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MARCHAUD'S TYPE INEQUALITIES FOR CONVOLUTION OF TWO PERIODICAL FUNCTIONS IN $L_p(\mathbb{T})$, I

Abstract

In the paper the upper estimations of smoothness L_r-module $\omega_m(h;\delta)_r$ of order m of the convolution $h = f * g$ of two 2π periodic functions $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$ are obtained by means of the product expression $\omega_l(f;\delta)_p\,\omega_k(g;\delta)_q$ of smoothness modules of these functions, where $m, l, k \in \mathbb{N}$, $p, q \in [1, \infty]$, $1/r = 1/p + 1/q - 1 \geq 0$, $\mathbb{T} = (-\pi, \pi]$. In particular, it is proved in the case $p, q \in (1, \infty)$ that the obtained estimations are exact in the terms of order on classes of convolutions with given majorants of smoothness modules of f and g under some regularity of the majorants in the case $m < l + k$ and under arbitrary majorants in the case $m \geq l + k$.

In what follows we use the following notation.

- $L_p(\mathbb{T})$, $1 \leq p < \infty$, is the space of all measurable 2π periodic functions $f : \mathbb{R} \to \mathbb{C}$ with finite L_p-norm $\|f\|_p = \left((2\pi)^{-1} \int_{\mathbb{T}} |f(x)|^p\,dx\right)^{1/p} < \infty$.
- $C(\mathbb{T}) \equiv L_\infty(\mathbb{T})$ is the space of all continuous 2π periodic functions with uniform norm $\|f\|_\infty \equiv \max\{|f(x)| : x \in \mathbb{T}\}$.
- $E_n(f)_p$ is the best approximation of a function f in the metric of $L_p(\mathbb{T})$ by the trigonometric polynomials of order $\leq n \in \mathbb{Z}_+$.
- $S_n(f;\cdot)$ is the partial sum of order $n \in \mathbb{Z}_+$ of the Fourier-Lebesgue series of a function $f \in L_1(\mathbb{T}) : S_n(f;x) = \sum_{|\nu|=0}^{n} c_\nu(f)\,e^{i\nu x}$, $x \in \mathbb{T}$.
- $\omega_l(f;\delta)_p$ is the smoothness module of order l of a function $f \in L_p(\mathbb{T})$:
  $\omega_l(f;\delta)_p = \sup\left\{\left\|\Delta_t^l f\right\|_p : t \in \mathbb{R},\ |t| \leq \delta\right\}$, $l \in \mathbb{N}$, $\delta \geq 0$, where $\Delta_t^l f(x) = \sum_{\nu=0}^{l} (-1)^{l-\nu} \binom{l}{\nu} f\ (x + \nu t)$, $x \in \mathbb{R}$.
- M_0 is the class of all sequences $\varepsilon = \{\varepsilon_n\}_{n=1}^{\infty}$ such that $0 < \varepsilon_n \downarrow 0$ $(n \uparrow \infty)$.
- $E_p[\varepsilon] = \{f \in L_p(\mathbb{T}) : E_{n-1}(f)_p \leq \varepsilon_n,\ n \in \mathbb{N}\}$ for $p \in [1, \infty]$ and $\varepsilon \in M_0$.
- $\Omega_l(0, \pi] \equiv \Omega_l$ is the class of all functions $\omega(\delta)$ defined on $(0, \pi]$ and satisfying the conditions: $0 < \omega(\delta) \downarrow 0$ $(\delta \downarrow 0)$ and $\delta^{-l}\omega(\delta) \downarrow (\delta \uparrow)$.
- $H_p^l[\omega] = \{f \in L_p(\mathbb{T}) : \omega_l(f;\delta)_p \leq \omega(\delta),\quad \delta \in (0, \pi]\}$.

The convolution $h = f * g$ of $f \in L_1(\mathbb{T})$ and $g \in L_1(\mathbb{T})$ is defined by the formula: $h(x) = (f * g)(x) = (1/2\pi) \int_{\mathbb{T}} f(x - y)\,g(y)\,dy$; it is known (see f.e. [1], v.1, § 2.1, pp.64-65, [2], v.1, § 3.1, pp.65-66) that the function h is defined almost everywhere, 2π periodic, measurable and $\|h\|_1 \leq \|f\|_1 \|g\|_1$ (whence it follows in particular that

$h = f * g \in L_1(\mathbb{T})$). The last statement is a particular case of the following result known as the W.Young's inequality (see, f.e. [1], v.1, Theorem (1.15), pp.67-68; [2], v.2, Theorem 13.6.1, pp.176-177; [2], v.1, Theorem 3.1.4, p.70, Theorem 3.1.6, p.72). Given $p \in (1,\infty)$, let $p' = p/(p-1)$, $p' = 1$ for $p = \infty$ and $p' = \infty$ for $p = 1$.

Theorem A. *Let* $p,q \in [1,\infty]$, $f \in L_p(\mathbb{T})$ *and* $g \in L_q(\mathbb{T})$, $h = f * g$, $1/r = 1/p + 1/q - 1$. *Then*

- *If* $1/r > 0$ *then* h *belongs to* $L_r(\mathbb{T})$ *and* $\|h\|_r \le \|f\|_p \|g\|_q$.
- *If* $1/r = 0$ *then* h *belongs to* $C(\mathbb{T}) \equiv L_\infty(\mathbb{T})$ *and* $\|h\|_\infty \le \|f\|_p \|g\|_{p'}$.

Recall that the Fourier coefficients $c_n(h)$ of $h = f * g$ of two arbitrary functions $f \in L_1(\mathbb{T})$ and $g \in L_1(\mathbb{T})$ are calculated by the formula (see [1], v.1, Theorem (1.5), p.64; [2], v.1, p.66, formula (3.1.5)) $c_n(h) = c_n(f * g) = c_n(f) \cdot c_n(g)$ for every $n \in \mathbb{Z}$.

The upper estimation of the smoothness module $\omega_k(\psi;\delta)_p$ of $\psi \in L_p(\mathbb{T})$ by means of $\omega_l(\psi;\delta)_p$ is called the *Marchaud inequality* (without derivatives) in $L_p(\mathbb{T})$, where $k,l \in \mathbb{N}$, $k < l$, $1 \le p \le \infty$. For the first time a similar estimation for the case of the real functions ψ continuous on $[0,1]$ with uniform norm $\|\psi\| = \max\{|\psi(x)| : x \in [0,1]\}$ appeared in [3], Section 2.4.21, Inequality (20), p. 374 (see also [4], Theorem 3.3.1, Inequality (15), p. 164; [5], Section 3.3.2, Inequalities (11) and (12) , pp. 117 and 119; [6], Proposition 3.1, p. 291; [7], Theorem 2.8.1, Inequality (8.2), p. 47).

$$\omega_k(\psi;\delta) \le C_1(k,l)\delta^k \left(\int_\delta^1 t^{-(k+1)} \omega_l(\psi;t)\,dt + \|\psi\| \right), \ \delta \in (0,1], \tag{1}$$

where $\omega_k(\psi;\delta) = \max\left\{ \left|\Delta_t^k \psi(x)\right| : 0 \le x \le 1 - kt, 0 \le t \le \delta \right\}$, $0 < \delta \le 1/k$.

Later, the other proof of (1) (with constant $1/l$ instead of 1 for the upper bound of the integral and for $\delta \le 1/2k$) was given in [8], Section 4.1, Inequality (34), p. 741, by applying an result of approximation of $\psi \in C[0,1]$ by piecewise polynomial functions (splines). The example of a function that shows that (1) is an exact estimation in the sense of order was given in [9], formula (5) and Example 5, pp. 195 and 198 (see also [4], Section 3.3, formula (21), p. 168, and Section 3.5, p. 191).

In the periodic case the estimation

$$\omega_k(\psi;\delta)_p \le C_2(k,l,p)\delta^k \left(\int_\delta^{2\pi} t^{-(\sigma k+1)} \omega_l^\sigma(\psi;t)_p\,dt \right)^{1/\sigma}, \ \delta \in (0,\pi], \tag{2}$$

is an analogue of the Marchaud inequality, where $\sigma = \sigma(p) = \min\{2,p\}$ under $p \in [1,\infty)$ and $\sigma(\infty) = 1$ (see [10], Theorem 3, Inequalities (27), p. 130, case $l = k+1$; also [6], Section 3, Inequality (3.10), p. 293; [7], Theorem 2.8.4, Inequalities (8.14) and (8.15), pp. 49 and 50).

Inequality 2 is a consequence of the estimation

$$\omega_k(\psi;\pi/n)_p \le C_3(k,l,p)n^{-k} \left(\sum\nolimits_{\nu=1}^n \nu^{\sigma k-1} \omega_l^\sigma(\psi;\pi/\nu)_p \right)^{1/\sigma}, \ n \in \mathbb{N}, \tag{3}$$

(see Remark 1). This estimation is received by applying the inequality

$$E_{n-1}(\psi)_p \le C_4(l)\omega_l(\psi;\pi/n)_p,\ n\in\mathbb{N}, \tag{4}$$

of the so called direct theorem "without derivatives" of the approximation theory of periodic functions in $L_p(\mathbb{T})$ in the inequality

$$\omega_k(\psi;\pi/n)_p \le C_5(k,p)n^{-k}\left(\sum_{\nu=1}^{n}\nu^{\sigma k-1}E_{\nu-1}^{\sigma}(\psi)_p\right)^{1/\sigma},\ n\in\mathbb{N}, \tag{5}$$

of the so called inverse theorem "without derivatives" of the approximation theory of periodic functions in $L_p(\mathbb{T})$.

The inequalities (4) and (5) are well known and given in many monographs on the approximation theory (see for instance [4], Sections 4.2 and 5.4; [5], Sections 5.1, 5.11, and 6.1; [7], Sections 7.1-7.3, and their references). The historic review of appearance of (4) and (5), and of their exactness in the sense of order on the classes $H_p^l[\omega]$ and $E_p[\varepsilon]$, respectively, are given by the author in [11] and [12].

The estimation (2) or the equivalent estimation (3) (see Remark 1) is exact in the sense of order on the class $H_p^l[\omega]$ for all $p\in[1,\infty]$, namely, for each $p\in[1,\infty]$ and $\omega\in\Omega_l$ there is an individual function $\psi_0(\cdot;p;\omega)\in L_p(\mathbb{T})$ with $\omega_l(\psi_0;\delta)_p\le\omega(\delta)$, $\delta\in(0,\pi]$, such that

$$\omega_k(\psi_0;\pi/n)_p \ge C_6(k,l,p)n^{-k}\left(\sum_{\nu=1}^{n}\nu^{\sigma k-1}\omega^{\sigma}(\pi/\nu)\right)^{1/\sigma},\ n\in\mathbb{N}. \tag{6}$$

The examples of functions for which (6) holds in the case $p=1$ and $p=\infty$ are given in [13], Lemma 3, p. 176 (see also [14], Lemma 5, p. 75, case $p=\infty$, and [15], Theorem 14, p. 28, case $p=1$). The corresponding example for $1\le p<\infty$ (for $p=1$ this example differs from the function in [13] and [15]) was given in [16], Proposition 1, Lemmas 1 and 2, p. 209. Note that the assertion of the validity of (6) in the integral form for all $p\in[1,\infty]$ and arbitrary $\omega\in\Omega_l$ was announced by the author in [17], Lemma 3, p. 1303. The complete proof of this assertion was given in [18], Lemma 3.8, p. 75. Examples of functions $\psi_0(\cdot;p;\omega)$ for (6) are also given by the author in [19-22].

In the present paper the upper estimations of $\omega_m(f*g;\delta)_r$ are obtained by the products $\omega_l(f;\delta)_p\omega_k(g;\delta)_q$, where $m,l,k\in\mathbb{N}$, $p,q\in[1,\infty]$ and $r=pq/(p+q-pq)\in[1,\infty]$. In the case $p,q\in(1,\infty)$ the exactness of obtained estimations in the sense of order is proved for the classes of convolutions with given majorants of smoothness modules of functions f and g under condition of some regularity of these majorants in the case $m<l+k$ and for arbitrary majorants in the case $m\ge l+k$.

The following statement is an analogue of the inverse theorem of the approximation theory for convolution of two periodic functions.

Theorem B ([12], Theorem 1). *Let* $p,q\in[1,\infty]$, $1/r=1/p+1/q-1\ge 0$, $f\in L_p(\mathbb{T})$, $g\in L_q(\mathbb{T})$, $h=f*g$, $m\in\mathbb{N}$. *Then*

(i) *If* $1/r>0$ *then* $h\in L_r(\mathbb{T})$, $r\in[1,\infty)$, *and, for* $\theta=\theta(r)=\min\{2,r\}$,

$$\omega_m(h;\pi/n)_r \le C_7(m,r)n^{-m}\left(\sum_{\nu=1}^{n}\nu^{\theta m-1}E_{\nu-1}^{\theta}(f)_p E_{\nu-1}^{\theta}(g)_q\right)^{1/\theta},\ n\in\mathbb{N}.$$

(ii) *If* $1/r=0$ *then* $h\in C(\mathbb{T})\equiv L_\infty(\mathbb{T})$, $q=p'$ *and*

$$\omega_m(h;\pi/n)_\infty \le C_7(m,r)n^{-m}\sum_{\nu=1}^{n}\nu^{m-1}E_{\nu-1}(f)_p E_{\nu-1}(g)_q,\ n\in\mathbb{N},$$

where $C_7(m,r)$ *is a positive constant depending only on* m *and* r.

Theorem 1. *Let* $p,q\in[1,\infty]$, $r=pq/(p+q-pq)\in[1,\infty]$, $f\in L_p(\mathbb{T})$, $g\in L_q(\mathbb{T})$, $h=f*g$, $k,l,m\in\mathbb{N}$, $\theta=\theta(r)=\min\{2,r\}$ *for* $r\in[1,\infty)$ *and* $\theta(\infty)=1$. *Then* $h\in L_r(\mathbb{T})$ *and the following estimations hold* $(n\in\mathbb{N})$*:*

(i) *for* $m<l+k$

$$\omega_m(h;\pi/n)_r\le C_8(k,l,m,r)\,n^{-m}\left(\sum_{\nu=1}^{n}\nu^{\theta m-1}\omega_l^\theta(f;\pi/\nu)_p\,\omega_k^\theta(g;\pi/\nu)_q\right)^{1/\theta};$$

(ii) *for* $m>l+k$

$$\omega_m(h;\pi/n)_r\le 2^{l+k}C_8(k,l,m,r)\,\omega_l(f;\pi/n)_p\,\omega_k(g;\pi/n)_q;$$

(iii) for $m=l+k$

$$\omega_m(h;\pi/n)_r\le 2^{m}C_8(k,l,m,r)\,\omega_l(f;\pi/n)_p\,\omega_k(g;\pi/n)_q\,(\ln(en))^{1/\theta},$$
$$\omega_{m+1}(h;\pi/n)_r\le 2^{m}C_8(k,l,m+1,r)\,\omega_l(f;\pi/n)_p\,\omega_k(g;\pi/n)_q.$$

Proof. In virtue of inequalities (i) and (ii) of Theorem B and (4), we have that

$$\begin{aligned}\omega_m(h;\pi/n)_r&\le C_7(m,r)\,n^{-m}\left(\sum\nolimits_{\nu=1}^{n}\nu^{\theta m-1}E_{\nu-1}^\theta(f)_p\,E_{\nu-1}^\theta(g)_q\right)^{1/\theta}\\&\le C_8n^{-m}\left(\sum\nolimits_{\nu=1}^{n}\nu^{\theta m-1}\omega_l^\theta(f;\pi/\nu)_p\,\omega_k^\theta(g;\pi/\nu)_q\right)^{1/\theta},\end{aligned}\tag{7}$$

whence the estimation (i) follows with $C_8=C_8(k,l,m,r)=C_7(m,r)\,C_4(l)C_4(k)$.

Further, applying well known property of smoothness L_p-module of order l (see for instance [5], p. 116, inequality (6))

$$\delta_2^{-l}\omega_l(f;\delta_2)_p\le 2^l\delta_1^{-l}\omega_l(f;\delta_1)_p \text{ for } 0<\delta_1<\delta_2\tag{8}$$

we obtain that

$$\begin{aligned}&n^{-m}\left(\sum\nolimits_{\nu=1}^{n}\nu^{\theta m-1}\omega_l^\theta(f;\pi/\nu)_p\,\omega_k^\theta(g;\pi/\nu)_q\right)^{1/\theta}\\&=n^{-m}\left(\sum\nolimits_{\nu=1}^{n}\left(\nu^l\omega_l(f;\pi/\nu)_p\right)^\theta\left(\nu^k\omega_k(g;\pi/\nu)_q\right)^\theta\nu^{\theta[m-(l+k)]-1}\right)^{1/\theta}\\&\le 2^{l+k}\omega_l(f;\pi/n)_p\,\omega_k(g;\pi/n)_q\,n^{-m+l+k}\left(\sum\nolimits_{\nu=1}^{n}\nu^{\theta[m-(l+k)]-1}\right)^{1/\theta}\\&\le\begin{cases}2^{l+k}\omega_l(f;\pi/n)_p\,\omega_k(g;\pi/n)_q & \textit{for } m>l+k,\\ 2^{l+k}\omega_l(f;\pi/n)_p\,\omega_k(g;\pi/n)_q\,(\ln(en))^{1/\theta} & \textit{for } m=l+k.\end{cases}\end{aligned}$$

Taking into account this estimation in (7), we have (ii) and the first estimation in (iii).

At last, applying (8) we have by (7) under $m=l+k$ that for $C_8=C_8(k,l,m+1,r)$

$$\begin{aligned}\omega_{m+1}(h;\pi/n)_r&\le C_8n^{-(m+1)}\left(\sum\nolimits_{\nu=1}^{n}\nu^{\theta(m+1)-1}\omega_l^\theta(f;\pi/\nu)_p\,\omega_k^\theta(g;\pi/\nu)_q\right)^{1/\theta}\\&\le 2^{l+k}C_8n^{-(m+1)+l+k}\omega_l(f;\pi/n)_p\,\omega_k(g;\pi/n)_q\left(\sum\nolimits_{\nu=1}^{n}\nu^{\theta-1}\right)^{1/\theta}\\&\le 2^{l+k}C_8\omega_l(f;\pi/n)_p\,\omega_k(g;\pi/n)_q,\end{aligned}$$

from that the second estimation in (iii) follows. Note that one can easily reach this by (ii) since $m+1>l+k$ if $m=l+k$.

Theorem 1 is proved.

Remark 1. The estimation (i) of Theorem 1 admits an equivalent formulation:

(i) *If* (i) *of Theorem* 1 *holds for some constant* $C_8=C_8\left(k,l,m,r\right)$, *then for every* $\delta\in(0,\pi]$

$$\omega_m\left(h;\delta\right)_r\le C_9\delta^m\left(\int_\delta^{2\pi}t^{-(\theta m+1)}\omega_l^\theta\left(f;t\right)_p\omega_k^\theta\left(g;t\right)_q dt\right)^{1/\theta} \tag{9}$$

with constant $C_9=C_9(k,l,m,r)<2^{2m}C_8$.

(ii) *If* (9) *holds for some constant* $C_9=C_9\left(k,l,m,r\right)$, *then for every* $n\in\mathbb{N}$

$$\omega_m\left(h;\pi/n\right)_r\le C_{10}n^{-m}\left(\sum_{\nu=1}^n\nu^{\theta m-1}\omega_l^\theta\left(f;\pi/\nu\right)_p\omega_k^\theta\left(g;\pi/\nu\right)_q\right)^{1/\theta}$$

with $C_{10}=C_{10}(k,l,m,r)<2^mC_9\left\{2^{\theta m-1}+\left(2^{\theta m}-1\right)2^{\theta(l+k)}/\left(\theta m2^{\theta m}\right)\right\}^{1/\theta}$.

Proof. For every $\delta\in(0,\pi]$ there exists $n\in\mathbb{N}$ such that $\pi/(n+1)<\delta\le\pi/n$. Put $\psi(\delta)=\omega_l\left(f;\delta\right)_p\omega_k\left(g;\delta\right)_q$, $\delta\in(0,\pi]$.

(i) Since $\psi(\delta)\uparrow(\delta\uparrow)$ then (for $\psi_n=\psi(\pi/n)$)

$$\begin{aligned}\int_\delta^\pi t^{-(\theta m+1)}\psi^\theta(t)dt &\ge \int_{\pi/n}^\pi t^{-(\theta m+1)}\psi^\theta(t)dt=\sum_{\nu=1}^{n-1}\int_{\pi/(\nu+1)}^{\pi/\nu}t^{-(\theta m+1)}\psi^\theta(t)dt\\ &\ge\left(\theta m\pi^{\theta m}\right)^{-1}\sum_{\nu=1}^{n-1}\left((\nu+1)^{\theta m}-\nu^{\theta m}\right)\psi_{\nu+1}^\theta\ge\pi^{-\theta m}\sum_{\nu=1}^{n-1}\nu^{\theta m-1}\psi_{\nu+1}^\theta\\ &=\pi^{-\theta m}\sum_{\nu=2}^n(\nu-1)^{\theta m-1}\psi_\nu^\theta\ge\pi^{-\theta m}2^{-(\theta m-1)}\sum_{\nu=2}^n\nu^{\theta m-1}\psi_\nu^\theta,\end{aligned}$$

and

$$\int_\pi^{2\pi}t^{-(\theta m+1)}\psi^\theta(t)dt\ge\psi^\theta(\pi)\int_\pi^{2\pi}t^{-(\theta m+1)}dt=(\theta m\pi^{\theta m}2^{\theta m})^{-1}(2^{\theta m}-1)\psi_1^\theta.$$

In virtue of estimations obtained we have that

$$\begin{aligned}\delta^{\theta m}\int_\delta^{2\pi}\frac{\psi^\theta(t)}{t^{\theta m+1}}dt &>(\pi/(n+1))^{\theta m}\left\{\int_{\pi/n}^\pi\frac{\psi^\theta(t)}{t^{\theta m+1}}dt+\int_\pi^{2\pi}\frac{\psi^\theta(t)}{t^{\theta m+1}}dt\right\}\\ &\ge(4n)^{-\theta m}\left\{2\sum_{\nu=2}^n\nu^{\theta m-1}\psi^\theta(\pi/\nu)+(\theta m)^{-1}(2^{\theta m}-1)\psi^\theta(\pi)\right\}\\ &>(4n)^{-\theta m}\left\{\sum_{\nu=2}^n\nu^{\theta m-1}\psi^\theta(\pi/\nu)+\psi^\theta(\pi)\right\}=(4n)^{-\theta m}\sum_{\nu=1}^n\nu^{\theta m-1}\psi^\theta(\pi/\nu),\end{aligned}$$

whence

$$\begin{aligned}\omega_m\left(h;\delta\right)_r\le\omega_m\left(h;\pi/n\right)_r &\le C_8(k,l,m,r)n^{-m}\left(\sum_{\nu=1}^n\nu^{\theta m-1}\psi^\theta(\pi/\nu)\right)^{1/\theta}\\ &\le C_9(k,l,m,r)\delta^m\left(\int_\delta^{2\pi}t^{-(\theta m+1)}\psi^\theta(t)dt\right)^{1/\theta},\quad\delta\in(0,\pi].\end{aligned}$$

(ii) Taking into account that

$$\begin{aligned}\int_\delta^\pi t^{-(\theta m+1)}\psi^\theta(t)dt &<\int_{\pi/(n+1)}^\pi t^{-(\theta m+1)}\psi^\theta(t)dt=\sum_{\nu=1}^n\int_{\pi/(\nu+1)}^{\pi/\nu}t^{-(\theta m+1)}\psi^\theta(t)dt\\ &\le\left(\theta m\pi^{\theta m}\right)^{-1}\sum_{\nu=1}^n\left((\nu+1)^{\theta m}-\nu^{\theta m}\right)\psi_\nu^\theta\le\pi^{-\theta m}2^{\theta m-1}\sum_{\nu=1}^n\nu^{\theta m-1}\psi_\nu^\theta\end{aligned}$$

and

$$\int_{\pi}^{2\pi} t^{-(\theta m+1)}\psi^{\theta}(t)dt \le \psi^{\theta}(2\pi)\int_{\pi}^{2\pi} t^{-(\theta m+1)}dt = (\theta m\pi^{\theta m}2^{\theta m})^{-1}(2^{\theta m}-1)\psi^{\theta}(2\pi)$$
$$\le (\theta m\pi^{\theta m}2^{\theta m})^{-1}(2^{\theta m}-1)2^{\theta(l+k)}\psi_1^{\theta},$$

we have that

$$\delta^{\theta m}\int_{\delta}^{2\pi}\tfrac{\psi^{\theta}(t)}{t^{\theta m+1}}dt < (\pi/n)^{\theta m}\left\{\int_{\pi/(n+1)}^{\pi}\tfrac{\psi^{\theta}(t)}{t^{\theta m+1}}dt + \int_{\pi}^{2\pi}\tfrac{\psi^{\theta}(t)}{t^{\theta m+1}}dt\right\}$$
$$\le n^{-\theta m}\left\{2^{\theta m-1}\sum_{\nu=1}^{n}\nu^{\theta m-1}\psi_{\nu}^{\theta} + (\theta m 2^{\theta m})^{-1}(2^{\theta m}-1)2^{\theta(l+k)}\psi_1^{\theta}\right\}$$
$$\le \left\{2^{\theta m-1} + (\theta m 2^{\theta m})^{-1}(2^{\theta m}-1)2^{\theta(l+k)}\right\} n^{-\theta m}\sum_{\nu=1}^{n}\nu^{\theta m-1}\psi^{\theta}(\pi/\nu),$$

whence

$$\omega_m(h;\pi/n)_r \le \omega_m(h;2\pi/(n+1))_r \le \omega_m(h;2\delta)_r \le 2^m\omega_m(h;\delta)_r$$
$$\le 2^m C_9(k,l,m,r)\,\delta^m\left(\int_{\delta}^{2\pi} t^{-(\theta m+1)}\psi^{\theta}(t)\,dt\right)^{1/\theta}$$
$$\le C_{10}(k,l,m,r)\,n^{-m}\left(\sum\nolimits_{\nu=1}^{n}\nu^{\theta m-1}\psi^{\theta}(\pi/\nu)\right)^{1/\theta}.$$

The proof is complete.

Corollary 1. *Let* $\omega_l(f;\delta)_p = O(\delta^{\alpha})$, $\alpha\in(0,l]$, and $\omega_k(g;\delta)_q = O(\delta^{\beta})$, $\beta\in(0,k]$, $\delta\in(0,\pi]$, *in the conditions of Theorem 1 for* $m<l+k$. *Then*

$$\text{(i)}\ \ \omega_m(h;\delta)_r = \begin{cases} O\left(\delta^{\alpha+\beta}\right) & (\alpha+\beta<m), \\ O\left(\delta^m(\ln(\pi e/\delta))^{1/\theta}\right) & (\alpha+\beta=m), \\ O\left(\delta^m\right) & (\alpha+\beta>m). \end{cases}$$

(ii) $\omega_{m+1}(h;\delta)_r = O(\delta^m)$ *if* $\alpha+\beta=m$.

Proof of Corollary 1 is similar to the proof of upper [12], Theorem 2, pp. 27-28.

The following assertion shows that the logarithm multiplier is needless in the first estimation in (iii) of Theorem 1.

Theorem 2. *Let* $p,q\in[1,\infty]$, $r=pq/(p+q-pq)\in[1,\infty]$, $f\in L_p(\mathbb{T})$, $g\in L_q(\mathbb{T})$, $k,l\in\mathbb{N}$. *Then* $f*g\in L_r(\mathbb{T})$ *and the following estimation holds:*

$$\omega_{l+k}(f*g;\delta)_r \le C_{11}(l,k)\omega_l(f;\delta)_p\omega_k(g;\delta)_q, \quad \delta\in(0,\pi]. \tag{10}$$

Proof. Put $m=l+k$. For every $\delta\in(0,\pi]$ there is $n\in\mathbb{N}$ such that $\pi/(n+1)<\delta\le\pi/n$. Let $T_{n,p}(f)$ and $T_{n,q}(g)$ be the best approximation polynomials of f and g in $L_p(\mathbb{T})$ and $L_q(\mathbb{T})$, respectively. Hence $\|f-T_{n,p}(f)\|_p = E_n(f)_p$ and $\|g-T_{n,q}(g)\|_q = E_n(g)_q$. Since the smoothness module is semi-additive, we have that

$$\omega_m(f*g;\delta)_r \le \omega_m(f*g-T_{n,p}(f)*T_{n,q}(g);\delta)_r + \omega_m(T_{n,p}(f)*T_{n,q}(g);\delta)_r.$$

Let $\sigma_1 = \omega_m(f * g - T_{n,p}(f) * T_{n,q}(g);\delta)_r$. Since the convolution is commutative and distributive, we have that

$$f * g - T_{n,p}(f) * T_{n,q}(g) = (f - T_{n,p}(f)) * (g - T_{n,q}(g)) \\ + T_{n,p}(f) * (g - T_{n,q}(g)) + T_{n,q}(g) * (f - T_{n,p}(f)),$$

whence

$$\sigma_1 \le \omega_m((f - T_{n,p}(f)) * (g - T_{n,q}(g));\delta)_r \\ + \omega_m(T_{n,p}(f) * (g - T_{n,q}(g));\delta)_r + \omega_m(T_{n,q}(g) * (f - T_{n,p}(f));\delta)_r \\ = \sigma_{11} + \sigma_{12} + \sigma_{13}.$$

By Young's inequality (see Theorem A), we have that

$$\sigma_{11} = \omega_m((f - T_{n,p}(f)) * (g - T_{n,q}(g));\delta)_r \\ \le 2^m \|(f - T_{n,p}(f)) * (g - T_{n,q}(g))\|_r \\ \le 2^m \|f - T_{n,p}(f)\|_p \|g - T_{n,q}(g)\|_q = 2^m E_n(f)_p E_n(g)_q.$$

Applying Young's inequality and taking into account that

$$\omega_m(T_{n,p}(f);\delta)_p \le \omega_m(T_{n,p}(f) - f;\delta)_p + \omega_m(f;\delta)_p \\ \le 2^m \|f - T_{n,p}(f)\|_p + \omega_m(f;\delta)_p = 2^m E_n(f)_p + \omega_m(f;\delta)_p,$$

we obtain that

$$\sigma_{12} = \omega_m(T_{n,p}(f) * (g - T_{n,q}(g));\delta)_r \le \omega_m(T_{n,p}(f);\delta)_p \|g - T_{n,q}(g)\|_q \\ \le \left\{2^m E_n(f)_p + \omega_m(f;\delta)_p\right\} E_n(g)_q = 2^m E_n(f)_p E_n(g)_q + \omega_m(f;\delta)_p E_n(g)_q.$$

Similarly, we have that

$$\sigma_{13} = \omega_m(T_{n,q}(g) * (f - T_{n,p}(f));\delta)_r \le 2^m E_n(g)_q E_n(f)_p + \omega_m(g;\delta)_q E_n(f)_p.$$

By the estimations obtained for σ_{11}, σ_{12} and σ_{13}, we have that

$$\sigma_1 \le 2^m 3 E_n(g)_q E_n(f)_p + \omega_m(f;\delta)_p E_n(g)_q + \omega_m(g;\delta)_q E_n(f)_p. \tag{11}$$

Now we estimate $\sigma_2 = \omega_m(T_{n,p}(f) * T_{n,q}(g);\delta)_r$. We need the well known result of S. B. Stechkin (see [23], Inequality (3), p. 1511, case $p = \infty$; also [4], Theorem 5.2.1′, p. 217; [5], Section 4.4.8, pp. 228-230, case $p \in [1,\infty]$): *for every trigonometric polynomial T_n of order $n \in \mathbb{N}$, and for every $l \in \mathbb{N}$,*

$$\left\|T_n^{(l)}\right\|_p \le n^l (2\sin(n\eta/2))^{-l} \left\|\Delta_\eta^l T_n\right\|_p, \quad \eta \in (0, 2\pi/n). \tag{12}$$

Setting $T_n = T_{n,p}(f)$ and $\eta = \pi/n$ in (12), we obtain that

$$\left\|T_{n,p}^{(l)}(f)\right\|_p \le 2^{-l} n^l \left\|\Delta_{\pi/n}^l T_{n,p}(f)\right\|_p \le 2^{-l} n^l \omega_l(T_{n,p}(f);\pi/n)_p \\ \le 2^{-l} n^l \left\{2^l E_n(f)_p + \omega_l(f;\pi/n)_p\right\} \le n^l \left\{E_n(f)_p + \omega_l(f;\pi/(n+1))_p\right\},$$

whence (for $\pi/(n+1) < \delta \leq \pi/n$)

$$\left\| T_{n,p}^{(l)}(f) \right\|_p \leq n^l \left\{ E_n(f)_p + \omega_l(f;\delta)_p \right\}. \tag{13}$$

Similarly, we have that

$$\left\| T_{n,q}^{(k)}(g) \right\|_q \leq n^k \left\{ E_n(g)_q + \omega_k(g;\delta)_q \right\}. \tag{14}$$

Applying Young's inequality and taking into account (13) and (14), we obtain (for $m = l + k$) that

$$\begin{aligned} \sigma_2 &\leq \delta^m \left\| (T_{n,p}(f) * T_{n,q}(g))^{(m)} \right\|_r \\ &= \delta^m \left\| T_{n,p}^{(l)}(f) * T_{n,q}^{(k)}(g) \right\|_r \leq \delta^m \left\| T_{n,p}^{(l)}(f) \right\|_p \left\| T_{n,q}^{(k)}(g) \right\|_q \\ &\leq \delta^m n^{l+k} \left\{ E_n(f)_p + \omega_l(f;\delta)_p \right\} \left\{ E_n(g)_q + \omega_k(g;\delta)_q \right\}, \end{aligned}$$

whence (for $\pi/(n+1) < \delta \leq \pi/n$)

$$\sigma_2 \leq \pi^m \left\{ E_n(f)_p + \omega_l(f;\delta)_p \right\} \left\{ E_n(g)_q + \omega_k(g;\delta)_q \right\}. \tag{15}$$

By (11) and (15), we have that

$$\begin{aligned} \omega_m(f*g;\delta)_r &\leq (2^m 3 + \pi^m) E_n(f)_p E_n(g)_q + \left(2^{m-l} + \pi^m\right) \omega_l(f;\delta)_p E_n(g)_q \\ &+ \left(2^{m-k} + \pi^m\right) \omega_k(g;\delta)_q E_n(f)_p + \pi^m \omega_l(f;\delta)_p \omega_k(g;\delta)_q, \end{aligned}$$

whence applying (4) we obtain that

$$\begin{aligned} \omega_m(f*g;\delta)_r &\leq (2^m 3 + \pi^m) C_4(l)\omega_l(f;\pi/(n+1))_p C_4(k)\omega_k(g;\pi/(n+1))_q \\ &+ \left(2^k + \pi^m\right) \omega_l(f;\delta)_p C_4(k)\omega_k(g;\pi/(n+1))_q \\ &+ \left(2^l + \pi^m\right) \omega_k(g;\delta)_q C_4(l)\omega_l(f;\pi/(n+1))_p + \pi^m \omega_l(f;\delta)_p \omega_k(g;\delta)_q \\ &\leq C_{11}(l,k)\omega_l(f;\delta)_p \omega_k(g;\delta)_q, \end{aligned}$$

where $C_{11} = (2^m 3 + \pi^m) C_4(l)C_4(k) + 2^k C_4(k) + 2^l C_4(l) + \pi^m (C_4(k) + C_4(l) + 1)$. The proof of the theorem is complete.

Remark 2. The estimation (ii) of Theorem 1 immediately follows from (10). Indeed, for $h = f * g$ and $m > l + k$ we have that

$$\omega_m(h;\pi/n)_r \leq 2^{m-(l+k)} \omega_{l+k}(h;\pi/n)_r \leq 2^{m-(l+k)} C_{11}(l,k)\omega_l(f;\pi/n)_p \omega_k(g;\pi/n)_q.$$

Remark 3. In the case $p, q \in (1,\infty)$ there is the other proof of (10) which is based on well known inequalities of M. Riesz (see [1], v. 1, Theorem 7.6.4, p. 423; [2], v. 2, Theorem 12.10.1, p. 120; [24], Theorems 8.20.1 and 8.20.2, pp. 593-594; [5], Section 3.12, Inequality (20), p. 183, and Section 5.11, Inequality (6), p. 339). They are

$$\|S_n(\psi)\|_p \leq C_{12}(p) \|\psi\|_p \text{ and } \|\psi - S_n(\psi)\|_p \leq C_{13}(p) E_n(\psi)_p \tag{16}$$

for $1<p<\infty$, $\psi\in L_p(\mathbb{T})$, $n\in\mathbb{Z}_+$, where $C_{13}(p)=1+C_{12}(p)$. Note that the right inequality of (16) is a simple consequence of the left one:

$$\|\psi-S_n(\psi)\|_p=\|\psi-T_{n,p}(\psi)+S_n(T_{n,p}(\psi))-S_n(\psi)\|_p$$
$$\le\|\psi-T_{n,p}(\psi)\|_p+\|S_n(T_{n,p}(\psi)-\psi)\|_p\le C_{13}(p)E_n(\psi)_p.$$

Now we present the other proof of (10). Let $m=l+k$, $h=f*g$, and let $n\in\mathbb{N}$ be such that $\pi/(n+1)<\delta\le\pi/n$, where $\delta\in(0,\pi]$. In virtue of well known properties of smoothness modules we have that

$$\omega_m(h;\delta)_r\le\omega_m(h-S_n(h);\delta)_r+\omega_m(S_n(h);\delta)_r$$
$$\le 2^m\|h-S_n(h)\|_r+\delta^m\left\|S_n^{(m)}(h)\right\|_r=\sigma_3+\sigma_4.$$

Estimate σ_3. In virtue of $S_n(f)*g=f*S_n(g)=S_n(f*g)=S_n(f)*S_n(g)$, we have that

$$h-S_n(h)=f*g-S_n(f*g)=(f-S_n(f))*(g-S_n(g)),$$

whence, by Young's inequality and the right inequality of (16), we obtain that

$$\|h-S_n(h)\|_r=\|(f-S_n(f))*(g-S_n(g))\|_r$$
$$\le\|f-S_n(f)\|_p\|g-S_n(g)\|_q\le C_{13}(p)E_n(f)_pC_{13}(q)E_n(g)_q.$$

Therefore

$$\sigma_3=2^m\|h-S_n(h)\|_r\le 2^mC_{13}(p)C_{13}(q)E_n(f)_pE_n(g)_q.$$

Now estimate σ_4. Since $m=l+k$ and $S_n^{(m)}(h)=S_n^{(l+k)}(f*g)=S_n^{(l)}(f)*S_n^{(k)}(g)$, applying Young's inequality, (12) for $\eta=\pi/n$ and the left inequality of (16) sequentially, we have that

$$\left\|S_n^{(m)}(h)\right\|_r=\left\|S_n^{(l)}(f)*S_n^{(k)}(g)\right\|_r\le\left\|S_n^{(l)}(f)\right\|_p\left\|S_n^{(k)}(g)\right\|_q$$
$$\le 2^{-(l+k)}n^{l+k}\left\|\Delta^l_{\pi/n}S_n(f)\right\|_p\left\|\Delta^k_{\pi/n}S_n(g)\right\|_q$$
$$=2^{-m}n^m\left\|S_n\left(\Delta^l_{\pi/n}f\right)\right\|_p\left\|S_n\left(\Delta^k_{\pi/n}g\right)\right\|_q$$
$$\le 2^{-m}n^mC_{12}(p)C_{12}(q)\left\|\Delta^l_{\pi/n}f\right\|_p\left\|\Delta^k_{\pi/n}g\right\|_q$$
$$\le 2^{-m}n^mC_{12}(p)C_{12}(q)\omega_l(f;\pi/n)_p\,\omega_k(g;\pi/n)_q,$$

whence ($\delta\le\pi/n$)

$$\sigma_4=\delta^m\left\|S_n^{(m)}(h)\right\|_r\le 2^{-m}\pi^mC_{12}(p)C_{12}(q)\omega_l(f;\pi/n)_p\,\omega_k(g;\pi/n)_q.$$

Taking into account the estimations for σ_3 and σ_4 and applying (4), we obtain that ($\pi/(n+1)<\delta$)

$$\omega_m(h;\delta)_r$$
$$\le 2^mC_{13}(p)C_{13}(q)E_n(f)_pE_n(g)_q+2^{-m}\pi^mC_{12}(p)C_{12}(q)\omega_l(f;\pi/n)_p\,\omega_k(g;\pi/n)_q$$
$$\le 2^mC_{13}(p)C_{13}(q)C_4(l)C_4(k)\omega_l(f;\pi/(n+1))_p\,\omega_k(g;\pi/(n+1))_q$$
$$+2^{-m}\pi^mC_{12}(p)C_{12}(q)2^{l+k}\omega_l(f;\pi/(n+1))_p\,\omega_k(g;\pi/(n+1))_q$$
$$\le\{2^mC_{13}(p)C_{13}(q)C_4(l)C_4(k)+\pi^mC_{12}(p)C_{12}(q)\}\,\omega_l(f;\delta)_p\,\omega_k(g;\delta)_q,$$

whence

$$\omega_{l+k}\left(f*g;\delta\right)_r \le C_{14}(p,q,l,k)\omega_l\left(f;\delta\right)_p \omega_k\left(g;\delta\right)_q, \quad \delta\in(0,\pi],$$

where $C_{14}(p,q,l,k)=2^{l+k}C_{13}(p)C_{13}(q)C_4(l)C_4(k)+\pi^{l+k}C_{12}(p)C_{12}(q)$. The proof is complete.

Given numbers $\sigma\in[1,\infty)$ and $l\in\mathbb{N}$, we put

$$D^{(\sigma)}=\left\{\lambda=\{\lambda_n\}_{n=1}^{\infty}\in M_0:\sum\nolimits_{n=1}^{\infty} n^{-1}\lambda_n^{\sigma}<\infty\right\},$$
$$B^{(\sigma)}=\left\{\lambda=\{\lambda_n\}_{n=1}^{\infty}\in M_0:\left(\sum\nolimits_{\nu=n+1}^{\infty}\nu^{-1}\lambda_\nu^{\sigma}\right)^{1/\sigma}=O\left(\lambda_n\right),\ n\in\mathbb{N}\right\},$$
$$B_l^{(\sigma)}=\left\{\lambda=\{\lambda_n\}_{n=1}^{\infty}\in M_0:n^{-l}\left(\sum\nolimits_{\nu=1}^{n}\nu^{\sigma l-1}\lambda_\nu^{\sigma}\right)^{1/\sigma}=O\left(\lambda_n\right),\ \ n\in\mathbb{N}\right\}.$$

Given $\alpha\in(0,\infty)$, let $M_0(\alpha)$ be the set of all sequences $\lambda\in M_0$ such that $n^{\alpha}\lambda_n\downarrow$ $(n\uparrow)$. If $\lambda\in M_0(\alpha)$ then in virtue of estimations

$$\begin{gathered}\left(\sum\nolimits_{n=1}^{\infty}n^{-1}\lambda_n^{\sigma}\right)^{1/\sigma}=\left(\sum\nolimits_{n=1}^{\infty}(n^{\alpha}\lambda_n)^{\sigma}n^{-\sigma\alpha-1}\right)^{1/\sigma}\\ \le\lambda_1\left(\sum\nolimits_{n=1}^{\infty}n^{-\sigma\alpha-1}\right)^{1/\sigma}\le\lambda_1\left(1+(\sigma\alpha)^{-1}\right)^{1/\sigma},\\ \left(\sum\nolimits_{\nu=n+1}^{\infty}\nu^{-1}\lambda_\nu^{\sigma}\right)^{1/\sigma}=\left(\sum\nolimits_{\nu=n+1}^{\infty}(\nu^{\alpha}\lambda_\nu)^{\sigma}\nu^{-\sigma\alpha-1}\right)^{1/\sigma}\\ \le(n+1)^{\alpha}\lambda_{n+1}\left(\sum\nolimits_{\nu=n+1}^{\infty}\nu^{-\sigma\alpha-1}\right)^{1/\sigma}\le(n+1)^{\alpha}\lambda_{n+1}(\sigma\alpha)^{-1/\sigma}n^{-\alpha}\\ \le2^{\alpha}(\sigma\alpha)^{-1/\sigma}\lambda_{n+1}\le2^{\alpha}(\sigma\alpha)^{-1/\sigma}\lambda_n,\quad n\in\mathbb{N},\end{gathered}$$

one has the inclusions $M_0(\alpha)\subset D^{(\sigma)}$ and $M_0(\alpha)\subset B^{(\sigma)}$ for every $\sigma\in[1,\infty)$. Besides, it is obvious that $B^{(\sigma)}\subset D^{(\sigma)}$, $\sigma\in[1,\infty)$.

Lemma C ([11], Lemma 2). *Let* $p\in(1,\infty)$, $p'=p/(p-1)$, $l\in\mathbb{N}$ *and* $\lambda=\{\lambda_n\}\in M_0$. *Then the function* $f_0(x;p;\lambda)=\sum_{n=1}^{\infty}n^{-1/p'}\lambda_n e^{inx}$ *for* $x\in\mathbb{T}$, *satisfies the following conditions:*

(i) $f_0\in L_p(\mathbb{T})$ *for* $\lambda\in D^{(p)}$.

(ii) $E_{n-1}\left(f_0\right)_p=O\left(\lambda_n\right),\ n\in\mathbb{N}$, *for* $\lambda\in B^{(p)}$.

(iii) $\omega_l\left(f_0;\pi/n\right)_p=O\left(\lambda_n\right),\ n\in\mathbb{N}$, *for* $\lambda\in B_l^{(\sigma)}\cap B^{(p)}$ *and* $\sigma=\min\{2,p\}$.

Note that in (iii) of Lemma C one can put $\sigma=1$ since $B_l^{(1)}\subset B_l^{(\sigma)}$.

In what follows we need the following statements [12].

Lemma D ([12], Lemma 2). *Let* $r\in(1,2]$, $\psi\in L_r(\mathbb{T})$ *with Fourier series* $\psi(x)\sim\sum_{n\in\mathbb{Z}}c_n(\psi)e^{inx}$ *and* $m\in\mathbb{N}$. *Then*

$$n^{-m}\left(\sum\nolimits_{\nu=1}^{n}\nu^{rm+r-2}\left|c_\nu(\psi)\right|^r\right)^{1/r}\le C_{15}(m,r)\omega_m(\psi;\pi/n)_r,\quad n\in\mathbb{N}.$$

Lemma E ([12], Lemma 3). *Let* $\psi\in L_2(\mathbb{T})$ *have the Fourier series* $\psi(x)\sim\sum_{n=0}^{\infty}c_n(\psi)e^{inx}$ *and* $m\in\mathbb{N}$. *Then*

$$n^{-m}\left(\sum\nolimits_{\nu=1}^{n}\nu^{2m-1}E_{\nu-1}^2\left(\psi\right)_2\right)^{1/2}\le C_{16}(m)\omega_m(\psi;\pi/n)_2,\quad n\in\mathbb{N}.$$

Lemma F ([12], Lemma 4). *Let* $\psi \in C(\mathbb{T})$ *have the Fourier series* $\psi(x) \sim \sum_{n=1}^{\infty} c_n(\psi)e^{inx}$ *with* $c_n(\psi) \geq 0$ *for every* $n \in \mathbb{N}$*, and let* $m \in \mathbb{N}$*. Then*

(i) $\omega_m(\psi;\pi/n)_\infty \geq \omega_m(\mathrm{Re}\,\psi;\pi/n)_\infty \geq C_{17}(m)n^{-\varkappa}\sum_{\nu=1}^{n}\nu^{\varkappa}c_\nu(\psi)$, $n \in \mathbb{N}$, *where* $\varkappa = m + (1-(-1)^m)/2 = \begin{cases} m & \text{if } m \text{ is even,} \\ m+1 & \text{if } m \text{ is odd.}\end{cases}$

(ii) $\omega_m(\psi;\pi/n)_\infty \geq \omega_m(\mathrm{Im}\,\psi;\pi/n)_\infty \geq C_{18}(m)n^{-\varkappa}\sum_{\nu=1}^{n}\nu^{\varkappa}c_\nu(\psi)$, $n \in \mathbb{N}$, *where* $\varkappa = m + (1+(-1)^m)/2 = \begin{cases} m+1 & \text{if } m \text{ is even,} \\ m & \text{if } m \text{ is odd.}\end{cases}$

Lemma 1. *Let* $p,q \in (1,\infty)$, $r = pq/(p+q-pq) \in (1,\infty]$, $\theta = \theta(r) = \min\{2,r\}$ *for* $r \in (1,\infty)$ *and* $\theta(\infty) = 1$, $m \in \mathbb{N}$, $\lambda = \{\lambda_n\}_{n=1}^{\infty} \in M_0(\alpha)$ *and* $\varepsilon = \{\varepsilon_n\}_{n=1}^{\infty} \in M_0(\beta)$ *for some* $\alpha,\beta \in (0,\infty)$*. Then there are functions* $f_0(\cdot;p;\lambda) \in L_p(\mathbb{T})$ *and* $g_0(\cdot;q;\varepsilon) \in L_q(\mathbb{T})$ *such that*

(i) $E_{n-1}(f_0)_p \leq C_{19}(p,\alpha)\lambda_n$, $E_{n-1}(g_0)_q \leq C_{19}(q,\beta)\varepsilon_n$, $n \in \mathbb{N}$.

(ii) $\omega_m(f_0 * g_0;\pi/n)_r \geq C_{20}(m,r)n^{-m}\left(\sum_{\nu=1}^{n}\nu^{\theta m-1}\lambda_\nu^\theta\varepsilon_\nu^\theta\right)^{1/\theta}$, $n \in \mathbb{N}$.

Proof. First we consider the case $1 < r \leq 2$. For $p,q \in (1,\infty)$ ($p' = p/(p-1)$, $q' = q/(q-1)$), let

$$f_0(x;p;\lambda) = \sum_{n=1}^{\infty} n^{-1/p'}\lambda_n e^{inx}, \quad g_0(x;q;\varepsilon) = \sum_{n=1}^{\infty} n^{-1/q'}\varepsilon_n e^{inx}, \quad x \in \mathbb{T}.$$

Since $\lambda \in M_0(\alpha) \subset B^{(p)} \subset D^{(p)}$ and $\varepsilon \in M_0(\beta) \subset B^{(q)} \subset D^{(q)}$, in virtue of (i) and (ii) of Lemma C we have $f_0 \in L_p(\mathbb{T})$, $E_{n-1}(f_0)_p \leq C_{19}(p,\alpha)\lambda_n$ and $g_0 \in L_q(\mathbb{T})$, $E_{n-1}(g_0)_q \leq C_{19}(q,\beta)\varepsilon_n$, $n \in \mathbb{N}$. By Theorem A, the convolution

$$h_0(x) = (f_0 * g_0)(x) = \sum_{n=1}^{\infty} n^{-(1/p'+1/q')}\lambda_n\varepsilon_n e^{inx}, \quad x \in \mathbb{T},$$

belongs to $L_r(\mathbb{T})$ for $r = pq/(p+q-pq)$. Since $r - 1 - r(1/p'+1/q') = 0$,

$$C_{15}(m,r)\omega_m(h_0;\pi/n)_r \geq n^{-m}\left(\sum_{\nu=1}^{n}\nu^{rm+r-2}\left|c_\nu(h_0)\right|^r\right)^{1/r}$$
$$= n^{-m}\left(\sum_{\nu=1}^{n}\nu^{rm+r-2-r(1/p'+1/q')}\lambda_\nu^r\varepsilon_\nu^r\right)^{1/r} = n^{-m}\left(\sum_{\nu=1}^{n}\nu^{rm-1}\lambda_\nu^r\varepsilon_\nu^r\right)^{1/r},$$

by Lemma D, whence the estimation (ii) follows in the case $1 < r \leq 2$.

Consider now the case $2 < r < \infty$. Let

$$f_0(x;\lambda) = \sum_{\nu=0}^{\infty}\lambda_{2^\nu}e^{i2^\nu x}, \quad g_0(x;\varepsilon) = \sum_{\nu=0}^{\infty}\varepsilon_{2^\nu}e^{i2^\nu x}, \quad x \in \mathbb{T}.$$

Taking into account that $\lambda \in M_0(\alpha)$ and $\varepsilon \in M_0(\beta)$, we have that

$$\sum_{\nu=0}^{\infty}\lambda_{2^\nu}^2 = \sum_{\nu=0}^{\infty}(2^{\nu\alpha}\lambda_{2^\nu})^2 2^{-2\nu\alpha} \leq \lambda_1^2\sum_{\nu=0}^{\infty}2^{-2\nu\alpha} = \lambda_1^2 2^{2\alpha}\left(2^{2\alpha}-1\right)^{-1} < \infty,$$

$$\sum_{\nu=0}^{\infty}\varepsilon_{2^\nu}^2 = \sum_{\nu=0}^{\infty}(2^{\nu\beta}\varepsilon_{2^\nu})^2 2^{-2\nu\beta} \leq \varepsilon_1^2\sum_{\nu=0}^{\infty}2^{-2\nu\beta} = \varepsilon_1^2 2^{2\beta}\left(2^{2\beta}-1\right)^{-1} < \infty.$$

In virtue of [1], v. 1, Theorem 8.20, p. 345, lacuna trigonometric series considered converge almost everywhere and are Fourier series of their sums f_0 and g_0, respectively, and, for every $p,q \in (1,\infty)$,

$$\|f_0(\cdot;\lambda)\|_p \leq C_{21}(p)(\sum_{\nu=0}^{\infty}\lambda_{2^\nu}^2)^{1/2} \leq C_{21}(p)2^\alpha\left(2^{2\alpha}-1\right)^{-1/2}\lambda_1,$$

$$\|g_0(\cdot;\varepsilon)\|_q \leq C_{21}(q)(\sum_{\nu=0}^{\infty}\varepsilon_{2^\nu}^2)^{1/2} \leq C_{21}(q)2^\beta\left(2^{2\beta}-1\right)^{-1/2}\varepsilon_1.$$

Hence $f_0 \in L_p(\mathbb{T})$ and $g_0 \in L_q(\mathbb{T})$. Since for every $n \in \mathbb{N}$ there exists $s \in \mathbb{Z}_+$ such that $2^s \le n < 2^{s+1}$ ($\Rightarrow 2^s < n+1 \le 2^{s+1}$), we have by [1], v. 1, Theorem 8.20, p. 345, that

$$\begin{aligned}E_n(f_0)_p &\le E_{2^s}(f_0)_p \le \|f_0(\cdot) - S_{2^s}(f_0;\cdot)\|_p = \left\|\sum\nolimits_{\nu=s+1}^{\infty} \lambda_{2^\nu} e^{i2^\nu x}\right\|_p \\ &\le C_{21}(p)\left(\sum\nolimits_{\nu=s+1}^{\infty} \lambda_{2^\nu}^2\right)^{1/2} = C_{21}(p)\left(\sum\nolimits_{\nu=s+1}^{\infty} (2^{\nu\alpha}\lambda_{2^\nu})^2 2^{-2\nu\alpha}\right)^{1/2} \\ &\le C_{21}(p)2^{(s+1)\alpha}\lambda_{2^{s+1}}\left(\sum\nolimits_{\nu=s+1}^{\infty} 2^{-2\nu\alpha}\right)^{1/2} \\ &= C_{21}(p)2^{\alpha}\left(2^{2\alpha}-1\right)^{-1/2}\lambda_{2^{s+1}} \le C_{21}(p)2^{\alpha}\left(2^{2\alpha}-1\right)^{-1/2}\lambda_{n+1},\end{aligned}$$

whence $E_n(f_0)_p \le C_{21}(p)2^{\alpha}\left(2^{2\alpha}-1\right)^{-1/2}\lambda_{n+1}$ for every $n \in \mathbb{N}$. Since $E_0(f_0)_p \le \|f_0\|_p \le C_{21}(p)2^{\alpha}\left(2^{2\alpha}-1\right)^{-1/2}\lambda_1$, we obtain that $E_{n-1}(f_0)_p \le C_{19}(p,\alpha)\lambda_n$ for every $n \in \mathbb{N}$. Similarly, $E_{n-1}(g_0)_q \le C_{19}(q,\beta)\varepsilon_n$ for every $n \in \mathbb{N}$. By the formula above for Fourier coefficients of convolution,

$$h_0(x) = (f_0 * g_0)(x) = \sum\nolimits_{\nu=0}^{\infty} \lambda_{2^\nu}\varepsilon_{2^\nu} e^{i2^\nu x}, \quad x \in \mathbb{T}.$$

Since $p, q \in (1,\infty)$, $h_0 \in L_r(\mathbb{T})$ for $r = pq/(p+q-pq) \in (1,\infty]$ and, since $r > 2$, $h_0 \in L_2(\mathbb{T})$. Let as above $s \in \mathbb{Z}_+$ be such that $2^s \le n < 2^{s+1}$ ($\Rightarrow n \le 2^{s+1}-1$). Clearly we have that

$$E_0^2(h_0)_2 = \sum\nolimits_{\nu=0}^{\infty} \lambda_{2^\nu}^2\varepsilon_{2^\nu}^2 \ge \lambda_1^2\varepsilon_1^2, \quad E_{2^s}^2(h_0)_2 = \sum\nolimits_{\nu=s+1}^{\infty} \lambda_{2^\nu}^2\varepsilon_{2^\nu}^2 \ge \lambda_{2^{s+1}}^2\varepsilon_{2^{s+1}}^2$$

for $s \in \mathbb{Z}_+$. Taking into account these estimations, we obtain that

$$\begin{aligned}&\sum_{\nu=1}^{n} \nu^{2m-1}\lambda_\nu^2\varepsilon_\nu^2 \le \sum_{\mu=0}^{s}\sum_{\nu=2^\mu}^{2^{\mu+1}-1} \nu^{2m-1}\lambda_\nu^2\varepsilon_\nu^2 \le \sum_{\mu=0}^{s}\lambda_{2^\mu}^2\varepsilon_{2^\mu}^2 \sum_{\nu=2^\mu}^{2^{\mu+1}-1} \nu^{2m-1} \\ &\le 2^{-1}\sum_{\mu=0}^{s} 2^{2m(\mu+1)}\lambda_{2^\mu}^2\varepsilon_{2^\mu}^2 = 2^{-1}\left\{2^{2m}\lambda_1^2\varepsilon_1^2 + 2^{4m}\lambda_2^2\varepsilon_2^2 + \sum_{\mu=2}^{s} 2^{2m(\mu+1)}\lambda_{2^\mu}^2\varepsilon_{2^\mu}^2\right\} \\ &= 2^{2m-1}\left\{\lambda_1^2\varepsilon_1^2 + 2^{2m}\lambda_2^2\varepsilon_2^2 + \sum_{\mu=1}^{s-1} 2^{2m(\mu+1)}\lambda_{2^{\mu+1}}^2\varepsilon_{2^{\mu+1}}^2\right\} \\ &\le 2^{2m-1}\left\{E_0^2(h_0)_2 + 2^{2m}E_1^2(h_0)_2 + \sum_{\mu=1}^{s-1} 2^{2m(\mu+1)}E_{2^\mu}^2(h_0)_2\right\} \\ &\le 2^{2m-1}\left\{E_0^2(h_0)_2 + 2^{2m}E_1^2(h_0)_2 + \frac{m2^{4m+1}}{2^{2m}-1}\sum_{\mu=1}^{s-1}\sum_{\nu=2^{\mu-1}+1}^{2^\mu} \nu^{2m-1}E_\nu^2(h_0)_2\right\} \\ &= 2^{2m-1}\left\{E_0^2(h_0)_2 + 2^{2m}E_1^2(h_0)_2 + \frac{m2^{4m+1}}{2^{2m}-1}\sum_{\nu=2}^{2^{s-1}} \nu^{2m-1}E_\nu^2(h_0)_2\right\} \\ &\le C_{22}(m)\sum_{\nu=1}^{2^s} \nu^{2m-1}E_{\nu-1}^2(h_0)_2 \le C_{22}(m)\sum_{\nu=1}^{n} \nu^{2m-1}E_{\nu-1}^2(h_0)_2,\end{aligned}$$

whence we obtain by Lemma E that for $r \in (2,\infty)$

$$n^{-m}\left(\sum_{\nu=1}^{n}\nu^{2m-1}\lambda_\nu^2\varepsilon_\nu^2\right)^{1/2}\le (C_{22}(m))^{1/2}n^{-m}\left(\sum_{\nu=1}^{n}\nu^{2m-1}E_{\nu-1}^2(h_0)_2\right)^{1/2}$$
$$\le (C_{22}(m))^{1/2}C_{16}(m)\omega_m(h_0;\pi/n)_2\le (C_{20}(m,2))^{-1}\omega_m(h_0;\pi/n)_r.$$

It follows from this estimation that (ii) holds for $r\in(2,\infty)$.

At last we consider the case $r=\infty$. In this case $1/p+1/q=1$, that is $q=p'$, and therefore $1/p'+1/q'=1$. Let $f_0(\cdot;p;\lambda)$ and $g_0(\cdot;q;\varepsilon)$ be functions such as in the case $1<r\le 2$, and $h_0=f_0*g_0$. By (i) of Lemma F for even m and (ii) of Lemma F for odd m, we have that

$$C_{23}(m)\omega_m(h_0;\pi/n)_\infty\ge n^{-m}\sum_{\nu=1}^{n}\nu^m c_\nu(h_0)$$
$$=n^{-m}\sum_{\nu=1}^{n}\nu^{m-(1/p'+1/q')}\lambda_\nu\varepsilon_\nu=n^{-m}\sum_{\nu=1}^{n}\nu^{m-1}\lambda_\nu\varepsilon_\nu,$$

whence the estimation (ii) follows with constant $C_{20}(m,\infty)=(C_{23}(m))^{-1}$ in the case $r=\infty$. Lemma 1 is proved.

Given $p,q\in[1,\infty]$ and $\lambda,\varepsilon\in M_0$, put

$$E_p[\lambda]*E_q[\varepsilon]=\{h=f*g: f\in E_p[\lambda], g\in E_q[\varepsilon]\}.$$

The following theorem shows that estimations (i) and (ii) of Theorem B are exact in the sense of order on classes $E_p[\lambda]*E_q[\varepsilon]$ in the case $p,q\in(1,\infty)$ under condition that $\lambda\in M_0(\alpha)$ and $\varepsilon\in M_0(\beta)$ for some $\alpha,\beta\in(0,\infty)$.

Theorem 3. *Let $p,q\in(1,\infty)$, $r=pq/(p+q-pq)\in(1,\infty]$, $\theta=\theta(r)=\min\{2,r\}$ for $r\in(1,\infty)$ and $\theta(\infty)=1$, $m\in\mathbb{N}$, $\lambda=\{\lambda_n\}\in M_0(\alpha)$, $\varepsilon=\{\varepsilon_n\}\in M_0(\beta)$, where $\alpha,\beta\in(0,\infty)$. Then*

$$\sup\{\omega_m(h;\pi/n)_r: h\in E_p[\lambda]*E_q[\varepsilon]\}\asymp n^{-m}\left(\sum_{\nu=1}^{n}\nu^{\theta m-1}\lambda_\nu^\theta\varepsilon_\nu^\theta\right)^{1/\theta}, n\in\mathbb{N}.$$

Proof. Indeed, the upper estimation for every $p,q\in[1,\infty]$ and $\lambda,\varepsilon\in M_0$ immediately follows by inequalites (i) and (ii) of Theorem B. The lower estimation is realized by function

$$h_0(\cdot;p,q;\lambda,\varepsilon)=(C_{19}(p,\alpha))^{-1}f_0(\cdot;p;\lambda)*(C_{19}(q,\beta))^{-1}g_0(\cdot;q;\varepsilon)\in E_p[\lambda]*E_q[\varepsilon]$$

in virtue of (ii) of Lemma 1.

Note that Theorem 3 in the case of scale of power majorants of sequences of the best approximations of functions forming the convolution was proved by author in [12], Theorem 2.

One says that a function $\omega\in\Omega_l$ satisfies (S)-*condition with parameter* γ (we write $\omega\in S(\gamma)$) if there is a number $\gamma\in(0,l)$ such that $\delta^{-\gamma}\omega(\delta)\uparrow(\delta\uparrow)$ and satisfies (S_l)-*condition with parameter* η (we write $\omega\in S_l(\eta)$) if there is a number $\eta\in(0,l)$ such that $\delta^{-(l-\eta)}\omega(\delta)\downarrow(\delta\uparrow)$. Put $\omega_n=\omega(\pi/n)$, $n\in\mathbb{N}$. Then conditions (S) and (S_l) admit equivalent formulations:

- $\omega\in S(\gamma)\iff n^\gamma\omega_n\downarrow(n\uparrow)\iff n_2^\gamma\omega_{n_2}\le n_1^\gamma\omega_{n_1}$ for every $n_1<n_2$.
- $\omega\in S_l(\eta)\iff n^{l-\eta}\omega_n\uparrow(n\uparrow)\iff n_1^{l-\eta}\omega_{n_1}\le n_2^{l-\eta}\omega_{n_2}$ for every $n_1<n_2$.

Note that if $\omega \in \Omega_l$ then $\{\omega_n\}_{n=1}^{\infty} \in M_0$ and $n^l \omega_n \uparrow (n \uparrow)$. Besides, it is obvious that $\omega \in S(\gamma) \implies \{\omega_n\} \in M_0(\gamma)$ for some $\gamma \in (0, l)$.

The conditions $(B^{(1)})$ and $(B_l^{(1)})$ (the conditions of N. K. Bari), (S) and (S_l) (conditions of S. B. Stechkin) appeared in [25], Section 2, (in the same place the series of equivalent conditions discussed).

The following implications are valid: $\omega \in S(\gamma) \implies \{\omega_n\} \in B^{(\sigma)}$, $\omega \in S_l(\eta) \implies \{\omega_n\} \in B_l^{(\sigma)}$ for every $\sigma \in [1, \infty)$. Indeed, the first implication follows by $S(\gamma) \subset M_0(\gamma) \subset B^{(\sigma)}$, $\gamma \in (0, l)$, and the second one follows from the following estimation $(\eta \in (0, l))$

$$\left(\sum_{\nu=1}^{n} \nu^{\sigma l-1} \omega_{\nu}^{\sigma}\right)^{1/\sigma} = \left(\sum_{\nu=1}^{n} \left(\nu^{l-\eta}\omega_{\nu}\right)^{\sigma} \nu^{\sigma\eta-1}\right)^{1/\sigma}$$
$$\leq n^{l-\eta}\omega_n \left(\sum_{\nu=1}^{n} \nu^{\sigma\eta-1}\right)^{1/\sigma} \leq C_{24}(\eta, \sigma) n^l \omega_n, \quad n \in \mathbb{N}.$$

Lemma 2. *Let $l, k, m \in \mathbb{N}$, $p, q \in (1, \infty)$, $r = pq/(p + q - pq) \in (1, \infty]$, $\theta = \theta(r) = \min\{2, r\}$ for $r \in (1, \infty)$ and $\theta(\infty) = 1$, $\omega \in S(\gamma_1) \cap S_l(\eta_1) \subset \Omega_l$, $\varphi \in S(\gamma_2) \cap S_k(\eta_2) \subset \Omega_k$. Then there are functions $f_0(\cdot; p; \omega) \in L_p(\mathbb{T})$ and $g_0(\cdot; q; \varphi) \in L_q(\mathbb{T})$ such that*

(i) $\omega_l(f_0; \delta)_p \leq C_{25}(l, p, \gamma_1, \eta_1)\omega(\delta)$ *and* $\omega_k(g_0, \delta)_q \leq C_{25}(k, q, \gamma_2, \eta_2)\varphi(\delta)$ *for* $\delta \in (0, \pi]$.

(ii) $\omega_m(f_0 * g_0; \pi/n)_r \geq C_{26}(m, r) n^{-m} \left(\sum\limits_{\nu=1}^{n} \nu^{\theta m-1} \omega^{\theta}(\pi/\nu)\varphi^{\theta}(\pi/\nu)\right)^{1/\theta}$, $n \in \mathbb{N}$.

Proof. Put $f_0(\cdot; p; \omega) = f_0(\cdot; p; \lambda)$ and $g_0(\cdot; q; \varphi) = g_0(\cdot; q; \varepsilon)$, where $\lambda = \{\omega_n\}$, $\varepsilon = \{\varphi_n\}$, $\omega_n = \omega(\pi/n)$, $\varphi_n = \varphi(\pi/n)$, $n \in \mathbb{N}$, and $f_0(\cdot; p; \lambda)$, $g_0(\cdot; q; \varepsilon)$ are functions considered in Lemma 1. Since $\omega \in S(\gamma_1) \implies \{\omega_n\} \in M_0(\gamma_1)$ and $\varphi \in S(\gamma_2) \implies \{\varphi_n\} \in M_0(\gamma_2)$, we have by Lemma 1 that $f_0(\cdot; p; \omega) \in L_p(\mathbb{T})$, $E_{n-1}(f_0)_p \leq C_{19}(p, \gamma_1)\omega_n$, $n \in \mathbb{N}$, *and* $g_0(\cdot; q; \varphi) \in L_q(\mathbb{T})$, $E_{n-1}(g_0)_q \leq C_{19}(q, \gamma_2)\varphi_n$, $n \in \mathbb{N}$. Taking into account that $\omega \in S_l(\eta_1) \iff n^{l-\eta_1}\omega_n \uparrow (n \uparrow)$ and in virtue of (5) we obtain that $(\sigma = \sigma(p) = \min\{2, p\})$

$$\omega_l(f_0; \pi/n)_p \leq C_5(l, p) n^{-l} \left(\sum_{\nu=1}^{n} \nu^{\sigma l-1} E_{\nu-1}^{\sigma}(f_0)_p\right)^{1/\sigma}$$
$$\leq C_5(l, p) C_{19}(p, \gamma_1) n^{-l} \left(\sum_{\nu=1}^{n} \nu^{\sigma l-1}\omega_{\nu}^{\sigma}\right)^{1/\sigma} \leq C_5(l, p) C_{19}(p, \gamma_1) C_{24}(\eta_1, \sigma)\omega_n,$$

whence $\omega_l(f_0; \pi/n)_p \leq C_{27}(l, p, \gamma_1, \eta_1)\omega_n$, $n \in \mathbb{N}$, and therefore

$$\omega_l(f_0; \delta)_p \leq 2^l C_{27}(l, p, \gamma_1, \eta_1)\omega(\delta), \quad \delta \in (0, \pi].$$

Similarly, taking into account that $\varphi \in S_k(\eta_2) \iff n^{k-\eta_2}\varphi_n \uparrow (n \uparrow)$, we obtain that

$$\omega_k(g_0; \delta)_q \leq 2^k C_{27}(k, q, \gamma_2, \eta_2)\varphi(\delta), \quad \delta \in (0, \pi].$$

At last, we have by (ii) of Lemma 1 that $(C_{26}(m, r) = C_{20}(m, r))$

$$\omega_m(f_0 * g_0; \pi/n)_r \geq C_{26}(m, r) n^{-m} \left(\sum_{\nu=1}^{n} \nu^{\theta m-1}\omega_{\nu}^{\theta}\varphi_{\nu}^{\theta}\right)^{1/\theta}, \quad n \in \mathbb{N}.$$

Lemma 2 is proved.

Given $p,q \in [1,\infty]$, $l,k \in \mathbb{N}$, and $\omega \in \Omega_l$, $\varphi \in \Omega_k$, we denote

$$H_p^l[\omega] * H_q^k[\varphi] = \left\{ h = f * g : f \in H_p^l[\omega], g \in H_q^k[\varphi] \right\}.$$

Theorem 4. *Let $p,q \in [1,\infty]$, $r = pq/(p+q-pq) \in [1,\infty]$, $\theta = \theta(r) = \min\{2,r\}$ for $r \in [1,\infty)$ and $\theta(\infty) = 1$, $l,k,m \in \mathbb{N}$, $\omega \in \Omega_l$, $\varphi \in \Omega_k$. Then ($n \in \mathbb{N}$)*

(i) *For arbitrary $h \in H_p^l[\omega] * H_q^k[\varphi]$ under constants $C_{28} = C_8(k,l,m,r)$ in the case $m < l+k$, $C_{29} = C_{11}(l,k)$ in the case $m = l+k$, $C_{29} = 2^{l+k} C_8(k,l,m,r)$ or $C_{29} = 2^{m-(l+k)} C_{11}(l,k)$ in the case $m > l+k$, there are the estimations:*

- $$\omega_m(h;\pi/n)_r \le C_{28} n^{-m} \left(\sum_{\nu=1}^{n} \nu^{\theta m-1} \omega^\theta(\pi/\nu) \varphi^\theta(\pi/\nu) \right)^{1/\theta} \quad \text{if } m < l+k;$$
- $\omega_m(h;\pi/n)_r \le C_{29} \omega(\pi/n) \varphi(\pi/n)$ *if* $m \ge l+k$.

(ii) *If $p,q \in (1,\infty)$ and $\omega \in S(\gamma_1) \cap S_l(\eta_1) \subset \Omega_l$, $\varphi \in S(\gamma_2) \cap S_k(\eta_2) \subset \Omega_k$ then there is an individual function $h_0 \in H_p^l[\omega] * H_q^k[\varphi]$ such that under constants $C_{30} = C_{30}(k,l,m,r,p,q,\gamma_1,\eta_1,\gamma_2,\eta_2)$ and $C_{31} = (\theta m)^{-1} C_{30}$*

- $$\omega_m(h_0;\pi/n)_r \ge C_{30} n^{-m} \left(\sum_{\nu=1}^{n} \nu^{\theta m-1} \omega^\theta(\pi/\nu) \varphi^\theta(\pi/\nu) \right)^{1/\theta} \quad \text{if } m < l+k;$$
- $\omega_m(h_0;\pi/n)_r \ge C_{31} \omega(\pi/n) \varphi(\pi/n)$ *if* $m \ge l+k$.

Proof. The first estimation in (i) follows from (i) of Theorem 1, the second one in (i) follows from (10) in the case $m = l+k$ and from (ii) of Theorem 1 and Remark 2 in the case $m > l+k$. The first estimation in (ii) holds for

$$h_0(\cdot;p,q;\omega,\varphi) = (C_{25}(l,p,\gamma_1,\eta_1))^{-1} f_0(\cdot;p,\omega) * (C_{25}(k,q,\gamma_2,\eta_2))^{-1} g_0(\cdot;q,\varphi)$$

by (ii) of Lemma 2. The second estimation in (ii) for the same h_0 follows by (ii) of Lemma 2 in virtue of monotonicity of $\omega \in \Omega_l$ and $\varphi \in \Omega_k$. Indeed,

$$\omega_m(h_0;\pi/n)_r \ge C_{30} n^{-m} \left(\textstyle\sum_{\nu=1}^{n} \nu^{\theta m-1} \omega^\theta(\pi/\nu) \varphi^\theta(\pi/\nu) \right)^{1/\theta}$$
$$\ge C_{30} n^{-m} \omega(\pi/n) \varphi(\pi/n) \left(\textstyle\sum_{\nu=1}^{n} \nu^{\theta m-1} \right)^{1/\theta} \ge (\theta m)^{-1} C_{30} \omega(\pi/n) \varphi(\pi/n),$$

where $C_{30} = C_{25}(l,p,\gamma_1,\eta_1) C_{25}(k,q,\gamma_2,\eta_2) C_{26}(m,r)$.

Corollary 2. *Let $p,q \in (1,\infty)$, $r = pq/(p+q-pq) \in (1,\infty]$, $\theta = \theta(r) = \min\{2,r\}$ for $r \in (1,\infty)$ and $\theta(\infty) = 1$, $l,k,m \in \mathbb{N}$, $m < l+k$, $\alpha \in (0,l)$, $\beta \in (0,k)$. Then for $\delta \in (0,\pi]$*

(i) $$\sup\left\{ \omega_m(h;\delta)_r : h \in H_p^l[\delta^\alpha] * H_q^k[\delta^\beta] \right\} \asymp \begin{cases} \delta^{\alpha+\beta} & (\alpha+\beta < m), \\ \delta^m (\ln(\pi e/\delta))^{1/\theta} & (\alpha+\beta = m), \\ \delta^m & (\alpha+\beta > m). \end{cases}$$

(ii) $\sup\left\{ \omega_{m+1}(h;\delta)_r : h \in H_p^l[\delta^\alpha] * H_q^k[\delta^\beta] \right\} \asymp \delta^m$ *if* $\alpha+\beta = m$.

Proof. The upper estimations in (i) and (ii) immediately follow from (i) of Theorem 4 (see also Corollary 1). The lower estimations in (i) and (ii) are realized by an individual function $h_0 \in H_p^l[\delta^\alpha] * H_q^k[\delta^\beta]$ in virtue of (ii) of Theorem 4 since $\omega(\delta) = \delta^\alpha \in S \cap S_l$ for $\alpha \in (0, l)$ and $\varphi(\delta) = \delta^\beta \in S \cap S_k$ for $\beta \in (0, k)$ (see the proof of lower estimations in Theorem 2 of [12], p. 28).

Remark 4. Theorem 4 shows in fact that inequalities (i) and (ii) of Theorem 1 and (10) are exact in the sense of order on classes $H_p^l[\omega] * H_q^k[\varphi]$ in the case $p, q \in (1, \infty)$ under condition that $\omega \in S \cap S_l$ and $\varphi \in S \cap S_k$. The last condition guarantees existence of individual functions $(C_{25}(l, p, \gamma_1, \eta_1))^{-1} f_0(\cdot; p, \omega) \in H_p^l[\omega]$ and $(C_{25}(k, q, \gamma_2, \eta_2))^{-1} g_0(\cdot; q, \varphi) \in H_q^k[\varphi]$ convolution of which gives the extremal function $h_0(\cdot; p, q; \omega, \varphi) \in H_p^l[\omega] * H_q^k[\varphi]$.

Remark 5. In the case $m \geq l + k$ we have by Theorem 4 that

$$\sup \left\{ \omega_m(h; \delta)_r : h \in H_p^l[\omega] * H_q^k[\varphi] \right\} \asymp \omega(\delta)\varphi(\delta), \quad \delta \in (0, \pi], \tag{17}$$

under the condition that $\omega \in S \cap S_l \subset \Omega_l$ and $\varphi \in S \cap S_k \subset \Omega_k$ for every $k, l \in \mathbb{N}$, $p, q \in (1, \infty)$ and $r = pq/(p + q - pq) \in (1, \infty]$. In this case (17) takes place for arbitrary functions $\omega \in \Omega_l$ and $\varphi \in \Omega_k$ without condition that $\omega \in S \cap S_l$ and $\varphi \in S \cap S_k$, but the lower estimation is realized by sequence $\{h_n(\cdot; p, q; \omega, \varphi)\}_{n=1}^\infty \subset H_p^l[\omega] * H_q^k[\varphi]$. Indeed, the upper estimation in (17) immediately follows by (ii) of Theorem 1 for $m > l + k$ (see also Remark 2), and by (10) for $m = l + k$. The lower estimation in (17) is realized by sequence (see [11], Lemma 1) of functions

$$h_n(\cdot; p, q; \omega, \varphi) = (C_{32}(l, p))^{-1} f_n(\cdot; p, \omega) * (C_{32}(k, q))^{-1} g_n(\cdot; q, \varphi), \quad n \in \mathbb{N},$$

in virtue of (4) and (ii) of Lemma 1 [11], namely

$$C_4(m)\omega_m(h_n; \pi/n)_r \geq E_{n-1}(h_n)_r \geq C_{33}(r)(C_{32}(l, p)C_{32}(k, q))^{-1}\omega(\pi/n)\varphi(\pi/n),$$

where $f_n(x; p, \omega) = n^{1/p-1}\omega(\pi/n)d_{4n}(x)$, $g_n(x; q, \varphi) = n^{1/q-1}\varphi(\pi/n)d_{4n}(x)$, $d_{4n}(x) = \sum_{\nu=1}^{4n} e^{i\nu x}$, $n \in \mathbb{N}$. Note that $\{f_n(\cdot; p, \omega)\} \subset L_p(\mathbb{T})$, $\{g_n(\cdot; q, \varphi)\} \subset L_q(\mathbb{T})$ and $\omega_l(f_n; \delta)_p \leq C_{32}(l, p)\omega(\delta)$, $\omega_k(g_n; \delta)_q \leq C_{32}(k, q)\varphi(\delta)$, $\delta \in (0, \pi]$.

Remark 6. **(i)** For existence of an individual function $h_0 = f_0 * g_0 \in H_p^l[\omega] * H_q^k[\varphi]$ which realizes the estimation $E_{n-1}(h_0)_r \geq C_{34}(r, l, k, p, q)\omega(\pi/n)\varphi(\pi/n)$, $n \in \mathbb{N}$, for $r \in (1, \infty)$, and for arbitrary $\omega \in \Omega_l$ and $\varphi \in \Omega_k$ (whence $\omega\varphi \in \Omega_{l+k}$), it is necessary that $\omega\varphi \in S_{l+k}(\eta)$ for some $\eta \in (0, l + k)$. Indeed, if such function exists, in virtue of the following inequality [26]

$$n^{-(l+k)} \left(\sum_{\nu=1}^{n} \nu^{\rho(l+k)-1} E_{\nu-1}^{\rho}(\psi)_r \right)^{1/\rho} \leq C_{35}(l + k, r)\omega_{l+k}(\psi; \pi/n)_r \tag{18}$$

(where $r \in (1, \infty)$ and $\rho = \max\{2, r\}$, $\psi \in L_r(\mathbb{T})$) and (10), we have that

$$\begin{aligned} &C_{34} n^{-(l+k)} \left(\sum_{\nu=1}^{n} \nu^{\rho(l+k)-1} \omega^\rho(\pi/\nu)\varphi^\rho(\pi/\nu) \right)^{1/\rho} \\ &\leq n^{-(l+k)} \left(\sum_{\nu=1}^{n} \nu^{\rho(l+k)-1} E_{\nu-1}^{\rho}(h_0)_r \right)^{1/\rho} \leq C_{35}\omega_{l+k}(h_0; \pi/n)_r \\ &\leq C_{35}C_{11}\omega_l(f_0; \pi/n)_p \omega_k(g_0; \pi/n)_q \leq C_{35}C_{11}\omega(\pi/n)\varphi(\pi/n), \quad n \in \mathbb{N}, \end{aligned}$$

whence ($n \in \mathbb{N}$)

$$n^{-(l+k)}\left(\sum\nolimits_{\nu=1}^{n}\nu^{\rho(l+k)-1}\omega^{\rho}(\pi/\nu)\varphi^{\rho}(\pi/\nu)\right)^{1/\rho}\leq C_{36}(r,l,k,p,q)\omega(\pi/n)\varphi(\pi/n).$$

Therefore $\omega\varphi \in B_{l+k}^{(\rho)}$, and this is equivalent to $\omega\varphi \in S_{l+k}(\eta)$ for some $\eta \in (0, l+k)$. Indeed, if $\omega\varphi \in S_{l+k}(\eta)$ for some $\eta \in (0, l+k)$ then clearly $\omega\varphi \in B_{l+k}^{(\rho)}$ for every $\rho \in [1,\infty)$ (see the argument before Lemma 2). On the other hand, if $\omega\varphi \in B_{l+k}^{(\rho)}$ for some $\rho \in [1,\infty)$ (in particular, for $\rho = \max\{2, r\}$), then $(\omega\varphi)^{\rho} \in B_{\rho(l+k)}^{(1)}$ is equivalent to $(\omega\varphi)^{\rho} \in S_{\rho(l+k)}(\xi)$ for some $\xi \in (0, \rho(l+k))$ (see [25], Lemma 2.3, case $\rho(l+k) \in \mathbb{N}$; the argument holds if $\rho(l+k) \notin \mathbb{N}$). It follows that $\omega\varphi \in S_{l+k}(\eta)$ for $\eta = \xi/\rho \in (0, l+k)$.

Note that if $\omega \in S_l(\eta_1) \subset \Omega_l$ and $\varphi \in S_k(\eta_2) \subset \Omega_k$ then $\omega\varphi \in S_{l+k}(\eta_1+\eta_2) \subset \Omega_{l+k}$; the converse does not hold in general.

(ii) For existence of an individual function $h_0 = f_0 * g_0 \in H_p^l[\omega] * H_q^k[\varphi]$ which realizes the estimation $\omega_m(h_0;\delta)_r \geq C_{37}(m,r,l,k,p,q)\omega(\delta)\varphi(\delta)$, $\delta \in (0,\pi]$, for $r \in (1,\infty)$ and $m > l+k$, and for arbitrary $\omega \in \Omega_l$ and $\varphi \in \Omega_k$ (whence $\omega\varphi \in \Omega_{l+k}$), it is necessary that $\omega\varphi \in S_{l+k}(\eta)$ for some $\eta \in (0, l+k)$.

The proof of this assertion is similar to the proof of (i). We only use instead of (18) the following estimation ($m > l+k$)

$$n^{-(l+k)}\left(\sum\nolimits_{\nu=1}^{n}\nu^{\rho(l+k)-1}\omega_m^{\rho}\left(\psi;\pi/\nu\right)_r\right)^{1/\rho}\leq C_{38}(m,l+k,r)\omega_{l+k}(\psi;\pi/n)_r. \quad (19)$$

For (19), it is necessary to take into account (5) for $k = m$, $p = r$, $\sigma = \theta = \min\{2, r\}$, to apply Hardy's inequality (see [27], Theorem 346, p. 308) for $r \neq 2$ (whence $\rho/\theta > 1$, $\rho(m-(l+k))+1 > 1$) and to change the summation order for $r = 2$ (whence $\rho/\theta = 1$), and to apply (18).

Remark 7. One can get the upper estimation $\omega_{l+k}(f*g;\delta)_r$ by means $\omega_l(f;\delta)_p\omega_k(g;\delta)_q$ in Theorem 2 for $p,q \in [1,\infty]$, $r = pq/(p+q-pq) \in [1,\infty]$ immediately by applying Young's inequality (see Theorem A). Indeed, if $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$ then in virtue of $\left\|\Delta_t^l f\right\|_p \leq 2^{l+1}\left\|f\right\|_p$ and $\left\|\Delta_t^k g\right\|_q \leq 2^{k+1}\left\|g\right\|_q$, we have that $\Delta_t^l f \in L_p(\mathbb{T})$ and $\Delta_t^k g \in L_q(\mathbb{T})$ for every $t \in \mathbb{R}$. Since $\Delta_t^l(e^{i\nu x}) = \left(e^{i\nu t}-1\right)^l e^{i\nu x}$ and $\Delta_t^k(e^{i\nu x}) = \left(e^{i\nu t}-1\right)^k e^{i\nu x}$, then ($x \in \mathbb{T}$, $t \in \mathbb{R}$)

$$\Delta_t^l f(x) \sim \sum_{\nu\in\mathbb{Z}\setminus\{0\}} c_\nu(f)\left(e^{i\nu t}-1\right)^l e^{i\nu x} \text{ and } \Delta_t^k g(x) \sim \sum_{\nu\in\mathbb{Z}\setminus\{0\}} c_\nu(g)\left(e^{i\nu t}-1\right)^k e^{i\nu x}.$$

Taking into account that the formula for calculating the Fourier coefficients of the convolution (see after Theorem A), we obtain that

$$\left(\Delta_t^l f * \Delta_t^k g\right)(x) \sim \sum\nolimits_{\nu\in\mathbb{Z}\setminus\{0\}} c_\nu(f)c_\nu(g)\left(e^{i\nu t}-1\right)^{l+k}e^{i\nu x} \sim \Delta_t^{l+k}\left(f*g\right)(x),$$

whence it follows that $\Delta_t^{l+k}(f*g) = \Delta_t^l f * \Delta_t^k g$ a.e. on $\mathbb{T}$. Applying Young's inequality (taking into account that $f*g \in L_r(\mathbb{T})$ and, therefore, $\Delta_t^{l+k}(f*g) \in L_r(\mathbb{T})$), we have that for $|t| \leq \delta$

$$\left\|\Delta_t^{l+k}\left(f*g\right)\right\|_r \leq \left\|\Delta_t^l f\right\|_p\left\|\Delta_t^k g\right\|_q \leq \omega_l(f;\delta)_p\omega_k(g;\delta)_q,$$

whence

$$\omega_{l+k}(f*g;\delta)_r \le \omega_l(f;\delta)_p \omega_k(g;\delta)_q, \quad \delta\in[0,\infty). \tag{20}$$

It is natural to call (20) the *Young inequality for smoothness modules.*

Remark 8. The proof of the upper estimation of $\omega_{l+k}(f*g;\delta)_r$ by means of $\omega_l(f;\delta)_p\omega_k(g;\delta)_q$ given in Theorem 2 has the aim to determine an amount of characteristics $E_n(f)_p$ and $E_n(g)_q$ in the expression of the estimation (see the estimation after (15)). The proof of this estimation given in Remark 7 (see (20)) doesn't provide such an information.

It should be noted that one can receive the estimation of σ_2 (see (15)) without (12) if apply (20) (see the proof of Theorem 2) in the following way:

$$\sigma_2 = \omega_{l+k}(T_{n,p}(f)*T_{n,q}(g);\delta)_r \le \omega_l(T_{n,p}(f);\delta)_p\omega_k(T_{n,q}(g);\delta)_q$$

$$\le \left(2^l E_n(f)_p + \omega_l(f;\delta)_p\right)\left(2^k E_n(g)_q + \omega_k(g;\delta)_q\right)$$

$$\le \left(2^l C_4(l)+1\right)\left(2^k C_4(k)+1\right)\omega_l(f;\delta)_p\omega_k(g;\delta)_q.$$

Note also that application of (20) allows us to receive immediately the following estimation (compare with the estimation σ_4 in Remark 3):

$$\omega_m(S_n(h);\delta)_r = \omega_{l+k}(S_n(f*g);\delta)_r = \omega_{l+k}(S_n(f)*S_n(g);\delta)_r$$

$$\le \omega_l(S_n(f);\delta)_p\,\omega_k(S_n(g);\delta)_q \le C_{12}(p)\,C_{12}(q)\,\omega_l(f;\delta)_p\omega_k(g;\delta)_q.$$

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