for a Schrodinger non-linear equation with pare imaginary coefficient in the non-linear part.

Optimal control problems for a Schrodinger nonlinear equation often arise in quantum mechanics, nucleur physics, non-linear optics, high-conductivity theory and in other fields of up-to-date physics and engineering.

In the present paper we consider an optimal control problem for a Schrodinger non-linear equation with pare imaginary coefficient in the non-linear part. It should be noted that such problems for a Schrodinger non-linear equation in other statements were investigated in the papers [1, 2] and others.

Let $l > 0$, $T > 0$ be the given numbers,

$$x \in (0,l)\,, t \in (0,T)\,, \Omega_t = (0,l) \times (0,t)\,, \Omega = \Omega_t\ .$$

It is required to minimize the functional

$$J_\alpha(v) = \int_\Omega |\psi_1(x,t) - \psi_2(x,t)|^2\, dxdt + \alpha \|v - \omega\|_H^2 \tag{1}$$

on the set $V \equiv \left\{ v = v(x) : v \in W_2^1(0,l)\,,\ \|v\|_{W_2^1(0,l)} \le b \right\}$ under conditions:

$$i\frac{\partial \psi_k}{\partial t} + a_0 \frac{\partial^2 \psi_k}{\partial x^2} - a(x)\,\psi_k - v(x)\,\psi_k + i a_1 |\psi_k|^2 \psi_k = f_k(x,t)\,, \quad (x,t) \in \Omega, \tag{2}$$

$$\psi_k(x,0) = \varphi_k(x)\,, \quad k = 1,2, \quad x \in (0,l) \tag{3}$$

$$\psi_1(0,t) = \psi_1(l,t) = 0, \quad t \in (0,T)\,, \tag{4}$$

$$\frac{\partial \psi_2(0,t)}{\partial x} = \frac{\partial \psi_2(l,t)}{\partial x} = 0, \quad t \in (0,T)\,, \tag{5}$$

where $i^2 = -1$, $a_0 > 0$, $a_1 > 0$, $b > 0$ are the given numbers, $a = a(x)$ is a bounded, measurable function satisfying the condition

$$\begin{gathered} 0 < \mu_0 \le a(x) \le \mu_1\ , \quad \left| \frac{da(x)}{dx} \right| \le \mu_2\ , \\ \forall x \in (0,l)\,, \quad \mu_0, \mu_1, \mu_2 = const > 0, \end{gathered} \tag{6}$$

and the functions $\varphi_k(x)$, $f_k(x,t)$, $k=1,2$ satisfy the conditions:

$$\varphi_1 \in \overset{\circ}{W}{}_2^2(0,l), \quad \varphi_2 \in W_2^2(0,l), \quad \frac{d\varphi_2(0)}{dx} = \frac{d\varphi_2(l)}{dx} = 0, \tag{7}$$

$$f_1 \in \overset{\circ}{W}{}_2^{1,1}(\Omega), \quad f_2 \in W_2^{1,1}(\Omega). \tag{8}$$

A problem on determination of the functions $\psi_k = \psi_k(x,t)$, $k=1,2$, from the conditions $(2)-(5)$ for the given $v \in V$ is said to be a reduced problem. Under the solution of this problem we'll understand the functions $\psi_k = \psi_k(x,t)$, $k=1,2$, belonging to

$$B_1 \equiv C^0\left([0,T], \overset{\circ}{W}{}_2^2(0,l)\right) \cap C^1([0,T], L_2(0,l))$$

and

$$B_2 \equiv C^0\left([0,T], W_2^2(0,l)\right) \cap C^1([0,T], L_2(0,l))$$

respectively and satisfying the conditions $(2)-(5)$ for almost all $x \in (0,l)$ and $\forall t \in [0,T]$. The reduced problem consists of two boundary value problems, i.e. first and second boundary value problems for a Schrodinger equation. It should be noted that boundary value problems for the equation (2) was earlier investigated in the papers $[1-3]$ and others. However, these results here as well are not sufficient for our goal. In the indicated papers a more wide class of admissible controls is a set from $W_\infty^1(0,l)$, but in our case a class of admissible controls is a set from a Hilbert space $W_2^1(0,l)$, that is wider than $W_\infty^1(0,l)$. Therefore, there again arises a necessity to study the correctness problem of the statement of boundary value problem $(2)-(5)$ with a coefficient from the set $V \subset W_2^1(0,l)$.

Using the Galerkin method and the proof method of the paper $[1,2,4,5]$ we can prove the validity of the statement:

Theorem 1. *Let* $a(x), \varphi_k(x), f_k(x,t), k=1,2$ *satisfy the conditions* $(6)-(8)$. *Then, reduced problem* $(2)-(5)$ *has a unique solution for each* $v \in V$, *has a unique solution* $\psi_1 \in B_1$ *and* $\psi_2 \in B_2$ *and the estimates:*

$$\|\psi_1(\cdot,t)\|_{\overset{\circ}{W}{}_2^2(0,l)} + \left\|\frac{\partial \psi_1(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} \le$$

$$\le M_1\left(\|\varphi_1\|_{\overset{\circ}{W}{}_2^2(0,l)} + \|f_1\|_{\overset{\circ}{W}{}_2^{1,1}(\Omega)} + \|\varphi_1\|^3_{\overset{\circ}{W}{}_2^1(0,l)} + \|f_1\|^3_{\overset{\circ}{W}{}_2^{1,0}(\Omega)}\right) \tag{9}$$

$$\|\psi_2(\cdot,t)\|_{W_2^2(0,l)} + \left\|\frac{\partial \psi_2(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} \le$$

$$\le M_2\left(\|\varphi_2\|_{W_2^2(0,l)} + \|f_2\|_{W_2^{1,1}(\Omega)} + \|\varphi_2\|^3_{W_2^1(0,l)} + \|f_2\|^3_{W_2^{1,0}(\Omega)}\right) \tag{10}$$

are valid for $\forall t \in [0,T]$ *where* M_1 *and* M_2 *are positive constants.*

In this paper we'll study correctness of the statement of the optimal control problem $(1)-(5)$. At first we show that for $\alpha > 0$ the considered oprimal contol problem has a unique solution.

Theorem 2. *Let the conditions of theorem 1 be fulfilled and $\omega \in W_2^1(0,l)$ be a given element. Then there exists such everywhere dense subset G of the space $W_2^1(0,l)$ that for any $\omega \in G$ for $\alpha > 0$ the optimal control problem $(1)-(5)$ has a unique solution.*

Proof. At first we prove continuity of the functional:

$$J_0(v) = \|\psi_1 - \psi_2\|^2_{L_2(\Omega)} \tag{11}$$

on the set V.

Let $\delta v \in W_2^1(0,l)$ be an increment of any element $v \in V$ such that $v + \delta v \in V$. Then $\psi_k = \psi_k(x,t) \equiv \psi_k(x,t;v)$, $k = 1,2$ - solution of the reduced problem $(2)-(5)$ for $v \in V$ gets an increment $\delta\psi_k = \delta\psi_k(x,t) \equiv \psi_k(x,t;v+\delta v)$, where $\psi_{k\delta} = \psi_{k\delta}(x,t) \equiv \psi_k(x,t;v+\delta v)$ is a solution of the reduced problem $(2)-(5)$ for $v + \delta v \in V$. It follows from the conditions $(2)-(5)$ that the functions $\Delta\psi_k = \Delta\psi_k(x,t)$, $k = 1,2$ are the solutions of the following boundary value problem:

$$i\frac{\partial\delta\psi_k}{\partial t} + a_0\frac{\partial^2\delta\psi_k}{\partial x^2} - a(x)\delta\psi_k - (v+\delta v)\delta\psi_k + ia_1\left(|\psi_{k\delta}|^2 + |\psi_k|^2\right)\delta\psi_k +$$

$$+ia_1\psi_{k\delta}\psi_k\delta\overline{\psi}_k = \delta v\psi_k(x,t;v), \quad (x,t) \in \Omega, \tag{12}$$

$$\delta\psi_k(x,0) = 0, \quad x \in (0,l), \quad k = 1,2, \tag{13}$$

$$\delta\psi_1(0,t) = \delta\psi_1(l,t) = 0, \quad t \in (0,T), \tag{14}$$

$$\frac{\partial\delta\psi_2(0,t)}{\partial x} = \frac{\partial\delta\psi_2(l,t)}{\partial x} = 0, \quad t \in (0,T) \tag{15}$$

where $\psi_k = \psi_k(x,t) \equiv \psi_k(x,t;v)$, $k = 1,2$ is a solution of the reduced problem $(2)-(5)$ for $v \in V$.

Now, let's estimate the solution of this boundary value problem. To this end we multiply the both parts of equations (12) by the function $\delta\overline{\psi}_k(x,t)$ and integrate the obtained relation in the domain Ω_t. As a result we have:

$$\int\limits_{\Omega_t}\left[i\frac{\partial\delta\psi_k}{\partial\tau}\delta\psi_k - a_0\left|\frac{\partial\delta\psi_k}{\partial x}\right|^2 - a(x)|\delta\psi_k|^2 - \right.$$

$$-(v+\delta v)|\delta\psi_k|^2 + ia_1\left(|\psi_{k\delta}|^2 + |\psi_k|^2\right)|\delta\psi_k|^2 +$$

$$\left.+ia_1\psi_{k\delta}\psi_k(\delta\psi_k)^2\right]dxd\tau = \int\limits_{\Omega_t}\delta v(x)\psi_k(x,\tau)\delta\psi_k(x,\tau)dxd\tau, \quad k = 1,2.$$

We the subtract from this equality its complex conjugation after simple transformations we get validity of the inequality:

$$\|\delta\psi_k(\cdot,t)\|^2_{L_2(0,l)} \leq 3a_1\int\limits_{\Omega_t}\left(|\psi_{k\delta}|^2 + |\psi_k|^2\right)|\delta\psi_k|^2 dxd\tau +$$

$$+2\int\limits_{\Omega_t}|\delta v(x)\psi(x,\tau)||\delta\psi_k(x,\tau)|\,dxd\tau,\quad k=1,2,\quad \forall t\in[0,T] \tag{16}$$

Using the estimates (9) and (10) we can establish the validity of the inequalities:

$$\|\psi_k\|_{L_\infty(\Omega)}\le M_3\ ,\quad \|\psi_{k\delta}\|_{L_\infty(\Omega)}\le M_3\ ,\quad k=1,2, \tag{17}$$

where $M_3>0$ is a constant. Considering these inequalities in (16), hence we get

$$\|\delta\psi_k(\cdot,t)\|^2_{L_2(0,l)}\le M_4\|\delta v\|^2_{L_2(0,l)}+M_5\int\limits_0^t\|\delta\psi_k(\cdot,\tau)\|^2_{L_2(0,l)}d\tau,\ \forall t\in[0,T].$$

Hence by the Gronwall lemma we get the validity of the estimate:

$$\|\delta\psi_k(\cdot,t)\|^2_{L_2(0,l)}\le M_6\|\delta v\|^2_{L_2(0,l)}\ ,\quad \forall t\in[0,T]. \tag{18}$$

Now, let's consider an increment of the functional $J_0(v)$ on the element $v\in V$. Obviously

$$\delta J_0(v)=J_0(v+\delta v)-J_0(v)=$$

$$=2\int\limits_{\Omega}\mathrm{Re}\left[(\psi_1(x,t)-\psi_2(x,t))\left(\delta\overline{\psi}_1(x,t)-\delta\overline{\psi}_2(x,t)\right)\right]dxdt+$$

$$+\|\delta\psi_1\|^2_{L_2(0,l)}+\|\delta\psi_2\|^2_{L_2(0,l)}-2\int\limits_{\Omega}\mathrm{Re}\left[\left(\delta\psi_1\delta\overline{\psi}_2\right)\right]dxdt \tag{19}$$

Hence by means of the Cauchy-Bunyakovskii inequality and estimates $(9),(10)$ and (18) we get the validity of the inequality:

$$|\delta J_0(v)|\le M_7\left(\|\delta v\|^2_{L_2(0,l)}+\|\delta v\|_{L_2(0,l)}\right). \tag{20}$$

Hence, it follows continuity of the functional $J_0(v)$ on any element $v\in V$, i.e. on the set V. By the structure of the set V it is a closed, bounded and convex set in $W_2^1(0,l)$ and the space $W_2^1(0,l)$ is a uniform convex space $[6]$, or $W_2^1(0,l)$ is a Hilbert space. Then by the known theorem [7] and by the lower boundedness and continuity on V of the functional $J_0(v)$ there exists everywhere dense sub-set G of the space $W_2^1(0,l)$ such that for $\forall\omega\in G$ at $\alpha>0$ the problem $(1)-(5)$ has a unique solution. Theorem 2 is proved.

Now, show that for $\alpha\ge 0$ and $\forall\omega\in W_2^1(0,l)$ the optimal control problem $(1)-(5)$ has even if one solution.

Theorem 3. *Let the conditions of theorem 2 be fulfilled and $\alpha\ge 0$ be a given number. Then the optimal control problem $(1)-(5)$ has even if one solution.*

Proof. Take a minimizing sequence $\{v^m\}\subset V$ for the functional $J_\alpha(v)$:

$$\lim_{m\to\infty}J_\alpha(v^m)=\inf_{v\in V}J_\alpha(v)=J_{\alpha^*}\ .$$

Assume $\psi_{km}(x,t) \equiv \psi_k(x,t\,;v^m)$, $k=1,2,\ m=1,2,..$. Since $\{v^m\}\subset V$, the reduced problem $(2)-(5)$ under the conditions of theorem 1 for each $m=1,2,...$ has a unique solution: $\psi_{1m}\in B_1$ and $\psi_{2m}\in B_2$ and the estimations:

$$\|\psi_{1m}(\cdot,t)\|_{\overset{\circ}{W}{}_2^2(0,l)} + \left\|\frac{\partial\psi_{1m}(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} \le M_8 \ , \tag{21}$$

$$\|\psi_{2m}(\cdot,t)\|_{W_2^2(0,l)} + \left\|\frac{\partial\psi_{2m}(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} \le M_9 \ , \tag{22}$$

$m=1,2,...$ are valid for $\forall t\in[0,T]$, where $M_8>0$ and $M_9>0$ are the right hand sides of estimates (9) and (10), respectively. These constants are independent of t and m.

Since V is a closed, bounded and convex set of the reflexive Banach space $W_2^1(0,l)$, this set is weakly compact in $W_2^1(0,l)$. Therefore, from the sequence $\{v^m\}$ we can extract a sub-sequence that for the simplicity of the statement again denote by $\{v^m\}$, that

$$v^m \to v \text{ weakly in } W_2^1(0,l) \tag{23}$$

as $m\to\infty$. As V is a closed and convex set, this set is weakly closed. Therefore $v\in V$. Besides, $W_2^1(0,l)$ is compactly embedded into $L_\infty(\Omega)$. Then for the sequence $\{v^m\}\subset V$ it is valid the limiting relation:

$$v^m \to v \ \text{ strongly in } \ L_\infty(0,l) \tag{24}$$

as $m\to\infty$.

It follows from the relations (21) and (22) that the sequences $\psi_{km}(x,t)$, $k=1,2$ are uniformly bounded in the norm of the spaces B_1 and B_2 , respectively. Then from these sequences we can extract such sub-sequences that for the simplicity of the statement again will be denoted by $\psi_{km}(x,t)$, $k=1,2$ and weakly in B_1 and B_2 converge to the functions $\psi_k(x,t)$, $k=1,2$, respectively. In other words, the sub-sequences:

$$\{\psi_{km}\},\ \left\{\frac{\partial\psi_{km}}{\partial x}\right\},\left\{\frac{\partial\psi_{km}}{\partial t}\right\},\left\{\frac{\partial^2\psi_{km}}{\partial x^2}\right\},\ k=1,2 \ \text{ as } m\to\infty.$$

weakly in $L_2(0,l)$ converge to the functions ψ_k , $\dfrac{\partial\psi_k}{\partial x}$, $\dfrac{\partial\psi_k}{\partial t}$, $\dfrac{\partial^2\psi_k}{\partial x^2}$, $k=1,2$ respectively, for each $t\in[0,T]$.

Verify that the limiting functions $\psi_k(x,t)$, $k=1,2$ satisfy the equations (2) for almost all $x\in(0,l)$ and for each $t\in[0,T]$. To this end we consider the following integral identities:

$$\int_0^1 \left[i\frac{\partial\psi_{km}(x,t)}{\partial t} + a_0\frac{\partial^2\psi_{km}(x,t)}{\partial x^2} - a(x)\,\psi_{km}(x,t) - v^m(x)\,\psi_{km}(x,t) + \right.$$

$$+ia_1\left|\psi_{km}(x,t)\right|^2\psi_{km}(x,t)-f_k(x,t)\Big]\,\overline{g}_k(x)\,dx=0,\quad k=1,2,\quad t\in[0,T] \qquad (25)$$

for any functions $g_k\in L_2(0,l)$, $k=1,2$.

Using the strong convergence of the sequence $\{v^m\}$ to v in $L_\infty(0,l)$, i.e. limiting relation (24) and weak convergence of sequences $\{\psi_{km}(x,t)\}$, $k=1,2$ to the functions $\psi_k(x,t)$, $k=1,2$ in $L_2(0,l)$ for each $t\in[0,T]$ we can establish the validity of limiting relations

$$\lim_{m\to\infty}\int_0^l v^m\psi_{km}(x,t)\,\overline{g}_k(x)\,dx=$$

$$=\int_0^l v(x)\,\psi_k(x,t)\,\overline{g}_k(x)\,dx,\quad k=1,2,\quad t\in[0,T] \qquad (26)$$

for any functions $g_k\in L_2(0,l)$, $k=1,2$.

Now, let's prove the validity of the following limiting relations:

$$\lim_{m\to\infty}\int_0^l ia_1\left|\psi_{km}(x,t)\right|^2\psi_{km}(x,t)\,\overline{g}_k(x)\,dx=$$

$$=\int_0^l ia_1\left|\psi_k(x,t)\right|^2\psi_k(x,t)\,\overline{g}_k(x)\,dx,\quad k=1,2 \qquad (27)$$

for $t\in[0,T]$ and for any functions $g_k\in L_2(0,l)$, $k=1,2$.

By the embedding theorem the spaces B_1 and B_2 are compactly embedded into the space $C^0([0,T],\ L_2(0,l))$. Therefore, the sequence $\{\psi_{km}(x,t)\}$, $k=1,2$ weakly converging in B_1 and B_2 , respectively, to the functions $\psi_k(x,t)$, $k=1,2$ will strongly converge to the space $C^0([0,T],\ L_2(0,l))$ i.e. the following limiting relations hold

$$\left\|\psi_{km}(\cdot,t)-\psi_m(\cdot,t)\right\|_{L_2(0,l)}\to 0\ \text{ as }\ m\to\infty \qquad (28)$$

uniformly with respect to $t\in[0,T]$. Then, it is clear that these sequences $\{\psi_{km}(x,t)\}$ uniformly with respect to $t\in[0,T]$ almost everywhere converge in $(0,l)$ to the functions $\psi_k(x,t)$, $k=1,2$ as $m\to\infty$. Besides, using the estimates $(21),(22)$ we can establish the validity of the inequalities:

$$\left\|\left|\psi_{km}(\cdot,t)\right|^2\psi_{km}(\cdot,t)\right\|_{L_2(0,l)}\le M_{10}\ ,\quad k=1,2,\quad m=1,2,... \qquad (29)$$

for $\forall t\in[0,T]$. Hence and by the known lemma [5, pp. 530-531] we get the validity of the limiting relations (27).

Thus, using the limiting relations $(26),(27)$ and also a weak convergence of sequences $\{\psi_{km}(x,t)\}$, $k=1,2$ in the spaces B_1 and B_2 , if we pass to limit in (25) as $m\to\infty$, we get the limiting functions $\psi_k=\psi_k(x,t)$, $k=1,2$ satisfy the conditions (2) for almost all $x\in(0,l)$ and for each $t\in[0,T]$.

Fulfilment of initial and boundary conditions is proved in a similar way as in the paper $[1,4]$.

Thus, we proved that the limiting functions $\psi_k = \psi_k(x,t)$, $k = 1,2$ are the solutions of the reduced problem $(2)-(5)$ from B_1, B_2 corresponding to the limiting function $v = v(x)$ from V. The estimates (9) and (10) that directly follow from (21) and (22) with passage to limit as $m \to \infty$, are valid for these solutions.

As the spaces B_1 and B_2 are compactly embedded in $L_2(\Omega)$ the limiting relations:

$$\psi_{km} \to \psi_k\ ,\ k = 1,2 \text{ strongly in } L_2(\Omega) \tag{30}$$

as $m \to \infty$ (neverthermore weakly), hold. Then, using the weak lower continuity of the norms of the spaces $L_2(\Omega)$ and $W_2^1(0,l)$ and the condition that $\alpha \geq 0$ from the form of the functional $J_\alpha(v)$ we establish weak lower semi continuity on the element $v \in V$.

Therefore

$$J_{\alpha^*} \leq J_\alpha(v) \leq \underline{\lim}_{m\to 0} J_\alpha(v^m) = J_{\alpha^*}\ .$$

Hence, it follows that $J_{\alpha^*} = J_\alpha(v)$, i.e. $v \in V$ is a solution of the optimal control problem $(1)-(5)$. Theorem 3 is proved.

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