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ON SCATTERING DATA FOR DISCONTINUOUS STURM-LIOUVILLE OPERATOR

Abstract

In the paper a total collection of eigenfunctions and scattering data is constructed, behavior of scattering function at infinity is studied for discontinuous Sturm-Liouville operator.

Consider a boundary value problem generated on a semi-axis $0 \le x < \infty$ by the differential equation

$$-y'' + q(x)\,y = \lambda^2 y, \tag{1}$$

boundary condition

$$y(0) = 0, \tag{2}$$

and discontinuity conditions at the point $a \in (0, \infty)$

$$\begin{aligned} y(a-0) &= \alpha y(a+0)\,, \\ y'(a-0) &= \alpha^{-1} y'(a+0)\,, \end{aligned} \tag{3}$$

where $\alpha > 0$, $\alpha \ne 1$. Assume that the function $q(x)$ is real and satisfies the condition

$$\int_0^{+\infty} x\,|q(x)|\,dx < +\infty. \tag{4}$$

Let's introduce the following notation

$$\sigma(x) = \int_x^{\infty} |q(t)|\,dt, \quad \sigma_1(x) = \int_x^{\infty} \sigma(t)\,dt.$$

In this paper we define scattering data of the problem (1)-(3) and study behavior of the scattering function at infinity. Such a problem without discontinuity conditions (3), i.e. when $\alpha = 1$, was considered in the paper [1] (see also [2]).

At first we prove the following useful lemma.

Lemma 1. *For all values of λ equation (1) has the solution $S(x, \lambda)$ satisfying conditions (3) and*

$$S(x, \lambda) = x\,[1 + o(1)]\,, \quad S'(x, \lambda) = 1 + o(1)\,, \quad x \to 0. \tag{5}$$

This solution is an entire function of λ *and for* $\operatorname{Im}\lambda \geq 0$ *satisfies the inequality*

$$\left|\lambda\left(S\left(x,\lambda\right)-S_0\left(x,\lambda\right)\right)e^{i\lambda x}\right| \leq c^2\left\{\sigma_1\left(0\right)-\sigma_1\left(\left|\lambda\right|^{-1}\right)\right\}e^{c\int\limits_0^x t|q(t)|dt}, \qquad c=\alpha^+ + \left|\alpha^-\right|, \tag{6}$$

where the function $S_0\left(x,\lambda\right)$ *is defined by the formula*

$$S_0\left(x,\lambda\right)=\begin{cases}\dfrac{\sin\lambda x}{\lambda}, & 0<x<a,\\[2ex] \alpha^+\dfrac{\sin\lambda x}{\lambda}+\alpha^-\dfrac{\sin\lambda\left(x-2a\right)}{\lambda}, & a<x<\infty,\ \ \alpha^{\pm}=\dfrac{1}{2}\left(\alpha\pm\alpha^{-1}\right).\end{cases}$$

Proof. It is easy to show that problem (1), (3), (5) is equivalent to the integral equation

$$S\left(x,\lambda\right)=S_0\left(x,\lambda\right)+\int\limits_0^x S_0\left(t,x,\lambda\right)q\left(t\right)S\left(t,\lambda\right)dt, \tag{7}$$

where

$$S_0\left(t,x,\lambda\right)=\begin{cases}\dfrac{\sin\lambda\left(x-t\right)}{\lambda}, & a<t<x,\ \text{or}\ t<x<a,\\[2ex] \alpha^+\dfrac{\sin\lambda\left(x-t\right)}{\lambda}+\alpha^-\dfrac{\sin\lambda\left(x+t-2a\right)}{\lambda}, & t<a<x.\end{cases}$$

For $\operatorname{Im}\lambda \geq 0$ we'll look for the solution of integral equation (7) in the form $S\left(x,\lambda\right)=xe^{-i\lambda x}z\left(x,\lambda\right)$. Then, for the function $z\left(x,\lambda\right)$ we get the equation

$$z\left(x,\lambda\right)=\frac{S_0\left(x,\lambda\right)e^{i\lambda x}}{x}+\int\limits_0^x\frac{S_0\left(t,x,\lambda\right)e^{i\lambda(x-t)}}{x}tq\left(t\right)z\left(t,\lambda\right)dt.$$

To solve this equation we apply the method of successive approximations. Then

$$z\left(x,\lambda\right)=\sum_{k=0}^{\infty}z_k\left(x,\lambda\right), \tag{8}$$

where

$$z_0\left(x,\lambda\right)=\frac{S_0\left(x,\lambda\right)e^{i\lambda x}}{x},$$

$$z_k\left(x,\lambda\right)=\int_0^x\frac{S_0\left(t,x,\lambda\right)e^{i\lambda(x-t)}}{x}tq\left(t\right)z_{k-1}\left(t,\lambda\right)dt.$$

Since for $\operatorname{Im}\lambda \geq 0$, $0\leq t\leq x$

$$\left|\frac{S_0\left(x,\lambda\right)e^{i\lambda x}}{x}\right| \leq \alpha^+ + \left|\alpha^-\right|, \quad \left|\frac{S_0\left(t,x,\lambda\right)e^{i\lambda(x-t)}}{x}\right| \leq \alpha^+ + \left|\alpha^-\right|,$$

then

$$|z_0(x,\lambda)| \le \alpha^+ + \left|\alpha^-\right| = c,$$

$$|z_k(x,\lambda)| \le c\int_0^x t\,|q(t)|\,|z_{k-1}(t,\lambda)|\,dt = \frac{c}{k!}\left[c\int_0^x t\,|q(t)|\,dt\right]^k,$$

and the series (8) uniformly converges in the domain $x \in [0,b]$, $\operatorname{Im}\lambda \ge 0$ for any $b > 0$ and its sum is an analytic function of λ for $\operatorname{Im}\lambda \ge 0$, continuous in the half-plane $\operatorname{Im}\lambda \ge 0$ and satisfies the inequality

$$|z(x,\lambda)| \le ce^{c\int_0^x t|q(t)|dt}.$$

Therefore, the function $S(x,\lambda) = xz(x,\lambda)e^{-i\lambda x}$ satisfies equations (7) and (1), is an analytic function of λ for $\operatorname{Im}\lambda > 0$, continuous in the closed half-plane $\operatorname{Im}\lambda \ge 0$ and satisfies the inequality

$$\left|S(x,\lambda)e^{i\lambda x}\right| \le cxe^{c\int_0^x t|q(t)|dt}. \tag{9}$$

It is proved in a similar way that equation (7) has a solution for $\operatorname{Im}\lambda \le 0$ and its solution $S(x,\lambda)$ is analytic with respect to λ for $\operatorname{Im}\lambda < 0$ and continuous for $\operatorname{Im}\lambda \le 0$. Thus, $S(x,\lambda)$ is a solution of equation (1), and with respect to λ it is an entire function. It is easy to see that the solution $S(x,\lambda)$ satisfies conditions (5).

Using equation (7), estimation (9), inequality $\left|\lambda S_0(t,x,\lambda)e^{i\lambda(x-t)}\right| \le \alpha^+ + |\alpha^-|$, for $\operatorname{Im}\lambda \ge 0$ we have

$$\left|\lambda\left(S(x,\lambda) - S_0(x,\lambda)\right)e^{i\lambda x}\right| \le \int_0^x \left|\lambda S_0(t,x,\lambda)e^{i\lambda(x-t)}\right| |q(t)| \left|S(t,\lambda)e^{i\lambda t}\right| dt \le$$

$$\le c\int_0^x |q(t)|\,cte^{c\int_0^t \xi|q(\xi)|d\xi}dt = c\left(e^{c\int_0^x \xi|q(\xi)|d\xi} - 1\right). \tag{10}$$

Hence, in particular, it follows that

$$\left|\lambda S(x,\lambda)e^{i\lambda x}\right| \le ce^{c\int_0^x t|q(t)|dt}. \tag{11}$$

Now, we establish the validity of estimation (6) for $\operatorname{Im}\lambda \ge 0$. Consider the cases $|\lambda|^{-1} < x$ and $|\lambda|^{-1} \ge x$ separately.

Let $\operatorname{Im}\lambda \ge 0$ and $|\lambda|^{-1} < x$. Then, by (7), (9) and (11)

$$\left|\lambda\left(S(x,\lambda) - S_0(x,\lambda)\right)e^{i\lambda x}\right| \le \int_0^{|\lambda|^{-1}} \left|\lambda S_0(t,x,\lambda)e^{i\lambda(x-t)}q(t)e^{i\lambda t}S(t,\lambda)\right| dt +$$

$$+\int\limits_{|\lambda|^{-1}}^{x}\left|\lambda S_0(t,x,\lambda)e^{i\lambda(x-t)}q(t)e^{i\lambda t}S(t,\lambda)\right|dt\le$$

$$\le\int\limits_0^{|\lambda|^{-1}}c|q(t)|cte^{c\int\limits_0^t s|q(s)|ds}dt+\int\limits_{|\lambda|^{-1}}^{x}\frac{c}{|\lambda|}|q(t)|ce^{c\int\limits_0^t s|q(s)|ds}dt\le$$

$$\le c^2\left\{\int\limits_0^{|\lambda|^{-1}}t|q(t)|dt+\frac{1}{|\lambda|}\int\limits_{|\lambda|^{-1}}^{x}|q(t)|dt\right\}e^{c\int\limits_0^x t|q(t)|dt}=$$

$$=c^2\left\{-t\sigma(t)\Big|_0^{|\lambda|^{-1}}+\int\limits_0^{|\lambda|^{-1}}\sigma(t)dt+\frac{1}{|\lambda|}\left\{\sigma\left(|\lambda|^{-1}\right)-\sigma(x)\right\}\right\}e^{c\int\limits_0^x t|q(t)|dt}\le$$

$$\le c^2\left\{\sigma_1(0)-\sigma_1\left(|\lambda|^{-1}\right)\right\}e^{c\int\limits_0^x t|q(t)|dt},$$

i.e. inequality (6) is proved for $|\lambda|^{-1}<x$.

For $|\lambda|^{-1}\ge x$ we have $\int\limits_0^x t|q(t)|dt\le\sigma_1(0)-\sigma_1\left(|\lambda|^{-1}\right)$. On the other hand, it follows from inequality (10) that

$$\left|\lambda\left(S(x,\lambda)-S_0(x,\lambda)\right)e^{i\lambda x}\right|\le c\left(e^{c\int\limits_0^x t|q(t)|dt}-1\right)\le$$

$$\le c^2\int\limits_0^x t|q(t)|dt\cdot e^{c\int\limits_0^x t|q(t)|dt}\le c^2\left\{\sigma_1(0)-\sigma_1\left(|\lambda|^{-1}\right)\right\}e^{c\int\limits_0^x t|q(t)|dt}.$$

The lemma is proved.

In the paper [3] it is proved that provided (3) equation (1) has a Iost type solution $e(x,\lambda)$, regular in λ in the upper half-plane $\operatorname{Im}\lambda>0$, continuous for $\operatorname{Im}\lambda\ge0$ and can be represented in the form

$$e(x,\lambda)=e_0(x,\lambda)+\int\limits_x^{+\infty}K(x,t)e^{i\lambda t}dt, \qquad (12)$$

where

$$e_0(x,\lambda)=\begin{cases}e^{i\lambda x}, & x>a\\ \alpha^+e^{i\lambda x}+\alpha^-e^{i\lambda(2a-x)}, & 0<x\le a,\end{cases}$$

and the function $K(x,\cdot)\in L_1(0,\infty)$.

Since the functions $e(x,\lambda)$ and $e(x,-\lambda)$ for real $\lambda \neq 0$ form a fundamental system of solutions of equation (1) with discontinuity conditions (3), then

$$S(x,\lambda) = \frac{1}{2i\lambda}\left\{-e(x,-\lambda)\,e(0,\lambda) + e(x,\lambda)\,e(0,-\lambda)\right\}.$$

Since $q(x)$ is real, $e(0,-\lambda) = \overline{e(0,\lambda)}$. Therefore for all real $\lambda \neq 0$ $e(0,\lambda) \neq 0$,

$$\frac{-2i\lambda S(x,\lambda)}{e(0,\lambda)} = e(x,-\lambda) - S(\lambda)\,e(x,\lambda), \tag{13}$$

where

$$S(\lambda) = \frac{e(0,-\lambda)}{e(0,\lambda)} = \overline{S(-\lambda)} = S^{-1}(-\lambda). \tag{14}$$

From representation (12) it follows that $e(0,\lambda) \to \alpha^{+}$ as $|\lambda| \to \infty$, $\operatorname{Im}\lambda > 0$. Therefore the zeros of the function $e(0,\lambda)$ form bounded and at most countable set whose unique limit point may be only a zero, since for real $\lambda \neq 0$, $e(0,\lambda) \neq 0$. Further, as in the case of absence of discontinuity condition (3) we can prove that all the zeros of $e(0,\lambda)$ in the upper half-plane (if they exist) lie on an imaginary axis, they are simple and they are of finite number (see [2]). We denote these zeros by $i\lambda_1$, $i\lambda_2$,...,$i\lambda_n$ $(0 < \lambda_1 < ... < \lambda_n)$. We introduce the following notation

$$m_k^{-2} = \int_0^{+\infty} |e(x,i\lambda_k)|^2\,dx, \quad k = 1,2,...,n. \tag{15}$$

Obviously, the functions

$$u(x,\lambda) = e(x,-\lambda) - S(\lambda)\,e(x,\lambda), \quad \lambda \in (0,\infty), \tag{16}$$

$$u_k(x) = m_k e(x,i\lambda_k), \quad k = 1,2,...,n, \tag{17}$$

are bounded solutions of problem (1)-(3). We can prove that they form a total collection of normed eigenfunctions of this problem. From formulas (16), (17) it follows that as $x \to \infty$ normed eigenfunctions satisfy the asymptotic relations

$$u(x,\lambda) = e^{-i\lambda x} - S(\lambda)\,e^{i\lambda x} + o(1), \quad \lambda \in (-\infty,\infty),$$

$$u_k(x) = m_k e^{-\lambda_k x}(1 + o(1)), \quad k = 1,2,...,n.$$

A collection of quantities

$$\{S(\lambda), \lambda_k, m_k\},$$

that completely determine the behaviour of normed eigen functions at infinity is said to be scattering data of boundary value problem (1), (2), (3) with real potential $q(x)$ satisfying condition (4).

Behaviour of the scattering function $S(\lambda)$ is determined by the following lemma.

Lemma 2. *The function $S_0(\lambda) - S(\lambda)$ is a Fourier transformation of some function $F_s(t)$, i.e.*

$$S_0(\lambda) - S(\lambda) = \int_{-\infty}^{\infty} F_s(t)\,e^{-i\lambda t}dt,$$

where $S_0(\lambda) = \dfrac{e_0(0,-\lambda)}{e_0(0,\lambda)} = \dfrac{\alpha^+ + \alpha^- e^{-2ia\lambda}}{\alpha^+ + \alpha^- e^{2ia\lambda}}$, *and* $F_s(t)$ *can be represented in the form* $F_s(t) = F_s^{(1)}(t) + F_s^{(2)}(t)$, *where* $F_s^{(1)}(\cdot) \in L_1(-\infty,\infty)$, $F_s^{(2)}(\cdot) \in L_2(-\infty,\infty)$ *and* $\sup\limits_{-\infty<t<\infty} \left|F_s^{(2)}(t)\right| < \infty$.

Proof. Denoting $K(0,t) = K(t)$ for brevity, we have

$$e(0,\lambda) = \alpha^+ + \alpha^- e^{2ia\lambda} + \int_0^\infty K(0,t)e^{i\lambda t}dt =$$

$$= \alpha^+ + \alpha^- e^{2ia\lambda} + \widetilde{K}(-\lambda), \qquad \widetilde{K}(\lambda) = \int_0^\infty K(t)e^{-i\lambda t}dt,$$

$$S_0(\lambda) - S(\lambda) = \frac{e_0(0,-\lambda)}{e_0(0,\lambda)} \cdot \frac{\widetilde{K}(-\lambda)}{e_0(0,\lambda) + \widetilde{K}(-\lambda)} -$$

$$- \frac{\widetilde{K}(\lambda)}{e_0(0,\lambda) + \widetilde{K}(-\lambda)}. \tag{18}$$

At first, we show that the function $\dfrac{\widetilde{K}(-\lambda)}{\alpha^+ + \alpha^- e^{2ia\lambda}}$ is a Fourier transformation of some summable function. Really, if we denote $K_+(t) = K(t)$, $t > 0$, $K_+(t) = 0$, $t < 0$, then the series

$$\sum_{n=0}^\infty (-1)^n \left(\frac{\alpha^-}{\alpha^+}\right)^n K_+(t - 2an)$$

converges in the space L_1 to some function $\varphi(\cdot) \in L_1(-\infty,+\infty)$, since

$$\left|\int_{-\infty}^\infty \sum_{n=0}^\infty (-1)^n \left(\frac{\alpha^-}{\alpha^+}\right)^n K_+(t-2an)\,dt\right| \le$$

$$\le \sum_{n=0}^{+\infty} \left|\frac{\alpha^-}{\alpha^+}\right|^n \int_{-\infty}^\infty |K_+(t-2an)|\,dt = \sum_{n=0}^\infty \left|\frac{\alpha^-}{\alpha^+}\right|^n \int_{-\infty}^\infty |K(t)|\,dt.$$

Therefore

$$\frac{\widetilde{K}(-\lambda)}{\alpha^+ + \alpha^- e^{2ia\lambda}} = \frac{1}{\alpha^+} \sum_{n=0}^{+\infty} \left(\frac{\alpha^-}{\alpha^+}\right)^n (-1)^n \int_{2an}^{+\infty} K(t-2an)e^{i\lambda t}dt =$$

$$= \frac{1}{\alpha^+} \int_{-\infty}^\infty \varphi(t)e^{i\lambda t}dt = \frac{1}{\alpha^+}\tilde{\varphi}(-\lambda).$$

Consequently

$$\frac{\widetilde{K}(\lambda)}{e_0(0,\lambda) + \widetilde{K}(-\lambda)} = \frac{\dfrac{1}{\alpha^+}\tilde{\varphi}(\lambda)}{1 + \dfrac{1}{\alpha^+}\tilde{\varphi}(-\lambda)},$$

$$\frac{\widetilde{K}(-\lambda)}{e_0(0,\lambda)+\widetilde{K}(-\lambda)}=\frac{\frac{1}{\alpha^+}\tilde{\varphi}(-\lambda)}{1+\frac{1}{\alpha^+}\tilde{\varphi}(-\lambda)}.$$

Now, we can write equality (18) in the form

$$S_0(\lambda)-S(\lambda)=\frac{\widetilde{\psi}^*(-\lambda)}{1+\frac{1}{\alpha^+}\tilde{\varphi}(-\lambda)}, \tag{19}$$

where the function $\psi(\lambda)$ is also a Fourier transformation of a summable function. In the sequel, we are to argue in the same way as in the case $\alpha=1$ (see [2]), i.e. we can rewrite equality (19) in the form

$$S_0(\lambda)-S(\lambda)=\widetilde{\psi}^*(-\lambda)\left[\left\{1+\left(1-\widetilde{h}\left(\lambda N^{-1}\right)\right)\frac{1}{\alpha^+}\tilde{\varphi}(-\lambda)\right\}^{-1}-1\right]+$$

$$+\widetilde{\psi}^*(-\lambda)-\widetilde{\psi}^*(-\lambda)\left\{\frac{1}{1+\left(1-\widetilde{h}(\lambda N^{-1})\right)\frac{1}{\alpha^+}\tilde{\varphi}(-\lambda)}-\frac{1}{1+\frac{1}{\alpha^+}\tilde{\varphi}(-\lambda)}\right\}, \tag{20}$$

where

$$\widetilde{h}(\lambda)=\begin{cases}1, & |\lambda|<1\\ 2-|\lambda|, & 1\le|\lambda|\le 2\\ 0, & |\lambda|>2\end{cases}$$

Since for sufficiently large N the function $\left\{1+\left(1-\widetilde{h}\left(\lambda N^{-1}\right)\right)\frac{1}{\alpha^+}\tilde{\varphi}(-\lambda)\right\}^{-1}-1$ is a Fourier transformation of a summable function, the sum of the first two addends of the right hand-side of relation (20) is also a Fourier transformation of some summable function $F_s^{(1)}(x)$. Notice that $\widetilde{h}\left(\lambda N^{-1}\right)=0$ for $|\lambda|>2N$, consequently the third addend in relation (20) also equals zero for $|\lambda|>2N$. Therefore it is a Fourier transformation of some bounded function $F_s^{(2)}(\cdot)\in L_2(-\infty,+\infty)$. Lemma 2 is proved.

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