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EXTREMUM PROBLEMS FOR DELAY DISCRETE INCLUSIONS WITH VARIABLE STRUCTURE

Abstract

In this work necessary extremum conditions are obtained for one discrete systems class. We consider extremum problems for delay discrete inclusions with variable structure. The formulation of this problem is new.

1. Introduction. The survey of problems, considered in this work may be found in [4, p.132]. This work is generalization of some results, obtaining in [5, p.89-94; 6, p.106-112]. The considered problem is reduced to the problem of the mathematical programming using the method given in [1, p. 3-55]. Further, we obtain necessary extremum conditions in the problem for delay discrete inclusions with variable structure using the nonsmooth analysis theory [3, p. 98-100].

2. The formulation of the problem. Let X, Y be Banach spaces, $a_t : X^2 \to 2^X$, $t = 0, 1, .., k-1$, $b_t : Y^2 \to 2^Y$, $t = k, k+1, ..., m-1$ the multivalued mappings, where 2^V denotes the set of all subsets of V. We denote $grF = \{(z, v) \in Z \times V : v \in F(z)\}$.

Let us consider the delay discrete inclusions with variable structure

$$\begin{gathered} x_{t+1} \in a_t(x_{t-\Delta}, x_t), \quad t = 0, 1, ..., k-1 \\ x_t = c(t) \quad \text{at} \quad t = -\Delta, -\Delta+1, ..., -1, 0 \\ y_{t+1} \in b_t(y_{t-h}, y_t), \quad t = k, k+1, ..., m-1 \\ y_t = G(x_t) \quad \text{at} \quad t = k-h, k-h+1, .., k \\ y_m \in C, \end{gathered} \tag{2.1}$$

where $c(t) \in X$ at $t = -\Delta, -\Delta+1, ..., -1, 0$, $C \subset Y, G : X \to Y$ - mapping, k, m, Δ, h- fixed natural numbers. As a trajectory (solution) $(\{x_t\}, \{y_v\})$ of the discrete inclusion (2.1) we understand the process

$$x_t, t = 1, ..., k-1, k, y_v, v = k+1, ..., m$$

for which (2.1) is satisfied.

Suppose, that

$$\Delta < k-1, \; h < \min\{k-1, m-k-1\},$$

$$g_t : X \to R, \; t = \overline{1, k}\,, \quad f_t : Y \to R, \; t = \overline{k+1, m}.$$

We denote $x = (x_1, ..., x_k)$, $y = (y_{k+1}, ..., y_m)$

Consider the minimization of the function:

$$F(x, y) = \sum_{t=1}^{k} g_t(x_t) + \sum_{t=k+1}^{m} f_t(y_t) \tag{2.2}$$

on the trajectories of (2.1) discrete inclusion. We note that as a trajectory (solution) of (2.1) we take the pairs (x, y), for which (2.1) is satisfied. Denote by D the set of solutions of the problem (2.1). We denote $s = m - k$ and define the sets in $X^k \times Y^s$ as

$$M_0 = \{(x_1, ..., x_k\ ,\ y_{k+1}, ..., y_m) \in X^k \times Y^s : x_1 \in a_0\left(c\left(-\Delta\right), c\left(0\right)\right)\},$$
$$M_1 = \{(x_1, ..., x_k\ ,\ y_{k+1}, ..., y_m) \in X^k \times Y^s : x_2 \in a_1\left(c\left(-\Delta+1\right), x_1\right)\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$M_\Delta = \{(x_1, ..., x_k\ ,\ y_{k+1}, ..., y_m) \in X^k \times Y^s : x_{\Delta+1} \in a_\Delta\left(c\left(0\right), x_\Delta\right)\},$$
$$M_{\Delta+1} = \{(x_1, ..., x_k\ ,\ y_{k+1}, ..., y_m) \in X^k \times Y^s : x_{\Delta+2} \in a_{\Delta+1}\left(x_1\ ,\ x_{\Delta+1}\right)\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$M_{k-1} = \{(x_1, ..., x_k\ ,\ y_{k+1}, ..., y_m) \in X^k \times Y^s : x_k \in a_{k-1}\left(x_{k-1-\Delta}\ ,\ x_{k-1}\right)\},$$
$$M_k = \{(x_1, ..., x_k\ ,\ y_{k+1}, ..., y_m) \in X^k \times Y^s : y_{k+1} \in b_k\left(G\left(x_{k-h}\right)\ , G\ \left(x_k\right)\right)\},$$
$$M_{k+1} = \{(x_1, ..., x_k\ ,\ y_{k+1}, ..., y_m) \in X^k \times Y^s : y_{k+2} \in b_{k+1}\left(G\left(x_{k+1-h}\right)\ , y_{k+1}\right)\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$M_{k+h} = \{(x_1, ..., x_k\ ,\ y_{k+1}, ..., y_m) \in X^k \times Y^s : y_{k+h+1} \in b_{k+h}\left(G\left(x_k\right)\ , y_{k+h}\right)\},$$
$$M_{k+h+1} = \{(x_1, ..., x_k\ ,\ y_{k+1}, ..., y_m) \in X^k \times Y^s : y_{k+h+2} \in b_{k+h+1}\left(y_{k+1}\ , y_{k+h+1}\right)\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$M_{m-1} = \{(x_1, ..., x_k\ ,\ y_{k+1}, ..., y_m) \in X^k \times Y^s : y_m \in b_{m-1}\left(y_{m-1-h}\ , y_{m-1}\right)\},$$
$$M_m = \{(x_1, ..., x_k\ ,\ y_{k+1}, ..., y_m) \in X^k \times Y^s : y_m \in C\}$$

Thus, the formulated problem will be reduced to the minimization of the function $F(x, y)$ on the set $D = \bigcap_{i=0}^{m} M_i$.

3. The solution of the problem. Let Z be a Banach space, E be a non-empty subset of Z. Consider the function $d_E : Z \to R$, defined as $d_E(z) = \inf\{\|z - \vartheta\| : \vartheta \in E\}$. Consider the generalized directional derivative z of the function φ in the point z$_0$:

$$\varphi^0(z_0; z) = \overline{\lim_{\substack{\vartheta \to z_0 \\ \lambda \downarrow 0}}} \frac{\varphi(\vartheta + \lambda z) - \varphi(\vartheta)}{\lambda}$$

If φ is a Lipschitz function in the neighbourhood of z_0, then $z \to \varphi^0(z_0; z)$ is a sublinear function. The generalized gradient of the function φ in the point z_0, denoted as $\partial\varphi(z_0)$ is the set of all linear continuous functionals $p \in Z^*$ such that $\varphi^0(z_0; z) \geq < p, z >$ for all $z \in Z$.

Suppose $z_0 \in E$. The vector $z \in Z$ is called the tangent to E in z_0, if $d_E^0(z_0; z) = 0$. The set of all tangents to E in z_0 is denoted as $T_E(z_0)$, i.e. $T_E(z_0) = \{z : d_E^0(z_0; z) = 0\}$. If $z_0 \in int\ E$, then $T_E(z_0) = Z$.

Define the normal cone to E at the point z_0 as a double cone to $T_E(z_0)$:

$$N_E(z_0) = \{z^* \in Z^* :< z^*, z > \leq 0 \ \text{ at } \ z \in T_E(z_0)\}$$

Furthermore denote by $\overline{z} = (\overline{x}, \overline{y}) = \left(\overline{x}_1, ..., \overline{x}_k\ ,\ \overline{y}_{k+1}, ..., \overline{y}_m\right)$ an optimal solution of the problem (2.1). From the corollary of theorem 2.4.5 [3, p.57] follow that

$$T_{M_0}(\overline{z}) = \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : x_1 \in T_{a0(c(-\Delta),c(0))}(\overline{x}_1)\},$$
$$T_{M_1}(\overline{z}) = \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : (x_1,x_2) \in T_{gra_1(c(-\Delta+1),\cdot)}(\overline{x}_1,\overline{x}_2)\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$T_{M_\Delta}(\overline{z}) = \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : (x_\Delta,x_{\Delta+1}) \in T_{gra_\Delta(c(0),\cdot)}(\overline{x}_\Delta,\overline{x}_{\Delta+1})\},$$
$$T_{M_{\Delta+1}}(\overline{z}) = \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : (x_1,x_{\Delta+1},x_{\Delta+2}) \in \in T_{gra_{\Delta+1}}(\overline{x}_1,\overline{x}_{\Delta+1},\overline{x}_{\Delta+2})\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$T_{M_{k-1}}(\overline{z}) = \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : (x_{k-1-\Delta}\ ,\ x_{k-1},\ x_k) \in \in T_{gra_{k-1}}(\overline{x}_{k-1-\Delta}\ ,\ \overline{x}_{k-1},x_k)\},$$
$$T_{M_k}(\overline{z}) = \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : (x_{k-h}\ ,x_k\ ,y_{k+1}) \in \in T_{grb_k(G(\cdot),G(\cdot))}\left(\overline{x}_{k-h}\ ,\overline{x}_k\ ,\overline{y}_{k+1}\right)\},$$
$$T_{M_{k+1}}(\overline{z}) = \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : (x_{k+1-h},y_{k+1},y_{k+2}) \in \in T_{grb_{k+1}(G(\cdot),\cdot)}\left(\overline{x}_{k+1-h}\ ,\overline{y}_{k+1}\ ,\overline{y}_{k+2}\right)\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$T_{M_{k+h}}(\overline{z}) = \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : (x_k\ ,y_{k+h}\ ,y_{k+h+1}) \in \in T_{grb_{k+h}(G(\cdot),\cdot)}\left(\overline{x}_k\ ,\overline{y}_{k+h}\ ,\overline{y}_{k+h+1}\right)\},$$
$$T_{M_{k+h+1}}(\overline{z}) = \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : (y_k\ ,y_{k+h+1}\ ,y_{k+h+2}) \in \in T_{grb_{k+h+1}}\left(\overline{y}_{k+1}\ ,\overline{y}_{k+h+1}\ ,\overline{y}_{k+h+2}\right)\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$T_{M_{m-1}}(\overline{z}) = \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : (y_{m-1-h}\ ,y_{m-1}\ ,y_m) \in \in T_{grb_{m-1}}\left(\overline{y}_{m-1-h}\ ,\overline{y}_{m-1}\ ,\overline{y}_m\right)\},$$
$$T_{M_m}(\overline{z}) = \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : y_m \in T_C(\overline{y}_m)\}.$$

Therefore

$$N_{M_0}(\overline{z}) = \{(x_1^*,0,...,0) \in X^{*^k} \times Y^{*^s} : x_1^* \in N_{a0(c(-\Delta),c(0))}(\overline{x}_1)\},$$
$$N_{M_1}(\overline{z}) = \{(x_1^*,x_2^*,0,...,0) \in X^{*^k} \times Y^{*^s} : (x_1^*,x_2^*) \in N_{gra_1(c(-\Delta+1),\cdot)}(\overline{x}_1,\overline{x}_2)\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$N_{M_\Delta}(\overline{z}) = \{\left(0,...,0,x_\Delta^*,x_{\Delta+1}^*,0,...,0\right) \in X^{*^k} \times Y^{*^s} : \left(x_\Delta^*,x_{\Delta+1}^*\right) \in N_{gra_\Delta(c(0),\cdot)}(\overline{x}_\Delta,\overline{x}_{\Delta+1})\},$$
$$N_{M_{\Delta+1}}(\overline{z}) = \{\left(x_1^*,0,...,0,x_{\Delta+1}^*,x_{\Delta+2}^*,0,...,0\right) \in X^{*^k} \times Y^{*^s} : \left(x_1^*,x_{\Delta+1}^*,x_{\Delta+2}^*\right) \in N_{gra_{\Delta+1}}(\overline{x}_1,\overline{x}_{\Delta+1},\overline{x}_{\Delta+2})\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$N_{M_{k-1}}(\overline{z}) = \{\left(0,...,0,x_{k-1-\Delta}^*,0,...,0,\ x_{k-1}^*,\ ,x_k^*,0,...,0\right) \in X^{*^k} \times Y^{*^s} : \left(x_{k-1-\Delta}^*,\ x_{k-1}^*,\ x_k^*\right) \in N_{gra_{k-1}}(\overline{x}_{k-1-\Delta},\ \overline{x}_{k-1},\ \overline{x}_k)\},$$
$$N_{M_k}(\overline{z}) = \{\left(0,...,0,x_{k-h}^*,0,...,0,\ x_k^*,\ y_{k+1}^*,0,...,0\right) \in X^{*^k} \times Y^{*^s} : \left(x_{k-h}^*,\ x_k^*,\ ,y_{k+1}^*\right) \in N_{grb_k(G(\cdot),G(\cdot))}\left(\overline{x}_{k-h}\ ,\overline{x}_k\ ,\overline{y}_{k+1}\right)\},$$
$$N_{M_{k+1}}(\overline{z}) = \{\left(0,...,0,x_{k+1-h}^*,0,...,0,\ y_{k+1}^*,\ y_{k+2}^*,0,...,0\right) \in X^{*^k} \times Y^{*^s} : \left(x_{k+1-h}^*,\ y_{k+1}^*,\ y_{k+2}^*\right) \in N_{grb_{k+1}(G(\cdot),\cdot)}\left(\overline{x}_{k+1-h}\ ,\overline{y}_{k+1}\ ,\overline{y}_{k+2}\right)\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$N_{M_{k+h}}(\overline{z})=\{\left(0,...,0,x_k^*,0,...,0,\ y_{k+h}^*,\ y_{k+h+1}^*,0,...,0\right)\in X^{*^k}\times Y^{*^s}:$$
$$\left(x_k^*,\ y_{k+h}^*,\ y_{k+h+1}^*\right)\in N_{grb_{k+h}(G(\cdot),\cdot)}\left(\overline{x}_k\ ,\overline{y}_{k+h}\ ,\overline{y}_{k+h+1}\right)\},$$
$$N_{M_{k+h+1}}(\overline{z})=\{\left(0,...,0,y_{k+1}^*,0,...,0,\ y_{k+h+1}^*,\ y_{k+h+2}^*,0,...,0\right)\ \in X^{*^k}\times Y^{*^s}:$$
$$\left(y_{k+1}^*,\ y_{k+h+1}^*,\ y_{k+h+2}^*\right)\in$$
$$\in N_{grb_{k+h+1}(G(\cdot),\cdot)}\left(\overline{y}_{k+1}\ ,\overline{y}_{k+h+1}\ ,\overline{y}_{k+h+2}\right)\},$$
....
$$N_{M_{m-1}}(\overline{z})=\{\left(0,...,0,y_{m-1-h}^*,0,...,0,\ y_{m-1}^*,\ y_m^*\right)\in X^{*^k}\times Y^{*^s}:$$
$$\left(y_{m-1-h}^*\ ,y_{m-1}^*\ ,y_m^*\right)\in N_{grb_{m-1}}\left(y_{m-1-h}^*\ ,y_{m-1}^*\ ,y_m^*\right)\},$$
$$N_{M_m}(\overline{z})=\{(0,...,0,...,0,\ \ y_m^*)\in X^{*^k}\times Y^{*^s}:y_m^*\in N_C(\overline{y}_m)\}.$$

The next corollary follows from the corollary of proposition 2.4.3 [3, p.55] .

Corollary 1. *If g_t is Lipschitz function in the neighbourhood of $\overline{x}_t$, $t=\overline{1,k}$ f_t is Lipschitz function in the neighbourhood of $\overline{y}_t$, $t=\overline{k+1,m}$, then*

$$0\in\partial F(\overline{z})+N_D(\overline{z}).$$

Corollary 2. *If the condition*

$$T_{M_0}(\overline{z})\cap\left(\bigcap_{i=1}^{m} int\ T_{M_i}(\overline{z})\right)\neq\varnothing\ ,$$

is satisfied there exists at least one hypertangent to M_i , $i=\overline{1,m}$ at the point $\overline{z}$ and g_t is Lipschitz function in the neighbourhood of $\overline{x}_t$, $t=\overline{1,k}$ f_t is Lipschitz function in the neighbourhood of $\overline{y}_t$, $t=\overline{k+1,m}$ then

$$0\in\partial F(\overline{z})+\sum_{i=0}^{m}N_{M_i}(\overline{z}).$$

The proof of corollary 2 follows from corollary 1 and theorem 2.9.8 [3, p.100].

Note that if M_i , $i=\overline{1,m}$ is a closed set, $X=R^{n_1}$ and $Y=R^{n_2}$, then the condition of existence of a hypertangent to M_i in $\overline{z}$ is redundant.

According to lemma 2.3.3 [3, p.43] we have

$$\partial F(\overline{x},\overline{y})\subset\sum_{t=1}^{k}\partial g_t(\overline{x}_t)+\sum_{t=k+1}^{m}\partial f_t(\overline{y}_t).$$

So taking into consideration corollary 2, we obtain

$$0\in\sum_{t=1}^{k}\partial_z g_t(\overline{x}_t)+\sum_{t=k+1}^{m}\partial_z f_t(\overline{y}_t)+\sum_{i=0}^{m}N_{M_i}(\overline{z}).\tag{3.1}$$

It is clear that vector $\overline{x}^*\in\partial F(\overline{z})$ has the view $\overline{x}^*=(x_{10}^*,...,x_{m0}^*)$, where $x_{t0}^*\in\partial g_t(\overline{x}_t)$, $t=1,k$, $x_{t0}^*\in\partial f_t(\overline{y}_t)$, $t=\overline{k+1,m}.$

Thus next theorem is proved.

Theorem 1. *If the conditions of corollary 2 are satisfied and $\overline{z}=(\overline{x},\overline{y})$ is an optimal solution of problem (2.1), (2.2), then there exist $x_{t0}^*\in\partial g_t(\overline{x}_t),t=\overline{1,k},x_{t0}^*\in\partial f_t(\overline{y}_t),t=\overline{k+1,m},x_1^*(0)\in N_{a0(c(-\Delta),c(0))}(\overline{x}_1),\left(x_t^*(t),x_{t+1}^*(t)\right)\in$*

$\in N_{gra_t(c(-\Delta+t),\cdot)}(\overline{x}_t\ ,\ \overline{x}_{t+1})\,, t=\overline{1,\Delta}, \left(x_t^*(\Delta+t)\,, x_{\Delta+t}^*(\Delta+t)\,, x_{\Delta+t+1}^*(\Delta+t)\in\right.$ $\in N_{gra_{\Delta+t}}(\overline{x}_t\ ,\ \overline{x}_{\Delta+t}\ ,\overline{x}_{\Delta+t+1})\,, t=\overline{1,k-1-\Delta}, \left(x_{k-h}^*(k)\,, x_k^*(k)\,, y_{k+1}^*(k)\right)\in$ $\in N_{grb_k(G(\cdot),G(\cdot))}\left(\overline{x}_{k-h}\ ,\ \overline{x}_k\ ,\overline{y}_{k+1}\right), \left(x_{k+t-h}^*(k+t)\,,\ y_{k+t}^*(k+t)\,,\ y_{k+t+1}^*(k+t)\right)\in$ $\in N_{grb_{k+t}(G(\cdot),\cdot)}\left(\overline{x}_{k+t-h},\ \overline{y}_{k+t}\ ,\overline{y}_{k+t+1}\right), t=\overline{1,h}\,, \left(y_{k+t}^*(k+h+t)\,, y_{k+h+t}^*(k+h+t)\,,\right.$ $\left.y_{k+h+t+1}^*(k+h+t)\right)\in N_{grb_{k+h+t}}\left(\overline{y}_{k+t}\ ,\ \overline{y}_{k+h+t}\ ,\overline{y}_{k+h+t+1}\right), t=\overline{1,m-1-k-h},$ $y_m^*(m)\in N_C(\overline{y}_m)$ *such that in the case* $h=\Delta$ *the relations are fulfilled:* $x_{t,0}^*+x_t^*(t-1)+x_t^*(t)+x_t^*(\Delta+t)=0$ *at* $t=\overline{1,k}$ *;* $x_{t,0}^*+y_t^*(t-1)+y_t^*(t)+y_t^*(t+\Delta)=0$ *at* $t=\overline{k+1,m-h-1}$*;* $x_{t,0}^*+y_t^*(t-1)+y_t^*(t)=0$ *at* $t=\overline{m-h,m}$*; in the case* $h<\Delta$ *the relations are fulfilled:* $x_{t,0}^*+x_t^*(t-1)+x_t^*(t)+x_t^*(\Delta+t)=0$ *at* $t=\overline{1,k-1-\Delta}$*;* $x_{t,0}^*+x_t^*(t-1)+x_t^*(t)=0$ *at* $t=\overline{k-\Delta,k-h-1}$*;* $x_{t,0}^*+x_t^*(t-1)+x_t^*(t)+x_t^*(t+h)=0$ *at* $t=\overline{k-h,k}$*;* $x_{t,0}^*+y_t^*(t-1)+y_t^*(t)+x_t^*(t+h)=0$ *at* $t=\overline{k+1,m-1-h}$*;* $x_{t,0}^*+y_t^*(t-1)+y_t^*(t)=0$ at $t=\overline{m-h,m}$ *in the case* $h>\Delta$ *the relations are fulfilled:* $x_{t,0}^*+x_t^*(t-1)+x_t^*(t)+x_t^*(\Delta+t)=0$ *at* $t=\overline{1,k-h-1}$ *;* $x_{t,0}^*+x_t^*(t-1)+x_t^*(t)+x_t^*(t+\Delta)+x_t^*(t+h)=0$ *at* $t=\overline{k-h,k-1-\Delta}$*;* $x_{t,0}^*+x_t^*(t-1)+x_t^*(t)+x_t^*(t+h)=0$ at $t=\overline{k-\Delta,k}$*;* $x_{t,0}^*+y_t^*(t-1)+y_t^*(t)+y_t^*(t+h)=0$ *at* $t=\overline{k+1,m-1-h}$ *;* $x_{t,0}^*+y_t^*(t-1)+y_t^*(t)=0$ at $t=\overline{m-h,m}$*.*

Let Z *be a Banach space,* $E\subset Z$*,* $\overline{z}\in E$*. We denote the set of all hypertangent vectors to* E *at the point* $\overline{z}$ *by* $I_E(\overline{z})$*.*

Theorem 2. *Let* $\overline{z}=\left(\overline{x}_1,...,\overline{x}_k\ ,\overline{y}_{k+1},...,\overline{y}_m\right)$ *be an optimal trajectory,* $I_{gra_t(c(-\Delta+t),\cdot)}(\overline{x}_t\ ,\ \overline{x}_{t+1})$ at $t=\overline{1,\Delta}$ *,* $I_{gra_t}(\overline{x}_{t-\Delta},\ \overline{x}_t\ ,\ \overline{x}_{t+1})$ *at* $t=\overline{\Delta+1,k-1}$*,* $I_{grb_k(G(\cdot),G(\cdot))}\left(\overline{x}_{k-h}\ ,\overline{x}_k\ ,\overline{y}_{k+1}\right)$*,* $I_{grb_t(G(\cdot),\cdot)}\left(\overline{x}_{t-h},\overline{y}_t,\overline{y}_{t+1}\right)$ *at* $t=\overline{k+1,k+h}$*,* $I_{grb_t}\left(\overline{y}_{t-h}\ ,\overline{y}_t\ ,\overline{y}_{t+1}\right)$ *at* $t=\overline{k+h+1,m-1}$ *and* $I_C(\overline{y}_m)$ *be a non-empty, the function* $g_t(\cdot)\,,\ t=\overline{1,k}$ *, satisfies to Lipschitz condition in the neighbourhood of* $\overline{x}_t$ *, the function* $f_t(\cdot)\,,\ t=\overline{k+1,m}$ *satisfies to Lipschitz condition in the neighbourhood of* $\overline{y}_t$ *. Then there exist* $x_{t0}^*\in\partial g_t(\overline{x}_t)$ *at* $t=\overline{1,k}$*,* $x_{t0}^*\in\partial f_t(\overline{y}_t)$ at $t=\overline{k+1,m}$*,* $x_1^*(0)\in N_{a0(c(-\Delta),c(0))}(\overline{x}_1)$*,* $\left(x_t^*(t)\,,\ x_{t+1}^*(t)\right)\in N_{gra_t(c(-\Delta+t),\cdot)}(\overline{x}_t\ ,\overline{x}_{t+1})$ at $t=\overline{1,\Delta}$ *,* $\left(x_t^*(\Delta+t)\,,\ x_{\Delta+t}^*(\Delta+t)\,,\ x_{\Delta+t+1}^*(\Delta+t)\right)\in N_{gra_{\Delta+t}}(\overline{x}_t\ ,\overline{x}_{\Delta+t},\overline{x}_{\Delta+t+1})$ at $t=\overline{1,k-1-\Delta}$ *,* $\left(x_{k-h}^*(k)\,,\ x_k^*(k)\,,\ y_{k+1}^*(k)\right)\in N_{grb_k(G(\cdot),G(\cdot))}\left(\overline{x}_{k-h}\ ,\overline{x}_k\ ,\overline{y}_{k+1}\right)$*,* $\left(x_{k+t-h}^*(k+t)\,, y_{k+t}^*(k+t)\,, y_{k+t+1}^*(k+t)\right)\in N_{grb_{k+t}(G(\cdot),\cdot)}\left(\overline{x}_{k+t-h}\ ,\ \overline{y}_{k+t}\ ,\overline{y}_{k+t+1}\right)$ at $t=\overline{1,h}$ *,* $\left(y_{k+t}^*(k+h+t)\,, y_{k+h+t}^*(k+h+t)\,,\ y_{k+h+t+1}^*(k+h+t)\right)\in$ $\in N_{grb_{k+h+t}}\left(\overline{y}_{k+t}\ ,\ \overline{y}_{k+h+t}\ ,\overline{y}_{k+h+t+1}\right)$ at $t=\overline{1,m-1-k-h}$*,* $y_m^*(m)\in N_C(\overline{y}_m)$ *and the number* λ *equals zero or* -1*, such that not all are equal to zero simultaneously and in the case* $h=\Delta$ *the relations are fulfilled:* $x_t^*(t-1)+x_t^*(t)+x_t^*(\Delta+t)\in\lambda\partial g_t(\overline{x}_t)$ *at* $t=\overline{1,k}$*;* $y_t^*(t-1)+y_t^*(t)+y_t^*(\Delta+t)\in\lambda\partial f_t(\overline{y}_t)$ *at* $t=\overline{k+1,m-h-1}$*,* $y_t^*(t-1)+y_t^*(t)\in\lambda\partial f_t(\overline{y}_t)$ at $t=\overline{m-h,m}$ *; in the case* $h<\Delta$ *the relations are fulfiled:* $x_t^*(t-1)+x_t^*(t)+x_t^*(\Delta+t)\in\lambda\partial g_t(\overline{x}_t)$*,* $t=\overline{1,k-1-\Delta}$*;* $x_t^*(t-1)+x_t^*(t)\in\lambda\partial g_t(\overline{x}_t)$ *at* $t=\overline{k-\Delta,k-h-1}$*,* $x_t^*(t-1)+x_t^*(t)+x_t^*(t+h)\in$ $\in\lambda\partial g_t(\overline{x}_t)$ at $t=\overline{k-h,k}$*,* $y_t^*(t-1)+y_t^*(t)+y_t^*(t+h)\in\lambda\partial f_t(\overline{y}_t)$ at $t=\overline{k+1,m-1-h}$*,* $y_t^*(t-1)+y_t^*(t)\in\lambda\partial f_t(\overline{y}_t)$ *at* $t=\overline{m-h,m}$*; in the case* $h>\Delta$ *the relations are fulfiled:* $x_t^*(t-1)+x_t^*(t)+x_t^*(\Delta+t)\in\lambda\partial g_t(\overline{x}_t)$ at $t=\overline{1,k-h-1}$*;* $x_t^*(t-1)+x_t^*(t)+x_t^*(t+\Delta)+x_t^*(t+h)\in\lambda\partial g_t(\overline{x}_t)$ *at* $t=\overline{k-h,k-1-\Delta}$*,* $x_t^*(t-1)+x_t^*(t)+x_t^*(t+h)\in\lambda\partial g_t(\overline{x}_t)$ at $t=\overline{k-\Delta,k}$*,* $y_t^*(t-1)+y_t^*(t)+y_t^*(t+h)\in\lambda\partial f_t(\overline{y}_t)$ at $t=\overline{k+1,m-1-h}$*,* $y_t^*(t-1)+y_t^*(t)\in\lambda\partial f_t(\overline{y}_t)$ *at* $t=\overline{m-h,m}$ *.*

Proof. It is straightforward to check, that

$$
\begin{aligned}
I_{M_t}(\overline{z}) &= \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : (x_t,\ x_{t+1}) \in \\
&\in I_{gra_t(c(-\Delta+t),\cdot)}(\overline{x}_t,\ \overline{x}_{t+1})\} \text{ at } t=\overline{1,\Delta};\\
I_{M_t}(\overline{z}) &= \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : (x_{t-\Delta},\ x_t\ ,\ x_{t+1}) \in \\
&\in I_{gra_t}(\overline{x}_{t-\Delta},\ \overline{x}_t\ ,\ \overline{x}_{t+1})\} \quad \text{at } t=\overline{\Delta+1,k-1};\\
I_{M_k}(\overline{z}) &= \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : \\
&\quad (x_{k-h},x_k,y_{k+1}) \in I_{grb_k(G(\cdot),G(\cdot))}\left(\overline{x}_{k-h},\overline{x}_k,\overline{y}_{k+1}\right)\},\\
I_{M_t}(\overline{z}) &= \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : \\
&\quad (x_{t-h}\ ,y_t\ ,y_{t+1}) \in I_{grb_t(G(\cdot),\cdot)}\left(\overline{x}_{t-h}\ ,\overline{y}_t\ ,\overline{y}_{t+1}\right)\}\\
&\quad \text{at } t=\overline{k+1,k+h};\\
I_{M_t}(\overline{z}) &= \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : \\
&\quad (y_{t-h}\ ,y_t\ ,y_{t+1}) \in I_{grb_{k+h+1}}\left(\overline{y}_{t-h}\ ,\overline{y}_t\ ,\overline{y}_{t+1}\right)\},\\
&\quad \text{at } t=\overline{k+h+1,m-1}\ ;\\
I_{M_m}(\overline{z}) &= \{(x_1,...,x_k\ ,\ y_{k+1},...,y_m) \in X^k \times Y^s : y_m \in I_C(\overline{y}_m)\}.
\end{aligned}
$$

If $T_{M_0}(\overline{z}) \cap \left(\bigcap_{i=1}^{m} I_{M_i}(\overline{z})\right) = \varnothing$, then according to lemma 5.11 [2, p.37] we can find the linear functionals $\omega_i^* \in N_{M_i}(\overline{z})$, $i=0,1,...,m$, not all of which are equal to zero, such that $\omega_0^*+\omega_1^*+...+\omega_m^*=0$. Then we obtain that at $\lambda=0$ the statement of theorem 2 be satisfied.

Let $T_{M_0}(\overline{z}) \cap \left(\bigcap_{i=1}^{m} I_{M_i}(\overline{z})\right) \neq \varnothing$. Then the conditions of corollary 2 be satisfied. Then from theorem 1 follow, that at $\lambda=1$ the statement of theorem 2 be satisfied. The theorem is proved.

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