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INFLUENCE OF SMALL RANDOM DEVIATIONS FROM LINEAR FORM OF A PRE-FRACTURE ZONE ON CRACKING IN A CIRCULAR DISK

Abstract

A problem of mechanics on cracking in a solid circular disk is solved for the case when a coutour of prefailure strip has roughness (small deviations from linear form). On the base of perturbations method an effective design method is developed for the solution of a problem on cracking in a disk with indicated prefailure strip condition for defining limiting value of external loading is obtained on the basis of the obtained solution.

In real materials surfaces of prefailure strips (zones of weakened interparticle bonds of a material) have unevenness and curvatures because of structural and technological factors.

Problem statement. Let's consider a metalloc circular disk in operation process. As a disk will be loaded with force and thermal loads in operation process, there will arise prefailure jones in a material that we sumulate as domains of weakened interparticle bonds. Prefailure strip formed in disk exploiation process is assumed to be close to a linear form admitting only small deviations of strip line from straight one. We'll simulate the disk by a real brittle body in accordance with Leonov-Panasyuk moded [1]. In deformation process at some points of the disk there may appear thezones at which Hook's law for disk's material is not fulfilled, i.e. in these zones stresses exceed limit of elasticity of the material.

Since the indicated zones (material's interlayers) are small in comparison with the remaining part of the disk, then according to Leonov-Panasyuk madel we can remove them mentally and substitute them for sections whose surfaces interact by some law corresponding to the action of removed material where a disk's material is deformed beyond elasticuty point. In operation process there may arise prefailure zones (domains of weakened interparticle bonds), where a material will underge plastic deformation.

In the case under consideration arise of embryonic crac is a process of prefailure zone to the domain of ruined bonds between surfaces of disk's material.

Therewith the dimension of prefailure zone is not know beforehand and should be determined in solving a problem of fracture mechanics on nucleation of crack type deficiency.

As numerical experimental investigations show at initial stage prefailure zones represent narrow strerched layers. Let's consider a problem of mechanics on nucleation of crack in an entire circular disk assuming that a prefailure strip contour has roughness (small deviations from a linear form) (fig.1). As the centre of prefailure strip we locate an origin of local coordinate system $x_1O_1y_1$. Choice of a coordinate system and denotation are in fig.1.

The prefailure strip is assumed to be close to linear form assuming only small deviations of a strip from straight line $y_1 = 0$. Equation of prefailure strip zone is accepted in the form

$$y_1 = f_1(x_1) \qquad |x_1| \le l \tag{1}$$

Assume that plastic flow holds in prefailure strip for constant stress.

On the basis of accepted assumption on the form of prefailure strip line, the functions $f_1(x_1)$ and $f_1'(x_1)$ are small quantities.

Refer the disk to a polar system of coordinates $r\theta$ having choosen the origin of coordinates at the center of a circle L of radius R (fig.1.). Let's consider some arbitrary realization of rough surface of prefailure strip.

The boundary conditions of the considered problem of fracture mechanics on cracking in a metallic circular disk will be of the form:

$$\sigma_r = N(\theta); \qquad \tau_{r\theta} = T(\theta) \qquad \text{for} \quad r = R \tag{2}$$

on the surfaces prefailure zone

$$\sigma_n = \sigma_s \ ; \qquad \tau_{nt} = \tau_s \ , \tag{3}$$

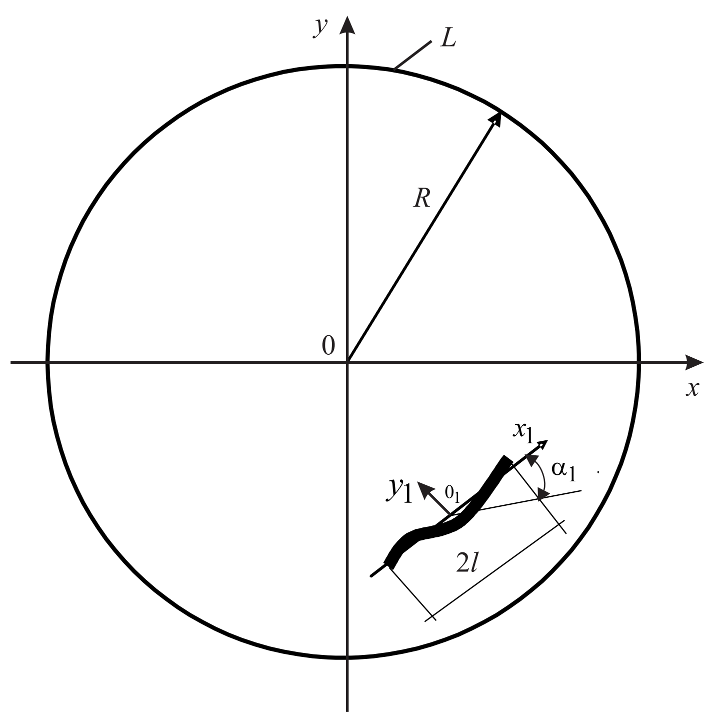

Fig. 1.

where σ_s is a tension yield point of disk's material; τ_s is a shear yield point of the material.

Solution of the boundary value problem. We represent the boundary conditions (3) on a surfaces prefailure strip in the following form

$$-\sigma_n^+ = \sigma_n^- = \sigma_{y_1}\cos^2\theta_1 + \sigma_{x_1}\sin^2\theta_1 - 2\tau_{x_1y_1}\sin\theta_1\cos\theta_1 \tag{4}$$

$$-\tau_{nt}^+ = \tau_{nt}^- = (\sigma_{y_1} - \sigma_{x_1})\sin\theta_1\cos\theta_1 - \tau_{x_1y_1}\left(\cos^2\theta_1 - \sin^2\theta_1\right)$$

where the sign "+" corresponds to the upper face of prefalure strip, "-" to the lower face of prefailure strip; θ_1 is an angle counted counter-clock wise from the axis O_1y_1 to external normal n of upper or lower face of prefailure strip.

Since the functions $f_1(x_1)$ and $f_1'(x_1)$ are small quantities, we can represent the function $f_1(x_1)$ in the form

$$f_1(x_1) = \varepsilon_1 H_2(x_1) \qquad |x| \le l, \tag{5}$$

where ε_1 is a small parameter.

We expand the stress tensor components σ_{x_1} , σ_{y_1} , $\tau_{x_1y_1}$ in a small parameter

$$\begin{aligned} \sigma_{x_1} &= \sigma_{x_1}^0 + \varepsilon_1\sigma_{x_1}^{(1)} + ... \\ \sigma_{y_1} &= \sigma_{y_1}^0 + \varepsilon_1\sigma_{y_1}^{(1)} + ... \\ \tau_{x_1y_1} &= \tau_{x_1y_1}^0 + \varepsilon_1\tau_{x_1y_1}^{(1)} + ... \end{aligned} \tag{6}$$

Expanding the expressions for stresses in the vicinity $y_1 = 0$ in series we find the values of stresses for $y_1 = f_1(x_1)$.

Applying the small parameter mathod, allowing for relations (6) we get the boundary conditions for $y_1 = 0,\ |x| \le l$:

at zero approximation

$$\sigma_r^{(0)} = N(\theta); \quad \tau_{r\theta}^{(0)} = T(\theta) \qquad \text{for} \qquad r = R \tag{7}$$

$$\sigma_{y_1}^{(0)} = \sigma_s; \quad \tau_{x_1y_1}^{(0)} = \tau_s \qquad \text{for} \qquad y_1 = 0,\ |x_1| \le l \tag{8}$$

Now, Let's pass to solution of the problem at zero approximation.

We can write boundary conditions (7) – (8) by means of Kolosov-Muskhileshvili [2] in the form of a boundary value problem for complex potentials $\Phi^{(0)}(z)$ and $\Psi^{(0)}(z)$ in the form

$$\Phi^{(0)}(\tau) + \overline{\Phi^{(0)}(\tau)} - e^{2i\theta}\left(\overline{\tau}\Phi'^{(0)}(\tau) + \Psi^{(0)}(\tau)\right) = N(\theta) - iT(\theta) \tag{9}$$

$$\Phi^{(0)}(t) + \overline{\Phi^{(0)}(t)} + \overline{t}\Phi'^{(0)}(t) + \Psi^{(0)}(t) = \sigma_s + i\tau_s \tag{10}$$

Here $\tau = R\ \exp(i\theta)$; t is an affix of points of surfaces prefailure zone.

We seek for the complex potential giving the solution of boundary value problem (9) – (10) in the form

$$\begin{aligned} \Phi^{(0)}(z) &= \Phi_0^{(0)}(z) + \Phi_1^{(0)}(z) + \Phi_2^{(0)}(z); \\ \Psi^{(0)}(z) &= \Psi_0^{(0)}(z) + \Psi_1^{(0)}(z) + \Psi_2^{(0)}(z); \end{aligned} \tag{11}$$

Here, when there is no prefailure strip, the potentials describe stress and displacements field in an continuous (defectless) disk at zero approximation.

We find the complex potentials $\Phi_1^{(0)}(z)$ and $\Psi_1^{(0)}(z)$ in the form

$$\Phi_1^{(0)}(z) = \frac{1}{2\pi}\int_{-l_1}^{l_1} \frac{g^0(t)\,dt}{t-z_1}; \tag{12}$$

$$\Psi_1^{(0)}(z) = \frac{1}{2\pi}e^{-2i\alpha_1}\int_{-l_1}^{l_1}\left[\frac{\overline{g^0(t)}}{t-z_1} - \frac{\overline{T}_1 e^{i\alpha_1}}{(t-z_1)^2}g^0(t)\right]dt\ ;$$

where $T_1 = te^{i\alpha_1} + z_1^0\ ;\ z_1 = e^{-i\alpha_1}\left(z - z_1^0\right)$

Here $g^0(t)$ is a desired function characterizing displacements when crossing through prefailure strip line. The unknown function $g^0(t)$ and complex potentials $\Phi_2^{(0)}(z)$ and $\Psi_2^{(0)}(z)$ should be determined from boundary conditions on the contour $L\,(r = R)$ and the surfaces prefailure strip.

On the contour L satisfying boundary condition of the disk by means of the functions $(11)-(12)$, after some transformations and calculations of appropriate integrals we find

$$\Phi_2^{(0)}(z) = \frac{1}{2\pi}\int_{-l}^{l}\left\{\left(\frac{1}{z\overline{T}_1 - 1} + \frac{1}{2}\right)\overline{T}_1 e^{i\alpha_1}g^0(t) + \right.$$

$$\left. + \left[\frac{T_1}{2} - \frac{z^2\overline{T}_1 - 2z + T_1}{\left(z\overline{T}_1 - 1\right)^2}\right]e^{-i\alpha_1}\overline{g^0(t)}\right\}dt\ ; \tag{13}$$

$$\Psi_2^{(0)}(z) = \frac{1}{2\pi}\int_{-l}^{l}\left[\frac{e^{i\alpha_1}\overline{T}_1^3}{\left(z\overline{T}_1 - 1\right)^2}g^0(t) + \right.$$

$$\left. + \left(z^2\overline{T}_1^2 + 4 - 3z\overline{T}_1 + zT_1\overline{T}_1^2 - 3T_1\overline{T}_1\right)\frac{\overline{T}_1 e^{-i\alpha_1}}{\left(z\overline{T}_1 - 1\right)^3}g^0(t)\right]dt\ ;$$

Now satisfying boundary condition (10) by the functions $(11)-(12)$ on the surfaces prefailure strip, after some transformations we get a complex singular integral equation for the unknown function $g^0(t)$

$$\int_{-l}^{l}\left[R(t,x_1)\,g^0(t) + S(t,x_1)\,\overline{g^0(t)}\right]dt = \pi\left[\sigma_s - i\tau_s + f_0^0(x_1)\right] \tag{14}$$

$$|x_1| \le l$$

Here $f_0^0(x_1) = -\left[\Phi_0^{(0)}(x_1) + \overline{\Phi_0^{(0)}(x_1)} + x_1\overline{\Phi_0^{\prime(0)}(x_1)} + \overline{\Psi_0^{(0)}(x_1)}\right]$

The quantities $R(t,x_1)$ and $S(t,x_1)$ are determined by the relations:

$$R(t,x_1) = \frac{e^{i\alpha_1}}{2}\left(\frac{1}{T_1 - X_1} + \frac{e^{-2i\alpha_1}}{\overline{T_1} - \overline{X_1}}\right) - \frac{e^{i\alpha_1}}{2}\left[\frac{X_1\overline{T}_1^{\,2}}{1 - X_1\overline{T}_1} + \right.$$

$$+\frac{\overline{X_1}^2 T_1-2\overline{X_1}+\overline{T_1}}{\left(1-T_1\overline{X_1}\right)^2}+e^{-2i\alpha_1}\frac{2X_1\left(T_1\overline{T_1}-1\right)+\overline{T_1}^2\left(\overline{X_1}+\overline{T_1}\right)\left(\overline{X_1}T_1-3\right)+4T_1}{\left(1-T_1\overline{X_1}\right)^3}\Bigg];$$

$$S\left(t,x_1\right)=\frac{e^{-i\alpha_1}}{2}\left(\frac{1}{\overline{T_1}-\overline{X_1}}+\frac{T_1-X_1}{\left(\overline{T_1}-\overline{X_1}\right)^2}e^{-2i\alpha_1}\right)-$$

$$-\frac{e^{-i\alpha_1}}{2}\left[\frac{T_1^2X_1}{1-T_1\overline{X_1}}+\frac{X_1^2\overline{T_1}-2X_1+T_1}{\left(1-X_1\overline{T_1}\right)^2}+\frac{T_1^2\left(X_1+T_1\right)e^{-2i\alpha_1}}{\left(1-T_1\overline{X_1}\right)^2}\right];$$

$$X_1=x_1e^{i\alpha_1}+z_1^0 \qquad (15)$$

In the case of internal prefailure zone, it is necessary to add additional equality

$$\int_{-l}^{l} g^0\left(t\right)dt=0 \qquad (16)$$

to the singular integral equation.

If we represent the unknown function $g^0\left(x_1\right)$ and loading function $f_0^0\left(x_1\right)$ in the form

$$g^0\left(x_1\right)=\upsilon^{(0)}\left(x_1\right)-iu^{(0)}\left(x_1\right);\qquad f^0\left(x_1\right)=\sigma_0^{(0)}\left(x_1\right)-i\tau_0^{(0)}\left(x_1\right),$$

then from one complex integral equation after separating real and imaginary parts we'll get two real integral equations for defining $\upsilon^{(0)}\left(x_1\right)$ and $u^{(0)}\left(x_1\right)$. We do the same with condition (16) and as a result get

$$\int_{-l}^{l}\upsilon^{(0)}\left(t\right)dt=0;\qquad \int_{-l}^{l}u^{(0)}\left(t\right)dt=0 \qquad (17)$$

The solution of integral equations are sought in a class of everywhere bounded functions (stresses).

Applying algebralization procedure [3, 4, 5, 6] instead of each singular integral equation provided (19) we get a finite algebraic system consisting of M equations for approximate values of desired functions $\upsilon^{(0)}\left(t_m\right)$ and $u^{(0)}\left(t_m\right)$ $(m=1,2,...,M)$ at hodes, respectively.

For closeness of the obtained algebraic equations we need equations expressing stress finiteness condition in the vicinity of the points $x_1=\pm l$. Writing the stress finiteness condition

$$\sum_{m=1}^{M}\left(-1\right)^{M+m}\upsilon^{(0)}\left(t_m\right)tg\frac{2m-1}{4M}\pi=0;\quad \sum_{m=1}^{M}\left(-1\right)^{m}\upsilon^{(0)}\left(t_m\right)ctg\frac{2m-1}{4M}\pi=0 \quad (18)$$

$$\sum_{m=1}^{M}\left(-1\right)^{M+m}u^{(0)}\left(t_m\right)tg\frac{2m-1}{4M}\pi=0;\quad \sum_{m=1}^{M}\left(-1\right)^{m}u^{(0)}\left(t_m\right)ctg\frac{2m-1}{4M}\pi=0$$

we get two closed finite algebraic systems. Because of unknown size of prefailure zone the systems of algebraic equations turned to be non-linear. For the given external loading the obtained systems allow to find stress-strain state of a circular disk in the presence of prefailure in a disk at zero approximation.

The solution of the problem in the first approximation was constructed in a similar way.

The successive approximations method was used to solve algebraic systems.

As a result of solution of the problem we find

$$g(x_1) = g^0(x_1) + \varepsilon_1 g^{(1)}(x_1); \tag{19}$$

$$l = l^{(0)} + \varepsilon_1 l^{(1)};$$

Using the solution of the problem on a plastic sliding strip we calculate the displacements on the surfaces of prefailure zone.

$$-\frac{1+k_0}{2\mu}\int\limits_{-l_1}^{x_1} g(x_1)\,dx_1 = \upsilon(x_1,0) - iu(x_1,0), \tag{20}$$

where $\upsilon(x_1,0) = \upsilon^+(x_1,0) - \upsilon^-(x_1,0) \qquad u(x_1,0) = u^+ - u^-$

Permutations on the of prefailure strip for $x_1 = x_0$ will be

$$-\frac{1+k_0}{2\mu}\int\limits_{-l}^{x_0} g(x_1)\,dx_1 = \upsilon(x_0,0) - iu(x_0,0) \tag{21}$$

Using variable substitution and substituting the integral by means of Gauss quadrature formula by the sum, we find

$$-\frac{1+k_0}{2\mu}\cdot\frac{\pi l}{M} g(x_1)\sum_{m=1}^{M_1} g(t_m) = \upsilon(x_0,0) - iu(x_0,0) \tag{22}$$

where M_1 is the number of nodes contained in the segment $(-l, x_0)$.

Considering the relations for $g^{(0)}(t_m)$ and $g^{(1)}(t_m)$ from (1.270) we find

$$\upsilon(x_0,0) = -\frac{1+k_0}{2\mu}\cdot\frac{\pi l}{M}\sum_{m=1}^{M_1}\left[\upsilon^{(0)}(t_m) + \varepsilon_1\upsilon^{(1)}(t_m)\right]$$

$$u(x_0,0) = -\frac{1+k_0}{2\mu}\cdot\frac{\pi l}{M}\sum_{m=1}^{M_1}\left[u^{(0)}(t_m) + \varepsilon_1 u^{(1)}(t_m)\right]$$

For modulus vector displacements on the surfaces of prefailure strip, for $x_1 = x_0$ we have

$$V_0 = \sqrt{(\upsilon^+ - \upsilon^-)^2 + (u^+ - u^-)^2} = \frac{1+k_0}{2\mu}\cdot\frac{\pi l_1}{M}\sqrt{A_1^2 + B_1^2}, \tag{23}$$

where

$$A = \sum_{m=1}^{M_1}\left[\upsilon^{(0)}(t_m) + \varepsilon_1\upsilon^{(1)}(t_m)\right]; \tag{24}$$

$$B=\sum_{m=1}^{M_1}\left[u^{(0)}\left(t_m\right)+\varepsilon_1 u^{(1)}\left(t_m\right)\right].$$

Thus, the condition defining limiting value of external load at which crack appears will be

$$\frac{1+k_0}{2\mu}\cdot\frac{\pi l_1}{M}\sqrt{A^2+B^2}=\delta_c \tag{25}$$

Solution analysis. Joint solution a nonlinear system of equations consisting of the system substituting integral equation with stress boundedness conditions at the points $x_1=\pm l$ and (25) allows to determine the size of prefailure zone, the values of the desired function $g\left(x_1\right)$ at the nodes, quantity of the extermal load at which crack appears in a circular metallic dick.

In calculations the function $f_1\left(x_1\right)$ describing the roughness of prefailure strip line was accepted to be a stationary random function with zero mean value $\left\langle f_1\left(x_1\right)\right\rangle=0$ and known dispersion $D\left[f_1\left(x_1\right)\right]$

$$f_1\left(x_1\right)=\sum_{n=0}^{\infty}\left(A_n\cos\frac{\pi n}{2l}x_1+B_n\sin\frac{\pi n}{2l}x_1\right),\qquad |x_1|\leq l,$$

where $A_n,\ B_n$ are non-corelated random variebles satisfying the condition

$$\langle A_n\rangle=\langle B_n\rangle=0,\qquad D\left(A_n\right)=D\left(B_n\right)=D_n$$

Correlation function $K\left(x_1\right)$ of the random function $f_1\left(x_1\right)$ may be represented in the form

$$K\left(x_1\right)=\sum_{n=0}^{\infty}D_n\cos\frac{\pi n}{2l}x_1$$

The case of two and several prefailure zones is considered similarly with obvious changes. If the end of a prefailure strip arrives at a circular surface of a disk, necessity of fulfilment of additional condition drops out.

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