MATHEMATICS

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STATIONARY CHARACTERISTICS OF THE SINGLE-SERVER QUEUEING LOSS SYSTEM WITH FALLING OUT POSSIBILITY

Abstract

The statistical-equilibrium state probabilities distribution and distribution of time of stay in a free state for the G/G/1/0 queueing system with falling out possibilities of the working and free server are obtained.

1. Introduction. Research ergodic properties of queueing systems type M/G successfully carried out by means of a method of embedded Markov chains [1, p. 96]. For systems of type G/G the proof of existence of limiting stationary process has appeared problematic. It is known only attempts of finding of stationary characteristics such queueing systems by means of semi-Markov processes with specially constructed phase space [2, chapt. 5].

In this work we assume existence of limiting stationary process and we set a problem of finding of stationary characteristics queueing system G/G/1/0 with falling out possibilities of the working and free server.

2. The general formulas for stationary probabilities. We study single-server queueing loss system with the stationary ordinary input flow of customers. The random variables T_λ (the interarrival time) and T_μ (the service time of one customer) are independent and arbitraryly distributed.

We assume that the working server can fail (to give up) through random time T_ν. We count this time from the moment of the beginning of service. In a free state the server too can become inaccessible to customers through time T_{ν_0} after end of service. In both cases restoration (repair) of the server begins immediately after its falling out. We designate random time of restoration through T_γ and T_{γ_0} respectively. The customer, which was served during the moment of an falling out of the server, leaves system not served.

Assume also that random variables T_λ, T_μ, T_ν, T_{ν_0}, T_γ and T_{γ_0} are independent in aggregate, arbitraryly distributed and have finite expectations m_λ, m_μ, m_ν, m_{ν_0}, m_γ and m_{γ_0} respectively.

Enter states numbering of system: s_0 is a state when the server is free; s_1 denotes that the server works and s_2 denotes that the server is on repair (is inaccessible).

At first consider a case when repeated failure of the free server is forbidden. We assume that after a state s_2 of inaccessibility of the server, which has come as a result of interruption of a state s_0 and random time T_{γ_0} proceeded, following transition of system to a state s_2 is possible only after its stay in a state s_1. It means, that the chain of change of states $s_0 \to s_2 \to s_0 \to s_2$ is impossible.

Such four variants of chains of change of states, which describe a cycle between two consecutive stays of system in a state s_1, are possible:

$$C_1:\ s_1 \to s_0 \to s_1; \qquad C_2:\ s_1 \to s_0 \to s_2 \to s_0 \to s_1;$$
$$C_3:\ s_1 \to s_2 \to s_0 \to s_1; \qquad C_4:\ s_1 \to s_2 \to s_0 \to s_2 \to s_0 \to s_1.$$

Durations of cycles are respectively equal:

$$\tau_1 = T_{\mu\nu} + T_{0\mu\nu0}; \quad \tau_2 = T_{\mu\nu} + T_{0\mu\nu0} + T_{\gamma0} + T_{0\gamma\mu};$$
$$\tau_3 = T_{\mu\nu} + T_{\gamma} + T_{0\nu\nu0}; \quad \tau_4 = T_{\mu\nu} + T_{\gamma} + T_{0\nu\nu0} + T_{\gamma0} + T_{0\gamma\nu}.$$

Here

$$T_{\mu\nu} = \min\{T_\mu, T_\nu\}, \quad T_{0\mu\nu0} = \min\{T_{0\mu}, T_{\nu0}\}, \quad T_{0\nu\nu0} = \min\{T_{0\nu}, T_{\nu0}\},$$

$T_{0\mu}$ is a time of stay in a state s_0 in case of chain C_1 for system with an interdiction of failure in a state s_0; $T_{0\nu}$ is a time of stay in a state s_0 in case of chain C_3 for system with an interdiction of failure in a state s_0; $T_{0\gamma\mu}$ is a time of stay in a state s_0 after a state s_2 in case of chain C_2; $T_{0\gamma\nu}$ is a time of stay in a state s_0 second time in case of chain C_4.

Analyze work of system on very big time interval T. We take for a reference mark the moment of arrival of the next customer in the free server.

Let $N(T)$ be the number of customers, which have arrived in system in time T, $N_{\text{serv}}(T)$ is a number of the customers served for this time, $N_{\text{iru}}(T)$ is a number of customers which service has been interrupted in connection with an falling out of the server, then for great values T we have such approached equality

$$T \approx N(T)m_\lambda \approx N_{\text{serv}}(T)\big(m_{\mu\nu} + m_{0\mu\nu0} + P_{\text{iru}\,0\mu}(T)(m_{\gamma0} + m_{0\gamma\mu})\big) + \\ + N_{\text{iru}}(T)\big(m_{\mu\nu} + m_\gamma + m_{0\nu\nu0} + P_{\text{iru}\,0\nu}(T)(m_{\gamma0} + m_{0\gamma\nu})\big), \tag{1}$$

where

$$m_{\mu\nu} = M(T_{\mu\nu}), \quad m_{0\mu\nu0} = M(T_{0\mu\nu0}), \quad m_{0\gamma\mu} = M(T_{0\gamma\mu}),$$
$$m_{0\nu\nu0} = M(T_{0\nu\nu0}), \quad m_{0\gamma\nu} = M(T_{0\gamma\nu}),$$
$$P_{\text{iru}\,0\mu}(T) = \frac{N_{\text{iru}\,0\mu}(T)}{N_{\text{serv}}(T)}, \qquad P_{\text{iru}\,0\nu}(T) = \frac{N_{\text{iru}\,0\nu}(T)}{N_{\text{serv}}(T)},$$

$N_{\text{iru}\,0\mu}(T)$ is a number of the served customers after which service the server has failed in a state s_0, $N_{\text{iru}\,0\nu}(T)$ is a number of customers after which interruption of service as a result of failure in a state s_1, the server has again failed in a state s_0, not having begun service of the new customer.

Enter a designation:

$$P_{\text{iru}}(T) = \frac{N_{\text{iru}}(T)}{N_{\text{serv}}(T) + N_{\text{iru}}(T)}. \tag{2}$$

If limiting stationary process exists, then $P_{\text{iru}} = \lim\limits_{T\to\infty} P_{\text{iru}}(T) = P\{T_\nu < T_\mu\}$ is a probability of interruption of service in connection with an falling out of the server; $P_{\text{iru}\,0\mu} = \lim\limits_{T\to\infty} P_{\text{iru}\,0\mu}(T) = P\{T_{\nu0} < T_{0\mu}\}$ is a probability of an falling out of the server in a state s_0 which follows after a state s_1, which proceeded up to the end of service; $P_{\text{iru}\,0\nu} = \lim\limits_{T\to\infty} P_{\text{iru}\,0\nu}(T) = P\{T_{\nu0} < T_{0\nu}\}$ is a probability of an falling out of the server in a state s_0, which follows after a state s_2.

Equality (1) is the more precisely, than longer time interval T. Having defined from (2) $N_{\text{iru}}(T)$, and then from (1) having found relation $N_{\text{serv}}(T)/N(T)$ and having passed in it to a limit at $T \to \infty$ we can define stationary value of probability of service for the customer, which has arrived in system (relative capacity queueing system):

$$P_{\text{serv}} = \lim_{T\to\infty} \frac{N_{\text{serv}}(T)}{N(T)} = \frac{m_\lambda(1-P_{\text{iru}})}{M_{\mu\nu\gamma}}, \tag{3}$$

where

$$M_{\mu\nu\gamma} = m_{\mu\nu} + (1-P_{\text{iru}})\big(m_{0\mu\nu0} + (m_{\gamma0}+m_{0\gamma\mu})P_{\text{iru}\,0\mu}\big)+ \\ +P_{\text{iru}}\big(m_\gamma + m_{0\nu\nu0} + (m_{\gamma0}+m_{0\gamma\nu})P_{\text{iru}\,0\nu}\big).$$

The kind of the right part (3) allows to approve, that the necessary condition of existence of limiting stationary process is existence of finite expectations of random variables T_λ, T_μ, $T_{\mu\nu}$, $T_{0\mu\nu0}$, $T_{0\nu\nu0}$, T_γ, T_{γ_0}, $T_{0\gamma\mu}$ and $T_{0\gamma\nu}$.

Coming back to equality (1), we see, that time interval T consists of the sum of lengths of intervals T_0 (an idle time of the server), T_1 (a busy time of the server) and T_2 (time of repair of the server), that is $T \approx \sum T_0 + \sum T_1 + \sum T_2$, where

$$\sum T_0 \approx N_{\text{serv}}(T)\big(m_{0\mu\nu0} + P_{\text{iru}\,0\mu}(T)m_{0\gamma\mu}\big)+ \\ +N_{\text{iru}}(T)\big(m_{0\nu\nu0} + P_{\text{iru}\,0\nu}(T)m_{0\gamma\nu}\big); \quad \sum T_1 \approx \big(N_{\text{serv}}(T)+N_{\text{iru}}(T)\big)m_{\mu\nu}; \\ \sum T_2 \approx N_{\text{serv}}(T)P_{\text{iru}\,0\mu}(T)m_{\gamma0} + N_{\text{iru}}(T)\big(m_\gamma + P_{\text{iru}\,0\nu}(T)m_{\gamma0}\big).$$

After transition to a limit at $T\to\infty$ in relations $\sum T_i/T$ $(i=\overline{1,3})$ we receive formulas for probabilities of states of limiting stationary process:

$$p_0 = \lim_{T\to\infty}\frac{\sum T_0}{T} = \frac{P_{\text{serv}}}{m_\lambda(1-P_{\text{iru}})}\times \\ \times\big((1-P_{\text{iru}})(m_{0\mu\nu0}+P_{\text{iru}\,0\mu}m_{0\gamma\mu}) + P_{\text{iru}}(m_{0\nu\nu0}+P_{\text{iru}\,0\nu}m_{0\gamma\nu})\big) = \\ = \frac{(1-P_{\text{iru}})(m_{0\mu\nu0}+P_{\text{iru}\,0\mu}m_{0\gamma\mu}) + P_{\text{iru}}(m_{0\nu\nu0}+P_{\text{iru}\,0\nu}m_{0\gamma\nu})}{M_{\mu\nu\gamma}}; \\ p_1 = \lim_{T\to\infty}\frac{\sum T_1}{T} = \frac{P_{\text{serv}}\,m_{\mu\nu}}{m_\lambda(1-P_{\text{iru}})} = \frac{m_{\mu\nu}}{M_{\mu\nu\gamma}}; \\ p_2 = \lim_{T\to\infty}\frac{\sum T_2}{T} = \frac{(1-P_{\text{iru}})P_{\text{iru}\,0\mu}m_{\gamma0} + P_{\text{iru}}(m_\gamma + P_{\text{iru}\,0\nu}m_{\gamma0})}{M_{\mu\nu\gamma}}. \tag{4}$$

If a input flow of customers is Poisson stationary that is random variable T_λ is exponentially distributed with parameter λ, then owing to absence of contagion random variables $T_{0\mu}$, $T_{0\nu}$, $T_{0\gamma\mu}$ and $T_{0\gamma\nu}$ are exponentially distributed too with parameter λ. Thus,

$$m_{0\gamma\mu} = m_{0\gamma\nu} = 1/\lambda, \quad T_{0\mu\nu0} = T_{0\nu\nu0} = T_{\lambda\nu0} = \min\{T_\lambda, T_{\nu0}\}, \\ m_{0\mu\nu0} = m_{0\nu\nu0} = m_{\lambda\nu0} = M(T_{\lambda\nu0}); \\ P_{\text{iru}\,0\mu} = P_{\text{iru}\,0\nu} = P_{\text{iru}\,0} = P\{T_{\nu0} < T_\lambda\},$$

and from equalities (3), (4) we receive generalization of formulas of Sevastyanov [3] for system M/G/1/0 with falling out possibility

$$P_{\text{serv}} = \frac{1-P_{\text{iru}}}{M_\lambda}; \qquad p_0 = \frac{P_{\text{serv}}(P_{\text{iru}\,0}+\lambda m_{\lambda\nu0})}{1-P_{\text{iru}}} = \frac{P_{\text{iru}\,0}+\lambda m_{\lambda\nu0}}{M_\lambda}; \\ p_1 = \frac{\lambda P_{\text{serv}} m_{\mu\nu}}{1-P_{\text{iru}}} = \frac{\lambda m_{\mu\nu}}{M_\lambda}; \qquad p_2 = 1-p_0-p_1,$$

where $M_\lambda = (\lambda m_{\gamma 0} + 1)P_{\text{iru}\,0} + \lambda(m_{\mu\nu} + m_{\lambda\nu 0} + P_{\text{iru}} m_\gamma)$.

3. The analysis of random variables $\mathbf{T_{0\mu}}$, $\mathbf{T_{0\nu}}$, $\mathbf{T_{0\gamma\mu}}$ and $\mathbf{T_{0\gamma\nu}}$. Random variables $T_{0\mu}$, $T_{0\nu}$, $T_{0\gamma\mu}$ and $T_{0\gamma\nu}$ have the general that they set an interval of time when the system is free, and this time interval begins after interval T_β busy or inaccessibility of the server. For simplification of the analysis of these random variables it is entered for them the general designation:

$$T_0 = \begin{cases} T_{0\mu}, & \text{if } T_\beta = T_{\mu\nu}; \\ T_{0\nu}, & \text{if } T_\beta = T_{\mu\nu} + T_\gamma; \\ T_{0\gamma\mu}, & \text{if } T_\beta = T_{\mu\nu} + T_{0\mu\nu 0} + T_{\gamma 0}; \\ T_{0\gamma\nu}, & \text{if } T_\beta = T_{\mu\nu} + T_\gamma + T_{0\nu\nu 0} + T_{\gamma 0}. \end{cases} \tag{5}$$

Time T_0 depends on number of customers, which arrive in system in time T_β. Let $T_\lambda^{(k)}$ be the k-multiple composition of random variables T_λ, $T_\lambda^{(0)} = 0$, $T_\lambda^{(1)} = T_\lambda$, and $q_k = P(A_k) = P\{T_\lambda^{(k-1)} \le T_\beta < T_\lambda^{(k)}\}$ $(k = 1, 2, \dots)$. Then

$$T_0 = \begin{cases} T_\lambda - T_\beta, & \text{with probability } q_1; \\ T_\lambda^{(2)} - T_\beta, & \text{with probability } q_2; \\ \dots\dots\dots & \\ T_\lambda^{(k)} - T_\beta, & \text{with probability } q_k; \\ \dots\dots\dots & \end{cases}$$

Thus, random variable T_0 depends on occurrence of one and only one of pairwise disjoint events A_k, $(k = 1, 2, \dots)$ which form total group. Therefore an expectation $m_0 = M(T_0)$ it is calculated under the formula of a total expectation:

$$m_0 = \sum_{k=1}^{\infty} P(A_k) M(T_0 | A_k) = \sum_{k=1}^{\infty} q_k (k m_\lambda - m_\beta) = m_\lambda S_q - m_\beta;$$
$$S_q = \sum_{k=1}^{\infty} k q_k, \quad m_\beta = M(T_\beta). \tag{6}$$

Here we consider that $\sum_{k=1}^{\infty} q_k = 1$, since events A_k $(k = 1, 2, \dots)$ form total group of pairwise disjoint events.

If numerical series S_q converges, then the expectation m_0 exists. Since $S_q = M(X) + 1$ where X is a stationary value of number of customers, which arrive in system in time T_β, then convergence of series S_q means that an average of customers, which arrive in system during inaccessibility of server T_β is finite. In the monography [2, p. 238] for a case when the input flow of customers is recurrent, and random variables T_λ and T_β have absolutely continuous distribution functions, $M(X)$ is found by $h_\lambda(t)$, which is a density of renewal function for random variable T_λ:

$$M(X) = \int_0^\infty h_\lambda(t) \big(1 - F_\beta(t)\big)\, dt,$$

where $F_\beta(t)$ is a distribution function of random variable T_β.

Thus, if distribution functions of random variables T_λ and T_β are absolutely continuous, then

$$S_q = 1 + \int\limits_0^\infty h_\lambda(t)\big(1 - F_\beta(t)\big)\,dt. \qquad (7)$$

Now consider a case when the input flow of customers is regular, that is $T_\lambda = T = const$, and the formula (7) is inapplicable for finding S_q. Directly calculate the sum of series S_q, then

$$S_q = \sum_{k=1}^{\infty} k q_k = 1 + q_2 + q_3 + \cdots = 1 + \sum_{k=1}^{\infty}(q_{k+1} + q_{k+2} + \cdots) = 1 + \sum_{k=1}^{\infty}(1 - S_k),$$

where $S_k = \sum\limits_{i=1}^{k} q_i$. Since

$$S_k = \sum_{i=1}^{k} P\{\,T_\lambda^{(i-1)} \le T_\beta < T_\lambda^{(i)}\} = \sum_{i=1}^{k} P\{\,(i-1)T \le T_\beta < iT\} =$$

$$= \sum_{i=1}^{k}\big(F_\beta(iT) - F_\beta((i-1)T)\big) = F_\beta(kT),$$

then for a regular input flow of customers we have

$$S_q = 1 + \sum_{k=1}^{\infty}\big(1 - F_\beta(kT)\big).$$

For calculation of expectations $m_{0\mu\nu 0}$, $m_{0\nu\nu 0}$, $m_{0\gamma\mu}$ and $m_{0\gamma\nu}$, which enter into relations (3) and (4) for stationary characteristics of system, it is necessary to know distributions of random variables $T_{0\mu}$ and $T_{0\nu}$. According to (5) for this purpose it is enough to find the distribution of random variable T_0, which is time of stay in a free state for usual system G/G/1/0.

Let T_β be the time of inaccessibility of system for customers (in case of system G/G/1/0 without an opportunity of falling out $T_\beta = T_\mu$, that is equal to a service time of one customer). If distribution functions of random variables T_λ and T_β are absolutely continuous, then according to [2, p. 238] density of distribution of random variable $T_{\beta+0} = T_\beta + T_0$ (an interval of time between arrivals of customers in free system) is defined by the formula

$$p_{\beta+0}(t) = p_\lambda(t)F_\beta(t) + \int\limits_0^t \int\limits_0^\tau h_\lambda(y)p_\beta(\tau)p_\lambda(t-y)\,dy\,d\tau,$$

where $p_\lambda(t)$, $p_\beta(t)$ are density of distribution of random variables T_λ and T_β respectively.

Assume that $p_\lambda(t) \ne 0$, $p_\beta(t) \ne 0 \;\forall\; t \in [0, \infty)$ and $p_0(t)$ is a density of distribution of random variable T_0. Then by the formula of convolution for densities of distribution of random variables T_β and T_0 we have

$$\int\limits_0^t p_0(\tau)p_\beta(t-\tau)\,d\tau = p_{\beta+0}(t). \qquad (8)$$

Let functions $q_{\beta}(t) = dp_{\beta}(t)/dt$, $q_{\beta+0}(t) = dp_{\beta+0}(t)/dt$ are sectionally continuous $\forall\ t \in [0, \infty)$ and let $p_{\beta}(0) \neq 0$. Then after differentiation of both parts of equality (8) with respect to a variable t, we obtain Volterra integral equation of the second genus concerning function $p_0(t)$:

$$p_{\beta}(0)p_0(t) + \int_0^t q_{\beta}(t-\tau)p_0(\tau)\,d\tau = q_{\beta+0}(t). \tag{9}$$

The solution of this equation according to [4, p. 96] is such:

$$p_0(t) = \frac{q_{\beta+0}(t)}{p_{\beta}(0)} - \frac{1}{p_{\beta}(0)}\int_0^t R_{\beta}(t-\tau)q_{\beta+0}(\tau)\,d\tau. \tag{10}$$

Here $R_{\beta}(t)$ is a resolvent kernel $q_{\beta}(t-\tau)$ of the equation (9). It is found or by means of Laplace transformation, having applied it to the equation (9), or a method of the iterated kernels:

$$R_{\beta}(t) = \sum_{k=1}^{\infty} q_{\beta}^{(k)}(t),$$

where $q_{\beta}^{(k)}(t)$ is a k-multiple convolution of function $q_{\beta}(t)$, $q_{\beta}^{(1)}(t) = q_{\beta}(t)$.

If random variable T_{β} is exponentially distributed with parameter β, then the equation (9) is such

$$p_0(t) - \int_0^t \beta e^{-\beta(t-\tau)}p_0(\tau)\,d\tau = \frac{q_{\beta+0}(t)}{\beta}.$$

In this case $R_{\beta}(t) = \beta$, and from (10) it is received:

$$p_0(t) = \frac{q_{\beta+0}(t)}{\beta} + p_{\beta+0}(t) - p_{\beta+0}(0).$$

In particular, if random variable T_{λ} is distributed under Erlang law of the second order with parameter λ, then

$$p_0(t) = \left(\frac{\lambda^2(2\lambda+\beta)}{(\lambda+\beta)^2} + \frac{\lambda^2\beta t}{(\lambda+\beta)}\right)e^{-\lambda t} - \frac{\lambda^2(2\lambda+\beta)}{(\lambda+\beta)^2}e^{-(2\lambda+\beta)t}, \quad t > 0.$$

Find the distribution of random variable T_0 for a case of a regular input flow of customers when the formula (10) is inapplicable for finding $p_0(t)$. If $T_{\lambda} = T = const$, then possible values of random variable T_0 are concentrated in the interval $[0, T]$, and

$$T_0 = \begin{cases} T - T_{\beta}, & \text{with probability } q_1; \\ 2T - T_{\beta}, & \text{with probability } q_2; \\ \dots\dots\dots & \\ kT - T_{\beta}, & \text{with probability } q_k; \\ \dots\dots\dots & \end{cases}$$

where $q_k = P(A_k) = P\{(k-1)T \le T_\beta < kT\}$ $(k = 1, 2, \dots)$. Define conditional densities of distribution T_0 provided that $T_\beta \in [(k-1)T, kT)$. We assume that random variable T_β is continuous, and its density of distribution $p_\beta(t) \neq 0$ $\forall\ t \in (0, \infty)$.

Functions of distribution of random variables $Y_k = kT - T_\beta$ it is defined as

$$F_{Y_k}(t) = 1 - F_\beta(kT - t), \quad 0 \le t \le kT \quad (k = 1, 2, \dots),$$

and their densities of distribution are respectively equal

$$p_{Y_k}(t) = p_\beta(kT - t), \quad 0 < t < kT \quad (k = 1, 2, \dots).$$

Therefore conditional densities of distribution of random variable T_0 provided that $T_\beta \in [(k-1)T,\ kT)$ it is defined such as

$$p_{0k}(t) = \frac{p_{Y_k}(t)}{N_k} = \frac{p_\beta(kT - t)}{N_k}, \quad 0 < t < T \quad (k = 1, 2, \dots),$$

where $N_k = \int\limits_0^T p_\beta(kT - t)\, dt$.

By the formula of total probability

$$F_0(t) = \sum_{k=1}^{\infty} F_{0k}(t) q_k, \qquad 0 \le t \le T,$$

where $F_0(t)$ is a distribution function of random variable T_0, nd $F_{0k}(t) = \int\limits_0^t p_{0k}(\tau)\, d\tau$. From here we receive the formula for definition of density of distribution of random variable T_0

$$p_0(t) = \sum_{k=1}^{\infty} p_{0k}(t) q_k = \sum_{k=1}^{\infty} \frac{p_\beta(kT - t)}{N_k} q_k, \quad 0 < t < T. \tag{11}$$

After transition to a variable of integration $\tau = kT - t$ in integral, which defines normalizing constant N_k, we receive that

$$N_k = \int\limits_0^T p_\beta(kT - t)\, dt = \int\limits_{(k-1)T}^{kT} p_\beta(\tau)\, d\tau = q_k.$$

Thus, from (11) we have:

$$p_0(t) = \sum_{k=1}^{\infty} p_\beta(kT - t), \qquad 0 < t < T. \tag{12}$$

If function $p_\beta(t) \neq 0$ only for $t \in (a, \infty)$, where $a > 0$, but the relation $p_\beta(kT - t)/N_k$ does not depend on k, then we receive from (11):

$$p_0(t) = \frac{p_\beta(kT - t)}{N_k}, \qquad 0 < t < T. \tag{13}$$

For example, if the density of distribution of random variable T_β looks like $p_\beta(t)=\beta e^{-\beta(t-a)}$, $t>a\geq 0$, then

$$N_k=\int\limits_0^T \beta e^{-\beta(kT-t-a)}\,dt=e^{-\beta((k-1)T-a)}-e^{-\beta(kT-a)},$$

and by means of (13) we receive:

$$p_0(t)=\frac{\beta e^{-\beta(T-t)}}{1-e^{-\beta T}},\qquad 0<t<T.$$

The same result is received, if random variable T_β is exponentially distributed with parameter β.

It is convenient to use the formula (12) in particular if random variable T_β is distributed under generalized Erlang law of the arbitrary order. For example, when we have generalized Erlang law of the second order with parameters β_1 and β_2, then

$$p_0(t)=\frac{\beta_1\beta_2}{\beta_2-\beta_1}\left(\frac{e^{-\beta_1(T-t)}}{1-e^{-\beta_1 T}}-\frac{e^{-\beta_2(T-t)}}{1-e^{-\beta_2 T}}\right),\qquad 0<t<T.$$

4. An example of calculation of stationary characteristics of system. Let the input flow of customers is regular ($T_\lambda=\ T=const$), the random variables T_μ, T_ν, $T_{\nu 0}$ are exponentially distributed with parameters μ, ν, ν_0, respectively, and intervals of repair time are determined: $T_\gamma=\ const$, $T_{\gamma 0}=\ const$. We shall describe on this example the sequence of finding of parameters, which are present at formulas (3) and (4) for stationary characteristics of system.

All over again, considering relations (5), by the formula (13) we find distribution of random variables $T_{0\mu}$ and $T_{0\nu}$:

$$p_{\,0\mu}(t)=p_{\,0\nu}(t)=\frac{\alpha e^{-\alpha(T-t)}}{1-e^{-\alpha T}},\quad \alpha=\mu+\nu;\qquad 0<t<T.$$

Further we calculate expectations of random variables $T_{0\mu\nu 0}$ and $T_{0\nu\nu 0}$:

$$m_{\,0\mu\nu 0}=m_{\,0\nu\nu 0}=\int\limits_0^\infty tp_{\,0\mu}(t)\big(1-F_{\nu 0}(t)\big)\,dt+\int\limits_0^\infty tp_{\,\nu 0}(t)\big(1-F_{0\mu}(t)\big)\,dt=$$
$$=\frac{1}{1-e^{-\alpha T}}\left(\frac{e^{-\alpha T}+(\alpha T-\nu_0 T-1)e^{-\nu_0 T}}{\alpha-\nu_0}+\frac{1-(\nu_0 T+1)e^{-\nu_0 T}}{\nu_0}\right).$$

The following step this is a finding of expectations $m_{\,0\gamma\mu}$ and $m_{\,0\gamma\nu}$. Again considering (5), from (6) we receive:

$$m_{\,0\gamma\mu}=m_{\,\lambda}S_{q\gamma\mu}-(m_{\,\mu\nu}+m_{\,0\mu\nu 0}+m_{\,\gamma 0}),$$
$$m_{\,0\gamma\nu}=m_{\,\lambda}S_{q\gamma\nu}-(m_{\,\mu\nu}+m_{\,\gamma}+m_{\,0\nu\nu 0}+m_{\,\gamma 0}),$$

where

$$S_{q\gamma\mu}=\sum_{k=1}^\infty kq_{\,\gamma\mu k},\qquad S_{q\gamma\nu}=\sum_{k=1}^\infty kq_{\,\gamma\nu k},$$
$$q_{\,\gamma\mu k}=P\{(k-1)T\leq T_{\mu\nu}+T_{0\mu\nu 0}+T_{\gamma 0}<kT\},$$
$$q_{\,\gamma\nu k}=P\{(k-1)T\leq T_{\mu\nu}+T_\gamma+T_{0\nu\nu 0}+T_{\gamma 0}<kT\}.$$

For distributions, which we consider in this example,

$$m_\lambda = T, \quad m_\gamma = T_\gamma, \quad m_{\gamma 0} = T_{\gamma 0},$$

$$S_{q\gamma\mu} = s + e^{-\alpha(sT - T_{\gamma 0})} + \frac{\alpha e^{-\alpha T}\left(e^{-\alpha(sT-T_{\gamma 0})} - e^{(\alpha-\nu_0)(sT-T_{\gamma 0})}\right)}{(2\alpha-\nu_0)(1-e^{-\alpha T})} +$$

$$+\frac{\alpha\left(e^{-\nu_0(sT-T_{\gamma 0})} - e^{-\alpha(sT-T_{\gamma 0})}\right)}{(\alpha-\nu_0)(1-e^{-\alpha T})} + \frac{e^{-\alpha\left((s+1)T-T_{\gamma 0}\right)}}{1-e^{-\alpha T}} +$$

$$+\frac{\alpha e^{-\alpha\left((s+1)T-T_{\gamma 0}\right)}}{(1-e^{-\alpha T})^2}\left(\frac{e^{-\alpha T} - e^{-(\nu_0-\alpha)T}}{2\alpha-\nu_0} - \frac{1-e^{-(\nu_0-\alpha)T}}{\alpha-\nu_0}\right),$$

$$(s-1)T \le T_{\gamma 0} \le sT \qquad (s = 1, 2, \ldots).$$

The formula for $S_{q\gamma\nu}$ it is received from expression for $S_{q\gamma\mu}$, having replaced in this expression $T_{\gamma 0}$ on $T_\gamma + T_{\gamma 0}$.

At last we shall note that $m_{\mu\nu} = 1/(\mu+\nu)$, $P_{\text{iru}} = \nu/(\mu+\nu)$,

$$P_{\text{iru}\,0\mu} = P_{\text{iru}\,0\nu} = \frac{\alpha - \nu_0 + \nu_0 e^{-\alpha T} - \alpha e^{-\nu_0 T}}{(\alpha-\nu_0)(1-e^{-\alpha T})},$$

and stationary characteristics of system by formulas (3) and (4) are completely defined.

5. Absence of restrictions on number of failures of the free server. We consider only a case of the Poisson stationary input flow of customers when random variable T_λ is exponentially distributed with parameter λ. Then

$$m_\lambda = m_{0\gamma\mu} = m_{0\gamma\nu} = 1/\lambda, \quad T_{0\mu\nu 0} = T_{0\nu\nu 0} = T_{\lambda\nu 0} = \min\{T_\lambda, T_{\nu 0}\},$$

$$m_{0\mu\nu 0} = m_{0\nu\nu 0} = m_{\lambda\nu 0} = M(T_{\lambda\nu 0});$$

$$P_{\text{iru}\,0\mu} = P_{\text{iru}\,0\nu} = P_{\text{iru}\,0} = P\{T_{\nu 0} < T_\lambda\},$$

and the approached equality (1) is such

$$T \approx N(T)m_\lambda \approx N_{\text{serv}}(T)\left(m_{\mu\nu} + m_{\lambda\nu 0} + \pi_{\text{iru}\,0}(T)\right) +$$

$$+N_{\text{iru}}(T)\left(m_{\mu\nu} + m_\gamma + m_{\lambda\nu 0} + \pi_{\text{iru}\,0}(T)\right),$$

where

$$\pi_{\text{iru}\,0}(T) = P_{\text{iru}\,0}(T)\Big(m_{\gamma 0} + m_{\lambda\nu 0} + P_{\text{iru}\,0}(T)\big(m_{\gamma 0} + m_{\lambda\nu 0} +$$

$$+P_{\text{iru}\,0}(T)(m_{\gamma 0} + m_{\lambda\nu 0} + \cdots)\big)\Big) = (m_{\gamma 0} + m_{\lambda\nu 0})\sum_{k=1}^{\infty}\left(P_{\text{iru}\,0}(T)\right)^k =$$

$$= (m_{\gamma 0} + m_{\lambda\nu 0})\frac{P_{\text{iru}\,0}(T)}{1 - P_{\text{iru}\,0}(T)}.$$

For $T \to \infty$ from here we receive:

$$P_{\text{serv}} = \frac{1 - P_{\text{iru}}}{\lambda M_\infty}; \qquad p_0 = \frac{m_{\lambda\nu 0}}{M_\infty(1 - P_{\text{iru}\,0})};$$

$$p_1 = \frac{m_{\mu\nu}}{M_\infty}; \qquad p_2 = \frac{m_{\gamma 0}P_{\text{iru}\,0} + m_\gamma P_{\text{iru}}(1 - P_{\text{iru}\,0})}{M_\infty(1 - P_{\text{iru}\,0})}, \tag{14}$$

where

$$M_\infty = m_{\mu\nu} + m_{\lambda\nu 0} + m_\gamma P_{\rm iru} + (m_{\gamma 0} + m_{\lambda\nu 0})\frac{P_{\rm iru\,0}}{1-P_{\rm iru\,0}}, \qquad P_{\rm iru} = P\{T_\nu < T_\mu\}.$$

In case of exponentially distributions of random variables T_μ, T_ν, $T_{\nu 0}$, T_γ and $T_{\gamma 0}$ formulas (14) are the same as the relations received in [5, p. 362].

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