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ON SOLVABILITY OF SECOND ORDER OPERATOR-DIFFERENTIAL EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS

Abstract

In the present paper we obtain sufficient conditions providing solvability of a periodic boundary value problem for a class of second order operator-differential equations. These conditions are expressed only by the properties of coefficients of the given operator-differential equation. Notice that when obtaining main results the estimations of the norms of intermediate derivatives operators are also given.

In a separable Hilbert space H we consider the following periodic boundary value problem

$$P\left(d/dt\right)u=-\left(\frac{d}{dt}-\omega_1 A\right)\left(\frac{d}{dt}-\omega_2 A\right)u+\sum_{j=0}^{2}A_{2-j}\frac{d^j u}{dt^j}=f\left(t\right),\quad t\in\left(0,1\right)\quad (1)$$

$$u\left(0\right)=u\left(1\right),\quad u'\left(0\right)=u'\left(1\right).\qquad (2)$$

The following conditions are assumed to be fulfilled for operator coefficients:

1) A is a positive-definite self-adjoint operator;

2) ω_1, ω_2 are real numbers, $\omega_1<0$, $\omega_2>0$;

3) the operators $B_j=A_jA^{-j}$ $(j=0,1,2)$ are bounded in H.

The right-hand side of equation (1), i.e. $f\left(t\right)$ belongs to the space $L_2\left(\left(0,1\right);H\right)$, where the Hilbert space $L_2\left(\left(0,1\right);H\right)$ is defined as follows:

$$L_2\left(\left(0,1\right);H\right)=\left\{f:\int_0^1\left\|f\left(t\right)\right\|_H^2dt=\left\|f\right\|^2_{L_2\left(\left(0,1\right);H\right)}\right\};$$

and $u\left(t\right)\in W_2^2\left(\left(0,1\right);H\right)$, where the Hilbert space

$$W_2^2\left(\left(0,1\right);H\right)=\left\{u:u''\in L_2\left(\left(0,1\right);H\right),\quad A^2u\in L_2\left(\left(0,1\right);H\right)\right\}$$

with the norm

$$\left\|u\right\|_{W_2^2\left(\left(0,1\right);H\right)}=\left(\left\|u''\right\|^2_{L_2\left(\left(0,1\right);H\right)}+\left\|A^2u\right\|^2_{L_2\left(\left(0,1\right);H\right)}\right)^{1/2}.$$

Definition 1. If the vector-function $u\left(t\right)\in W_2^2\left(\left(0,1\right);H\right)$ satisfies equation (1) almost everywhere on $\left(0,1\right)$ it will be said to be a regular solution of equation (1).

Definition 2. Let a regular solution of equation (1) satisfy the boundary values in the sense

$$\lim_{\substack{t\to+0\\0<t<1}}u\left(t\right)=\lim_{\substack{t\to1-0\\0<t<1}}u\left(t\right),\quad\lim_{\substack{t\to+0\\0<t<1}}u'\left(t\right)=\lim_{\substack{t\to1-0\\0<t<1}}u'\left(t\right),$$

where, in the first equality the limits are understood in the sense of the norm of the space $H_{3/2}$, in the second one $-$ in the sense of the norm of the space $H_{1/2}$, respectively.

And it holds the inequality

$$\|u\|_{W_2^2((0,1);H)} \leq const \, \|f\|_{L_2((0,1);H)} .$$

Then problem (1), (2) is said to be regular solvable.

Notice that problem (1), (2) for $\omega_1 = -1$, $\omega_1 = 1$ and A_j $(j = 0, 1, 2)$ are scalar coefficients, was considered in particular in the paper [1], for $\omega_1 = -1$, $\omega_2 = 1$, $A_j = 0$, $j = 0, 1, 2$ in the book [2].

Denote

$$W_2^2\left((0,1);H;\Pi\right) = \left\{u : u \in W_2^2\left((0,1);H\right),\ u(0) = u(1), u'(0) = u'(1)\right\}$$

and in this space consider the operators

$$P_0 u = -\left(\frac{d}{dt} - \omega_1 A\right)\left(\frac{d}{dt} - \omega_2 A\right) u, \qquad u \in W_2^2\left((0,1);H;\Pi\right)$$

$$P_1 u = \sum_{j=0}^{2} A_{2-j} u(j), \qquad u \in W_2^2\left((0,1);H;\Pi\right).$$

It holds

Lemma 1. *Let conditions 1) and 2) be fulfilled. Then the operator* $P_0 : W_2^2\left((0,1);H;\Pi\right) \to L_2\left((0,1);H\right)$ *is an isomorphism.*

Proof. Show that the equation $P_0 u = 0$ has only zero solution. General solution of the equation $P_0\left(d/dt\right) u(t) = 0$ from $W_2^2\left((0,1);H;\Pi\right)$ is of the form $u_0(t) = e^{\omega_1 t A}\varphi_1 + e^{\omega_2 (t-1) A}\varphi_2$, where φ_1, $\varphi_2 \in H_{3/2}$. It follows from the condition $u_0 \in W_2^2\left((0,1);H;\Pi\right)$ that

$$\varphi_1 + e^{-\omega_2 A}\varphi_2 = e^{\omega_1 A}\varphi_1 + \varphi_2,$$
$$\omega_1 A\varphi_1 + \omega_2 A e^{-\omega_2 A}\varphi_2 = \omega_1 A e^{\omega_1 A}\varphi_1 + \omega_2 A\varphi_2,$$

i.e.

$$\left(E - e^{\omega_1 A}\right)\varphi_1 - \left(E - e^{-\omega_2 A}\right)\varphi_2 = 0,$$
$$\omega_1\left(E - e^{\omega_1 A}\right)\varphi_1 - \omega_2\left(E - e^{-\omega_2 A}\right)\varphi_2 = 0.$$

Hence, it follows that $\left(E - e^{\omega_1 A}\right)\varphi_1 = 0$, $\left(E - e^{-\omega_2 A}\right)\varphi_2 = 0$. Consequently, from the invertibility of the operators $E - e^{\omega_1 A}$ and $E - e^{-\omega_2 A}$ it follows that $\varphi_1 = \varphi_2 = 0$, i.e. $u_0(t) = 0$. Now, let's show that the domain of definition of the operator P_0 coincides with $L_2\left((0,1);H\right)$, i.e. for any $f \in L_2\left((0,1);H\right)$ the equation $P_0 u = f$ has a regular solution $u \in W_2^2\left((0,1);H;\Pi\right)$.

It is easily seen that the vector-function

$$u_1(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} P_0(-i\xi)\left(\int_0^1 f(s)\, e^{i\xi(s-t)} ds\right) d\xi, \quad t \in R = (-\infty, \infty),$$

satisfies the equation $P_0\left(d/dt\right) u(t) = f(t)$ in $(0;1)$ almost everywhere and $u_1(t) \in W_2^2\left((-\infty,\infty);H;\Pi\right)$. Contraction of $u_1(t)$ on $(0;1)$ we denote by $\psi(t)$. Then

$\psi(t) \in W_2^2((0,1);H)$ and $\psi(0)$, $\psi(1) \in H_{3/2}$, $\psi'(0)$, $\psi'(1) \in H_{1/2}$ (see [3, p.29]). Then a general solution of the equation $P_0(d/dt)\,u(t) = f(t)$ is of the form:

$$u(t) = \psi(t) + e^{\omega_1 tA}\varphi_1 + e^{\omega_2(t-1)A}\varphi_2,$$

where $\varphi_1, \varphi_2 \in H_{3/2}$. Choose φ_1 and φ_2 so that $u(t) \in W_2^2((0,1);H;\Pi)$. To this end, for φ_1, φ_2 we get the system of equations

$$\begin{cases} \psi(0) + \varphi_1 + e^{-\omega_2 A}\varphi_2 = \psi(1) + e^{\omega_1 A}\varphi_1 + \varphi_2, \\ \psi'(0) + \omega_1 A\varphi_1 + \omega_2 Ae^{-\omega_2 A}\varphi_2 = \psi'(1) + \omega_1 Ae^{\omega_1 A} + \omega_2 A\varphi_2, \end{cases}$$

or

$$\left(E - e^{\omega_1 A}\right)\varphi_1 - \left(E - e^{-\omega_2 A}\right)\varphi_2 = \psi(1) - \psi(0),$$

$$\omega_1\left(E - e^{\omega_1 A}\right)\varphi_1 - \omega_2\left(E - e^{-\omega_2 A}\right)\varphi_2 = A^{-1}\left(\psi'(1) - \psi'(0)\right).$$

Let $x_1 = \left(E - e^{\omega_1 A}\right)\varphi_1$, $x_2 = \left(E - e^{-\omega_2 A}\right)\varphi_2$, then

$$\begin{cases} x_1 - x_2 = \psi(1) - \psi(0) \in H_{3/2}, \\ \omega_1 x_1 - \omega_2 x_2 = A^{-1}\left(\psi_1'(1) - \psi'(0)\right) \in H_{3/2}. \end{cases}$$

Hence, we have:

$$x_1 = \frac{1}{\omega_2 - \omega_1}\left(\omega_2\left(\psi(1) - \psi(0)\right) - A^{-1}\left(\psi_1'(1) - \psi'(0)\right)\right) \in H_{3/2},$$

$$x_2 = \frac{1}{\omega_2 - \omega_1}\left(-A^{-1}\left(\psi_1'(1) - \psi'(0)\right) + \omega_1\left(\psi(1) - \psi(0)\right)\right) \in H_{3/2}.$$

Then

$$\varphi_1 = \left(E - e^{\omega_1 A}\right)^{-1} x_1 \in H_{3/2}, \quad \varphi_2 = \left(E - e^{-\omega_2 A}\right)^{-1} x_2 \in H_{3/2}.$$

Thus, $u(t) \in W_2^2((0,1);H;\Pi)$. From the theorem on intermediate derivatives (see [3], p.23) it follows that

$$\|P_0 u\|_{L_2((0,1);H)} \le const\,\|u\|_{W_2^2((0,1);H)}.$$

Then by the Banach theorem on the inverse operator P_0 has a bounded inverse. The lemma is proved.\From this lemma and the theorem on intermediate derivatives it follows that the following numbers are finite:

$$N_0 = \sup_{0 \ne u \in W_2^2((0,1);H;\Pi)} \left\|A^2 u\right\|_{L_2} \|P_0 u\|_{L_2}^{-1}, \tag{3}$$

$$N_1 = \sup_{0 \ne u \in W_2^2((0,1);H;\Pi)} \left\|Au'\right\|_{L_2} \|P_0 u\|_{L_2}^{-1}, \tag{4}$$

$$N_2 = \sup_{0 \ne u \in W_2^2((0,1);H;\Pi)} \left\|u''\right\|_{L_2} \|P_0 u\|_{L_2}^{-1}. \tag{5}$$

Estimate these numbers.

Lemma 2. *Let conditions 1) and 2) be fulfilled. Then the estimates*

$$N_0 \le c_0 = \frac{1}{|\omega_1 \omega_2|}, \tag{6}$$

$$N_1 \leq c_1 = \frac{1}{\omega_2 + |\omega_1|}, \tag{7}$$

$$N_2 \leq c_2 = 1 \tag{8}$$

hold.

Proof. Let $u \in W_2^2\left((0,1);H;\Pi\right)$. Then $u(0) = u(1)$, $u'(0) = u'(1)$. Obviously

$$\|P_0 u\|^2_{L_2((0,1);H)} = \|-\left(d/dt - \omega_1 A\right)\left(d/dt - \omega_2 A\right)\|_{L_2((0,1);H)} =$$

$$\left\|-u'' + (\omega_1 + \omega_2) Au' + |\omega_1 \omega_2| A^2 u\right\|^2_{L_2((0,1);H)} =$$

$$\left\|u''\right\|^2_{L_2((0,1);H)} + |\omega_1 + \omega_2|^2 \left\|Au'\right\|^2_{L_2((0,1);H)} + |\omega_1\omega_2|^2 \left\|A^2 u\right\|^2_{L_2((0,1);H)} -$$

$$-2(\omega_1 + \omega_2) \operatorname{Re}\left(u'', Au'\right)_{L_2((0,1);H)} + 2|\omega_1\omega_2|(\omega_1 + \omega_2) \operatorname{Re}\left(Au', A^2 u\right)_{L_2((0,1);H)} -$$

$$-2|\omega_1\omega_2| \operatorname{Re}\left(u'', A^2 u\right)_{L_2((0,1);H)}. \tag{9}$$

By integrating by parts we get:

$$\left(u'', Au'\right)_{L_2((0,1);H)} = \int_0^1 \left(u'', Au'\right)_H dt = \left(\left\|u'(1)\right\|^2_{H_{1/2}} - \left\|u'(0)\right\|^2_{H_{1/2}}\right) -$$

$$-\int_0^1 \left(Au', u''\right)_H dt = -\left(Au', u''\right)_{L_2((0,1);H)},$$

i.e.

$$2 \operatorname{Re}\left(u'', Au'\right)_{L_2((0,1);H)} = 0. \tag{10}$$

Similarly we have:

$$\left(u'', Au'\right)_{L_2((0,1);H)} = \int_0^1 \left(u'', A^2 u\right)_H dt = \left(A^{1/2} u'(t), A^{3/2} u(t)\right)_H \Big|_0^1 -$$

$$-\int_0^1 \left(Au', Au'\right)_H dt = \left\|Au'\right\|^2_{L_2((0,1);H)}. \tag{11}$$

We get in a similar way that

$$2 \operatorname{Re}\left(Au', A^2 u\right)_{L_2((0,1);H)} = 0. \tag{12}$$

Allowing for equalities (10)-(12) in (9), we get:

$$\|P_0 u\|^2_{L_2((0,1);H)} = \left\|u''\right\|^2_{L_2((0,1);H)} + |\omega_1 + \omega_2|^2 \left\|Au'\right\|^2_{L_2((0,1);H)} +$$

$$+|\omega_1\omega_2| \left\|A^2 u\right\|_{L_2((0,1);H)} + 2|\omega_1\omega_2| \left\|Au'\right\|^2_{L_2((0,1);H)}. \tag{13}$$

From (13) it follows that

$$\left\|u''\right\|_{L_2((0,1);H)} \leq \|P_0 u\|_{L_2((0,1);H)},$$

i.e.

$$N_2 \leq 1 = c_2.$$

Similarly

$$|\omega_1\omega_2|^2 \left\| A^2 u \right\|^2_{L_2((0,1);H)} \leq \|P_0 u\|^2_{L_2((0,1);H)},$$

i.e.

$$\left\| A^2 u \right\|_{L_2((0,1);H)} \leq \frac{1}{|\omega_1\omega_2|} \|P_0 u\|_{L_2((0,1);H)}$$

or

$$N_0 \leq \frac{1}{|\omega_1\omega_2|} = c_0.$$

Now, let's estimate the number N_1. For $u \in W_2^2\left((0,1);H;\Pi\right)$ we have:

$$\left\|Au'\right\|^2_{L_2((0,1);H)} = \int_0^1 \left(Au', Au'\right)_H dt = \left(A^{1/2}u'(t), A^{3/2}u(t)\right)_H \Big|_0^1 -$$

$$-\int_0^1 \left(u'', A^2 u\right)_H dt = -\left(u'', A^2 u\right)_{L_2((0,1);H)} =$$

$$= -\frac{1}{|\omega_1\omega_2|} \left(u'', |\omega_1\omega_2| A^2 u\right)_{L_2((0,1);H)} \leq$$

$$\leq \frac{1}{|\omega_1\omega_2|} \left\|u''\right\|_{L_2((0,1);H)} |\omega_1\omega_2| \left\|A^2 u\right\|_{L_2((0,1);H)} \leq$$

$$\leq \frac{1}{2|\omega_1\omega_2|} \left(\left\|u''\right\|^2_{L_2((0,1);H)} + |\omega_1\omega_2|^2 \left\|A^2 u\right\|^2_{L_2((0,1);H)} \right).$$

Here, considering equality (13) we have:

$$\left\|Au'\right\|^2_{L_2((0,1);H)} = \frac{1}{2|\omega_1\omega_2|} \left(\|P_0 u\|^2_{L_2((0,1);H)} - \left(\omega_1^2 + \omega_2^2 + 2|\omega_1\omega_2| + \right.\right.$$

$$\left.\left. + 2|\omega_1\omega_2|\right) \left\|Au'\right\|^2_{L_2((0,1);H)} \right) \leq \frac{1}{2|\omega_1\omega_2|} \left(\|P_0 u\|^2_{L_2((0,1);H)} - \right.$$

$$\left. - \left(\omega_1^2 + \omega_2^2\right) \left\|Au'\right\|^2_{L_2((0,1);H)} \right).$$

Hence we have:

$$\left\|Au'\right\|_{L_2((0,1);H)} \leq \frac{1}{|\omega_1| + \omega_2} \|P_0 u\|_{L_2((0,1);H)},$$

i.e.

$$N_1 \leq \frac{1}{|\omega_1| + \omega_2}.$$

The lemma is proved.

Now, let's prove the basic theorem.

Theorem. *Let conditions 1)-3) be fulfilled and it hold the inequality*

$$\alpha = \|B_0\| + \frac{1}{|\omega_1| + \omega_2} \|B_1\| + \frac{1}{|\omega_1\omega_2|} \|B_2\| < 1.$$

Then problem (1)-(2) is regularly solvable.

Proof. From lemma 1 it follows that the operator P_0^{-1} exists and is bounded. Then we'll write problem (1)-(2) in the form:

$$P_0u + P_1u = f,$$

$f \in L_2((0,1);H)$, $u \in W_2^2((0,1);H;\Pi)$ and make substitution $P_0u = \upsilon \in \in L_2((0,1);H)$. Then we get the following equation in the space $L_2((0,1);H)$;

$$\vartheta + P_1P_0^{-1}\upsilon = f.$$

Since by lemma 2

$$\left\|P_1P_0^{-1}\upsilon\right\|_{L_2((0,1);H)} = \|P_1u\|_{L_2((0,1);H)} \leq \left\|A_0u''\right\|_{L_2((0,1);H)} + \\ + \left\|A_1\frac{du}{dt}\right\|_{L_2((0,1);H)} + \|A_2u\|_{L_2((0,1);H)} \leq \\ \leq \|B_0\|\left\|u''\right\|_{L_2((0,1);H)} + \|B_1\|\left\|A_1\frac{du}{dt}\right\|_{L_2((0,1);H)} + \|B_2\|\|A_2u\|_{L_2((0,1);H)} \leq \\ \leq \|B_0\|\|P_0u\|_{L_2((0,1);H)} + \frac{1}{|\omega_1|+\omega_2}\|B_1\|\|P_0u\|_{L_2((0,1);H)} + \\ + \frac{1}{|\omega_1\omega_2|}\|B_2\|\|P_0u\|_{L_2((0,1);H)} = \left(\|B_0\| + \frac{1}{|\omega_1|+\omega_2}\|B_1\| + \right. \\ \left. + \frac{1}{|\omega_1\omega_2|}\|B_2\|\right)\|P_0u\|_{L_2((0,1);H)} = \alpha\|\vartheta\|_{L_2((0,1);H)},$$

then the norm of the operator $\left\|P_1P_0^{-1}\right\|_{L_2((0,1);H)\to L_2((0,1);H)} < 1$.

Then

$$\upsilon = \left(E + P_1P_0^{-1}\right)^{-1}f \qquad \text{and} \qquad u = P_0^{-1}\left(E + P_1P_0^{-1}\right)^{-1}f.$$

Hence it follows that

$$\|u\|_{W_2^2((0,1);H;\Pi)} \leq const\ \|f\|_{L_2((0,1);H)}.$$

The theorem is proved.

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